

SCRIPTORES LOGARITHMICI.

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SCRIPTORES LOGARITHMICI;

OR

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

LOGARITHMS,

MENTIONED IN DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW
EDITION OF SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS
CONNECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME V.

LONDON:

PRINTED BY R. WILKS, CHANCERY-LANE;

AND SOLD BY J. WHITE, FLEET-STREET.

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THE
P R E F A C E.

I NOW present to the Publick a fifth Volume of this Collection of Mathematical Tracts, intituled, *Scriptores Logarithmici*, of the contents of which I will here endeavour to give my Readers a full and distinct account.

Of the Contents of the First Tract in this Volume.

The first Tract in this volume is an Investigation of Sir Isaac Newton's famous Binomial Theorem in the case of the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$, or the reciprocal of $1+x^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, or the n th root of the binomial quantity $1+x$, in which quantity $1+x^{\frac{-1}{n}}$ the letter x denotes any quantity not greater than 1, and n denotes any whole number whatsoever. Now Sir Isaac Newton discovered that the same series which gives us the value of $1+x^m$, or of the m th power of the binomial quantity $1+x$, when the letter m denotes any whole number whatsoever, to wit, the series

$$1 + \frac{m}{1} \times x + \frac{m}{1} \times \frac{m-1}{2} \times x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times x^3 + \frac{m}{1} \times$$

VOL. V. a $\frac{m-1}{2}$

$\frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times x^5 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times x^6 + \&c$, would, by substituting $-\frac{1}{n}$ in its several terms instead of m , produce a series which

would be equal to the quantity $1 + x)^{-\frac{1}{n}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$. This, however, is

by no means an easy, or obvious, consequence of the former theorem; and yet Sir Isaac Newton has no where given us the least hint of any demonstration of it, and I am therefore inclined to think that he found it out only by conjecture and trial, and contented himself with the probability arising from the successful results of his trials as a ground for believing that his new theorem would be universally true, without seeking any further proof that it was so. His method of proceeding I conceive to have been as follows. He probably began by substituting $-\frac{1}{n}$ (the index of the power of the binomial quantity

$1+x$ in the quantity $1+x)^{-\frac{1}{n}}$) instead of m in the terms of the series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2} \times x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times x^4 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times x^5 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \frac{m-5}{6} \times x^6 + \&c$, which he already

knew to be equal to $1+x)^m$, and thereby obtained the series $1 - \frac{1}{n}x + \frac{+1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times x^5 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times \frac{5n+1}{6n} \times x^6 \&c. ad infinitum$, and conjectured that this

second series would be equal to the quantity $1+x)^{-\frac{1}{n}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$, or $\frac{1}{\sqrt[n]{1+x}}$. He then supposed x to be equal to some particular number that

was

was less than 1, as, for instance, to $\frac{3}{10}$, and the binomial quantity $1 + x$ to be consequently equal to $1 + \frac{3}{10}$, or 1.300,000,000, and extracted two, or three, different roots of this number 1.300,000,000, as, for example, it's square root, it's cube root, and it's fifth root, by the common methods employed for this purpose, and then divided 1 by each of these roots; by which divisions he would obtain quotients that would be equal to the three quantities $\frac{1}{\sqrt{2}\sqrt[10]{1.300}}$, $\frac{1}{\sqrt[3]{1.300}}$, and $\frac{1}{\sqrt[5]{1.300}}$ respectively, or to the three quantities $\frac{1}{\sqrt[2]{1+\frac{3}{10}}}$, $\frac{1}{\sqrt[3]{1+\frac{3}{10}}}$, and $\frac{1}{\sqrt[5]{1+\frac{3}{10}}}$, respectively, or to the three quantities $\frac{1}{1+\frac{3}{10}^{\frac{1}{2}}}$, $\frac{1}{1+\frac{3}{10}^{\frac{1}{3}}}$, and $\frac{1}{1+\frac{3}{10}^{\frac{1}{5}}}$, respectively, or to the three quantities $\frac{-1}{1+\frac{3}{10}^{\frac{1}{2}}}$, $\frac{-1}{1+\frac{3}{10}^{\frac{1}{3}}}$, and $\frac{-1}{1+\frac{3}{10}^{\frac{1}{5}}}$, respectively. He then, I presume, would compute the three different values of the second series above-mentioned, to wit, the series $1 - \frac{1}{n} \times x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \&c$, arising from a supposition that x was equal to $\frac{3}{10}$, and that n was, first, equal to 2, and, secondly, equal to 3, and, thirdly, equal to 5; and, when he found that these three values of the said series were, respectively, equal to the values he had before obtained of the three quantities $\frac{1}{\sqrt[2]{1+\frac{3}{10}}}$, $\frac{1}{\sqrt[3]{1+\frac{3}{10}}}$, and $\frac{1}{\sqrt[5]{1+\frac{3}{10}}}$, or $\frac{1}{1+\frac{3}{10}^{\frac{1}{2}}}$, $\frac{1}{1+\frac{3}{10}^{\frac{1}{3}}}$, and $\frac{1}{1+\frac{3}{10}^{\frac{1}{5}}}$, or $\frac{-1}{1+\frac{3}{10}^{\frac{1}{2}}}$, $\frac{-1}{1+\frac{3}{10}^{\frac{1}{3}}}$, and $\frac{-1}{1+\frac{3}{10}^{\frac{1}{5}}}$, he would conclude, that the said second series would also be equal to the quantity $\frac{-1}{1+x} \frac{1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$, or $\frac{1}{\sqrt[n]{1+x}}$, in all other cases, or when x was equal to any other quantity different

different from $\frac{3}{10}$, (that was less than 1,) and when n was equal to any other whole number whatsoever, different from 2, 3, and 5. The probability, however, of the truth of this general conclusion arising from the success of such trials in particular examples, is very different from the satisfactory knowledge of it by means of a strict mathematical demonstration, which (as the subject will admit of such a demonstration,) might reasonably have been expected to have been given us by its great Inventor. This defect I have therefore endeavoured to supply in this first tract of the present volume, by proceeding in the following manner.

In the first place, I have substituted $-\frac{1}{n}$ instead of m in the terms of the series $1 + \frac{m}{1} A x + \frac{m-1}{2} B x^2 + \frac{m-2}{3} C x^3 + \frac{m-3}{4} D x^4 + \frac{m-4}{5} E x^5 + \frac{m-5}{6} F x^6 + \frac{m-6}{7} G x^7 + \&c$, (which is equal to $(1+x)^m$, when m is any whole number whatsoever,) and have shewn that the Series resulting from such substitution would be $1 - \frac{1}{n} A x + \frac{n+1}{2n} \times B x^2 - \frac{2n+1}{3n} \times C x^3 + \frac{3n+1}{4n} \times D x^4 - \frac{4n+1}{5n} \times E x^5 + \frac{5n+1}{6n} \times F x^6 - \frac{6n+1}{7n} \times G x^7 + \&c$ *ad infinitum*. We are therefore to prove that this series will always be equal to the quantity $\frac{-1}{1+x}^{\frac{1}{n}}$, or $\frac{1}{1+x}^{\frac{1}{n}}$, or $\frac{1}{\sqrt[n]{1+x}}$, when the letter n represents any whole number whatsoever. This substitution of $-\frac{1}{n}$ instead of m in the first series $1 + \frac{m}{1} A x + \frac{m-1}{2} B x^2 + \frac{m-2}{3} C x^3 + \frac{m-3}{4} D x^4 + \&c$. is performed in Art. 2, pages 5 and 6.

In order to discover the series that will be equal to the quantity $\frac{-1}{1+x}^{\frac{1}{n}}$, or $\frac{1}{1+x}^{\frac{1}{n}}$, it is observed in Art. 3, page 7, that it has been already shewn in the second volume of this Collection of Tracts, called *Scriptores Logarithmici*,

page

page 237, art. 47, that the quantity $1 + x^{\frac{1}{n}}$, or the denominator of the fraction $\frac{1}{1 + x^{\frac{1}{n}}}$, is equal to the infinite Series $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \&c$, and therefore that the quantity $\frac{-1}{1 + x^{\frac{1}{n}}}$, or $\frac{1}{1 + x^{\frac{1}{n}}}$, will be equal to the fraction

$$\frac{1}{1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \&c, ad infinitum,}$$

or to the quotient of the division of 1 by the infinite Series $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \&c, ad infinitum$. Then it is

observed that, from the nature of the operation of Division, the first term of this Quotient must be 1, and the second, and third, and fourth, and other following terms of it must involve in them the quantities $x, x^2, x^3, \&c$, or the several integral powers of x in their natural order, without interruption, and consequently that the said Quotient will be a series of quantities of which the first term will be 1, and the second and other following terms will consist of certain numeral co-efficients multiplied into the several powers of x in their natural order, without any interruption, to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c, ad infinitum$; so that, if we denote the said numeral co-efficients, (which are at first unknown,) by the capital letters B, C, D, E, F, G, H, &c, the said Quotient will be equal to the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c, ad infinitum$, of which the second term Bx , and all the following terms, are to be connected with the first term 1 either by subtraction or addition, and consequently are to have either the sign — or the sign + prefixed to them. And

further it is shewn that, since the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1 + x^{\frac{1}{n}}}$, (to which the said Quotient, or infinite Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, is equal,) is less than 1, the second term Bx of the said Series must necessarily be subtracted from the first term, and consequently must be marked with the sign

sign —, and therefore the said Quotient will be equal to the Series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c, ad infinitum$, in which the co-efficients B, C, D, E, F, G, H, &c, are at first unknown, and the signs — and + that are to be prefixed to the third term Cx^2 and the following terms $Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, are at first undetermined. These observations are made in Art. 3, pages 7 and 8.

Then in Art. 4, pages 9 and 10, the operation of dividing 1 by the Series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} \times xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \&c ad infinitum$ is performed and exhibited, and the first five terms of the Quotient are found to be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n} \times xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4$; so that we may now conclude that the quantity $\frac{-1}{1+x\frac{1}{n}}$, or $\frac{1}{1+x\frac{1}{n}}$, is equal to the Series $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n} \times xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4, \&c$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n} \times Bx^2 - \frac{2n+1}{3n} \times Cx^3 + \frac{3n+1}{4n} \times Dx^4, \&c, ad infinitum$, which agrees with the Series derived before from the Series $1 + \frac{m}{1}x + \frac{m}{1} \times \frac{m-1}{2}x^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}x^3 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}x^4 + \&c$ (which is equal to $1+x\frac{m}{1}$), in the first five terms of the said Series. But, “what will be the values of the remaining terms after the first five, and which of the two signs + and — is to be prefixed to each of them,” remains still to be determined.

The latter of these two points is then determined in Art. 7, 8, and 9, in pages 13, 14, and 15, by observations made on the several steps of the foregoing operation of Division, and it is shewn that the signs to be prefixed to the following terms $Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, \&c ad infinitum$ will be alternately — and +, and consequently that the foregoing Quotient (which is equal

equal to $\frac{1}{1+x^n}$, or $\frac{1}{1+x^n}$,) will be equal to the Series $1 - \frac{1}{n} A x + \frac{n-1}{2n} B x^2 - \frac{2+1}{3n} C x^3 + \frac{3n+1}{4n} \times D x^4 - F x^5 + G x^6 - H x^7 + I x^8 - K x^9 + L x^{10} - \&c.$

Then, in order to make the reasonings used in Art. 8, &c to prove that the signs $-$ and $+$ are to be prefixed to the terms $F x^5$, $G x^6$, $H x^7$, $I x^8$, &c alternately, be the more easily understood, the single small letters $b, c, d, e, f, g, h, \&c$ are substituted in the infinite Series $1 + \frac{1}{n} x - \frac{1}{n} \times \frac{n-1}{2n} x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \&c$, (which is equal to $\frac{1}{1+x}$, and is the divisor of 1 in the foregoing operation of division,) instead of the several coefficients of $x, x^2, x^3, x^4, x^5, x^6, x^7$, &c in that series, so as to convert it into the Series $1 + b x - c x^2 + d x^3 - e x^4 + f x^5 - g x^6 + h x^7 - \&c$, and the operation of division is performed over-again in Art. 10, pages 16 and 17, with the said Series $1 + b x - c x^2 + d x^3 - e x^4 + f x^5 - g x^6 + \&c$ for a divisor; and by this operation, so performed, it appears that the Quotient of the division will be equal to the following Series, to wit,

$$\begin{array}{l}
 1 - b x + c \\
 \quad + b^2 \} \times x^3 \quad \left. \begin{array}{l} - d \\ - 2 b c \\ - b^3 \end{array} \right\} \times x^3 \quad \left. \begin{array}{l} + e \\ + 2 b d \\ + c^2 \\ + 3 b^2 c \\ + b^4 \end{array} \right\} \times x^4 \\
 \\
 \left. \begin{array}{l} - f \\ - 2 b e \\ - 2 c d \\ - 3 b^2 d \\ - 3 b c^2 \\ - 4 b^3 c \\ - b^3 \end{array} \right\} \times x^5 \quad \left. \begin{array}{l} + g \\ + 2 b f \\ + 2 c e \\ + 3 b^2 e \\ + d^2 \\ + 6 b c d \\ + 4 b^3 d \\ + c^3 \\ + 6 b^2 c^2 \\ + 5 b^4 c \\ + b^6 \end{array} \right\} \times x^6 - \&c. \quad \text{This manner of}
 \end{array}$$

exhibiting

exhibiting the said operation of division makes it very apparent that, to whatever number of terms the Quotient should be continued, the signs to be prefixed to them must be alternately — and +.

And afterwards in Art. 12, pages 19, 20, and 21, the same division of 1 by the Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + \&c$ is exhibited again in a conciser form by substituting the single capital letters B, C, D, E, F, G, &c instead of the co-efficients of $x, x^2, x^3, x^4, x^5, x^6, \&c$ in the last Quotient, by which it appears plainly that the said Quotient will be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6, \&c$, in which the signs — and + are prefixed to the second and third, and other following terms alternately, and likewise that the same alternate succession of the signs — and + must continue in all the following terms of the said Quotient, to whatever number of terms it may be continued.

By these different methods of exhibiting this operation of Division of 1 by the Series which is equal to $\frac{1}{1+x} \frac{1}{n}$ it seems to be fully and satisfactorily proved that, to whatever number of terms the Quotient of the said division, (which is equal to the quantity $1 + x \frac{-1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$;) be continued, it's second term Bx and the following terms Cx², Dx³, Ex⁴, Fx⁵, Gx⁶, &c will be marked with the sign — and the sign + alternately.

And in the third, or last, way of exhibiting the foregoing operation of division, it appears that the values of B, C, D, E, F, G, H, I, K, L, &c are equal to the following quantities; to wit,

- 1st, B is = b ;
- 2dly, C is = $c + bB$;
- 3dly, D is = $d + cB + bC$;
- 4thly, E is = $e + dB + cC + bD$;
- 5thly, F is = $f + eB + dC + cD + bE$;
- 6thly, G is = $g + fB + eC + dD + cE + bF$;
- 7thly, H is = $h + gB + fC + eD + dE + cF + bG$;
- 8thly, I is = $i + hB + gC + fD + eE + dF + cG + bH$;
- 9thly, K is = $k + iB + hC + gD + fE + eF + dG + cH + bI$;
- and, 10thly, L is = $l + kB + iC + hD + gE + fF + eG + dH + cI + bK$;

and

and so of the following co-efficients M, N, O, P, Q, &c, the law of the continuation of these quantities being sufficiently manifest. It would therefore be possible, by computing the values of these several compound quantities one after another, to obtain the values of as many of the co-efficients B, C, D, E, F, G, H, I, K, L, &c of the powers of x in the Series, or Quotient $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - \&c$ (which is equal to the quantity $\frac{1}{1+x^{\frac{1}{n}}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$) as we please. But the labour of making these computations after the first five terms $1 - Bx + Cx^2 - Dx^3 + Ex^4$ (which have been already found to be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4$), would be so great as to render this method of obtaining these values almost impracticable.

After this first method of discovering the terms of the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum*, (which is equal to the quantity $\frac{1}{1+x^{\frac{1}{n}}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$), by means of an operation of division, I proceed to investigate them by means of an operation of Multiplication in the following manner.

Since the quantity $\frac{1}{1+x^{\frac{1}{n}}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$, is equal to the quantity

the Series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \&c$, or

the Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$, and is also equal to the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, it follows that these two latter quantities must be equal to each other, and consequently that, if we multiply the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ into the Series $1 + bx - cx^2 + dx^3 - ex^4 +$

$fx^5 - gx^6 + bx^7 - \&c$, the product of the said multiplication will be equal to 1. Now this product will be equal to the following compound quantity, consisting of several horizontal lines of terms placed one under the other, to wit, the compound quantity

$$\left. \begin{array}{l} 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c \\ - Bx - bBx^2 + cBx^3 - dBx^4 + eBx^5 - fBx^6 + gBx^7 - \&c \\ + Cx^2 + bCx^3 - cCx^4 + dCx^5 - eCx^6 + fCx^7 - \&c \\ - Dx^3 - bDx^4 + cDx^5 - dDx^6 + eDx^7 - \&c \\ + Ex^4 + bEx^5 - cEx^6 + dEx^7 - \&c \\ - Fx^5 - bFx^6 + cFx^7 - \&c \\ + Gx^6 + bGx^7 - \&c \\ - Hx^7 - \&c \end{array} \right\} \text{Therefore}$$

this compound quantity will be equal to 1. And hence it follows that we shall have the following equations for determining the values of B, C, D, E, F, G, H, &c, or the co-efficients of x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c in the terms of the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, which

is equal to the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1+x)^{\frac{1}{n}}}$, to wit,

1st, $B = b$,

2ndly, $C = c + bB$,

3dly, $D = d + cB + bC$,

4thly, $E = e + dB + cC + bD$,

5thly, $F = f + eB + dC + cD + bE$,

6thly, $G = g + fB + eC + dD + cE + bF$,

and 7thly, $H = b + gB + fC + eD + dE + cF + bG$,

and so of the rest; which are the very same equations which were obtained for this purpose by the third, or last, method of expressing the Quotient of the foregoing division of 1 by the Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$. So that this method of obtaining the values B, C, D, E, F, G, H, &c in the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, (which is equal to the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1+x)^{\frac{1}{n}}}$) by means of an

operation

operation of Multiplication, is a confirmation of the former Method of obtaining the said values by means of an operation of Division.

I then, in art. 16, 17, 18, &c. - - - 25, pages 25, 26, 27, &c. - - - 43, resolve the five first of the foregoing equations, to wit, the equations $B = b$, $C = c + bB$, $D = d + cB + bC$, $E = e + dB + cC + bD$, and $F = f + eB + dC + cD + bE$, and find the values of B , C , D , E , and F , to be as follows, to wit, B to be $(= b) = \frac{1}{n}$, and C to be $(= \frac{1}{n} \times \frac{n+1}{2n}) = \frac{n+1}{2n} \times B$, and D to be $(= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}) = \frac{2n+1}{3n} \times C$, and E to be $(= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}) = \frac{3n+1}{4n} \times D$, and F to be $(= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n}) = \frac{4n+1}{5n} \times E$. Therefore the first six terms of the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - \&c.$, (which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), will be $1 - \frac{1}{n} x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times x^5$, or $1 - \frac{1}{n} Ax + \frac{n+1}{2n} Bx^2 - \frac{2n+1}{3n} \times Cx^3 + \frac{3n+1}{4n} \times Dx^4 - \frac{4n+1}{5n} \times Ex^5$, of which the first five terms are the same that were found before by means of the operation of Division.

After investigating these six first terms of the Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, \&c.$ (which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), by this second Method, or by means of an operation of Multiplication, I have derived from the said Method a demonstration that in all the terms $Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, \&c.$ *ad infinitum*, that follow the first six terms $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5$, as well as in the five terms $Bx, Cx^2, Dx^3,$

Dx^3 , Ex^4 , and Fx^5 , the signs $+$ and $-$ will follow each other alternately, as had been before demonstrated from the first Method of investigation. This Demonstration is contained in Art. 29 and 30, pages 45, 46, 47, 48, and 49.

I then proceed to exhibit a third Method of investigating the terms of the Series $1 - Bx$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c, (which is equal to the quantity $\frac{1}{1+x} \frac{1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), without making use of the Series $1 + bx - cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + \&c$, or $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n+1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \frac{6n-1}{7n} Gx^7 - \&c$, (which is equal to $\frac{1}{1+x} \frac{1}{n}$), but only of the Series which is equal to an integral power of a residual quantity, which is the first and simplest case of Sir Isaac Newton's Theorem, which extends to residual quantities (such as $1 - x$ or $1 - y$), as well as to binomial quantities, such as $1 + x$. This third Method is as follows.

Since the whole of the Series 1 , Bx , Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c, or $1 - Bx$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c, (which is equal to the quantity $\frac{1}{1+x} \frac{1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), is less than it's first term 1 , it may be subtracted from the said first term. Let it be so subtracted. And the remainder will be the Series $+ Bx$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c, or Bx , Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c. Let this remainder, or new Series, be denoted by the Letter y . Then will $1 - y$ be equal to the Series $1 - Bx$, Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , &c, and consequently to the fraction $\frac{1}{1+x} \frac{1}{n}$. Therefore the n th power of the residual quantity $1 - y$ will be equal to the n th power of the fraction $\frac{1}{1+x} \frac{1}{n}$, that is, to the fraction $\frac{1}{1+x}$.

But,

But, by the residual Theorem in the case of integral powers, the n th power of $1 - y$ is equal to the Series $1 - \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}yy - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times y^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times y^5 + \&c$; and the fraction $\frac{1}{1+x}$ is equal to the Series $1 - x + x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum*, as will appear by the division of it's numerator 1 by it's denominator $1 + x$, without any reference to the binomial Theorem.

Therefore the Series $1 - \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}yy - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times y^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times y^5 + \&c$ will be equal to the Series $1 - x + x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum*; or, if, for the sake of brevity, we put $b = \frac{n}{1}$, and $c = \frac{n}{1} \times \frac{n-1}{2}$, and $d = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, and $e = \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $f, g, h, i, k, \&c$ for the co-efficients of $y^6, y^7, y^8, y^9, y^{10}, \&c$, we shall have the Series $1 - by + cyy - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + iy^8 - ky^9 + \&c$ equal to the Series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + \&c$ *ad infinitum*.

Further, since y is equal to the Series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, it follows that yy will be equal to the square of the said Series, and that y^3 will be equal to the cube of the said Series, and that $y^4, y^5, y^6, y^7, \&c$ will be equal to the fourth, fifth, sixth, seventh, and other following powers of the said Series. Let these powers of the said Series be raised by multiplication; and let the serieses thereby obtained, and which will be equal to $yy, y^3, y^4, y^5, y^6, y^7, \&c$, be inserted, instead of yy, y^3, y^4, y^5, y^6 , and $y^7, \&c$, respectively, in the terms $cyy, dy^3, ey^4, fy^5, gy^6, hy^7, \&c$ of the last equation $1 - by + cyy - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + \&c = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$, and let the said Series itself be inserted in the same equation instead of y in

y in the term by ; and we shall thereby convert the said equation into another very complicated equation of which the first term on each side of the mark of equality will be 1, and all the other terms on both sides will be the powers of x in their regular order, to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, multiplied into certain numeral co-efficients derived from the co-efficients, $b, c, d, e, f, g, h, \&c$, (which are already known,) and the co-efficients $B, C, D, E, F, G, H, \&c$, which are not yet known, but may be determined by the resolution of several successive simple equations. The operations of raising the successive powers of the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, as far as the seventh power, by multiplication, are performed in Art. 34, pages 52, and 53; and the complicated equation arising from the before-mentioned substitution of this series and its powers instead of $y, y^2, y^3, y^4, y^5, y^6$, and y^7 , in the equation $1 - by + cy^2 - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + \&c = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$, is set down in Art. 35, page 54. And then the values of the three co-efficients B, C , and D , and the signs $-$ and $+$ that are to be prefixed to the three terms Bx, Cx^2 , and Dx^3 , in the Series 1, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (which is equal to the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$), are determined by the resolution of as many separate simple equations, in Art. 36, 37, 38, 39, 40, and 41, pages 55, 56, 57, - - - 67; and it is found that the four first terms of the said Series 1, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ (which is equal to the quantity $\frac{-1}{1+x^{\frac{1}{n}}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$), will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \frac{2n+1}{3n}Cx^3$, as they were found to be by the two former Methods of investigating them.

This third Method of investigating the terms of the Series 1, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (which is equal to the quantity $1 + x^{\frac{-1}{n}}$, or $\frac{1}{1+x^{\frac{1}{n}}}$) becomes, after the computation of the three or four first terms of it, intolerably laborious, on account of the great number of terms that are contained

tained in the several simple equations that are to be resolved in order to determine the values of the following co-efficients E, F, G, H, &c, which makes it almost impossible to resolve them. I therefore think this third Method of finding the terms of the said Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, less convenient than either of the two former methods.

But neither of these three methods of investigating the terms of the Series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$ (which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), will enable us to discover the law of the continuation of the co-efficients B, C, D, E, F, G, H, I, K, L, &c in the remaining terms of the Series that come after those which we have actually investigated by the resolution of several successive simple equations in the manner above-described. I have therefore, for this purpose, had recourse to a fourth method of investigating this Series, which I have derived from the method employed in the second Volume of this Collection of Tracts called *Scriptures Logarithmici*, page 237, Art. 47, to investigate the Series $1 + \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n} \times x^5 - &c \text{ ad infinitum}$, or $1 + \frac{1}{n} A x - \frac{n-1}{2n} B x^2 + \frac{2n-1}{3n} C x^3 - \frac{3n-1}{4n} D x^4 + \frac{4n-1}{5n} E x^5 - &c, \text{ ad infinitum}$, which is equal to the quantity $\frac{1}{1+x} \frac{1}{n}$, or $\sqrt[n]{1+x}$; which method was either the Method given by Mr. John Landen for that purpose in his residual Analysis, or derived from it.

This fourth Method of finding the terms of the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + &c$, (which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or $\frac{1}{1+x} \frac{1}{n}$), consists of a great number of Algebraïck operations and substitutions, which are set forth very fully and distinctly in Art. 46, 47, 48, and 49, pages

pages 70, 71, 72, 73, 74, 75, and 76, and which cannot easily be described intelligibly in a more summary manner of stating them, and from which we at last obtain the following equation; to wit, the simple Series $1 - \frac{1}{n} x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum* is = to the following compound series, consisting of two horizontal rows of terms placed one under the other, to wit, the Series

$$1 - 2nCx + 3nDx^2 - 4nEx^3 + 5nFx^4 - 6nGx^5 + 7nHx^6 - \&c \\ + x - 2nCx^2 + 3nDx^3 - 4nEx^4 + 5nFx^5 - 6nGx^6 + \&c,$$

in which the Law of the continuation of the terms in both the rows (to whatever number of terms the said rows may be continued,) is very manifest. And from this equation it is easy to derive the following separate simple equations for determining the values of the co-efficients C, D, E, F, G, H, I, K, L, M, &c, to whatever number of terms we may chuse to investigate the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + \&c$; to wit, the equations

$$\frac{1}{n} = 2nC - 1, \text{ by the resolution of which we may obtain the value of } C;$$

$$C = 3nD - 2nC, \text{ by the resolution of which we may obtain the value of } D;$$

$$D = 4nE - 3nD, \text{ by the resolution of which we may obtain the value of } E;$$

$$E = 5nF - 4nE, \text{ by the resolution of which we may obtain the value of } F;$$

$$F = 6nG - 5nF, \text{ by the resolution of which we may obtain the value of } G;$$

$$G = 7nH - 6nG, \text{ by the resolution of which we may obtain the value of } H;$$

$$H = 8nI - 7nH, \text{ by the resolution of which we may obtain the value of } I;$$

$$I = 9nK - 8nI, \text{ by the resolution of which we may obtain the value of } K;$$

$$K = 10nL - 9nK, \text{ by the resolution of which we may obtain the value of } L;$$

and

and $L = 11nM - 10nL$, by the resolution of which we may obtain the value of M ;

and as many more simple equations of the same kind as we may think necessary. And the values of $C, D, E, F, G, H, I, K, L, M$, &c, resulting from the resolutions of these equations, will be as follows; to wit, $C = \frac{1}{n} \times \frac{n+1}{2n}$, or $C = \frac{n+1}{2n} \times B$, and $D = \frac{2n+1}{3n} \times C$, $E = \frac{3n+1}{4n} \times D$, $F = \frac{4n+1}{5n} \times E$, $G = \frac{5n+1}{6n} \times F$, $H = \frac{6n+1}{7n} \times G$, $I = \frac{7n+1}{8n} \times H$, $K = \frac{8n+1}{9n} \times I$, $L = \frac{9n+1}{10n} \times K$, and $M = \frac{10n+1}{11n} \times L$, &c.

The four first of these equations are resolved in Art. 50, 51, 52, and 53, pages 77, 78, 79, 80, 81, and 82; and in the following articles 54 and 55, pages 82, 83, and 84, it is shewn that the Law of continuation, by which the four co-efficients C, D, E , and F , are generated from the co-efficient B , or $\frac{1}{n}$, and each other, "to wit, by multiplying B, C, D , and E into the generating fractions $\frac{n+1}{2n}, \frac{2n+1}{3n}, \frac{3n+1}{4n}$, and $\frac{4n+1}{5n}$, in which every new generating fraction is derived from that which immediately preceeds it by adding n to both it's numerator and it's denominator," will extend to all the following co-efficients F, G, H, I, K, L, M , &c *ad infinitum*. And hence it is concluded with certainty, in Art. 56, page 85, that the quantity $\frac{-1}{1+x}^{\frac{1}{n}}$, or $\frac{1}{1+x}^{\frac{1}{n}}$, will be equal to the Series $1 - \frac{1}{n} Ax + \frac{n+1}{2n} Bx^2 - \frac{2n+1}{3n} Cx^3 + \frac{3n+1}{4n} Dx^4 - \frac{4n+1}{5n} Ex^5 + \frac{5n+1}{6n} Fx^6 - \frac{6n+1}{7n} Gx^7 + \frac{7n+1}{8n} Hx^8 - \frac{8n+1}{9n} Ix^9 + \frac{9n+1}{10n} Kx^{10} - \frac{10n+1}{11n} Lx^{11} + \&c$ *ad infinitum*, or, in other words, that Sir Isaac Newton's binomial Theorem is true in the case of the quantity $\frac{-1}{1+x}^{\frac{1}{n}}$,

or the fraction $\frac{1}{1+x}^{\frac{1}{n}}$, as well as in the first and simplest case of $1+x^n$, or of any integral power of the binomial quantity $1+x$, and in the case of $\frac{1}{1+x}^{\frac{1}{n}}$, or $\sqrt[n]{1+x}$, or of any root of the said binomial quantity.

In this last, or fourth, method of obtaining the values of the co-efficients C, D, E, F, G, H, &c, it is evident that the latter co-efficients F, G, H, &c, are obtained with nearly the same ease as the first co-efficient C, to wit, by multiplying E, F, and G, &c, into the generating fractions $\frac{4n+1}{5n}$, $\frac{5n+1}{6n}$, $\frac{6n+1}{7n}$, &c, respectively, in the same manner as C is obtained by multiplying $\frac{1}{n}$, or B, into the generating fraction $\frac{n+1}{2n}$; whereas in the three former methods of obtaining the values of these co-efficients, and more especially in the third method, the labour of investigating every new co-efficient is much greater than that of investigating the co-efficient immediately preceeding it, and increases so fast that the difficulty of investigating the values of the two co-efficients G and H in the third Method would be almost insuperable. This fourth Method, therefore, of obtaining the values of these co-efficients is, in a practical view, very greatly superiour to either of the three former methods of obtaining them; though the processes employed in it to obtain the compound Series consisting of two rows of terms, which is set down at the end of Art. 49, are very numerous, and the reasonings that accompany them are seemingly somewhat remote and subtle. But the great facility with which the co-efficients, C, D, E, F, G, H, &c, (to whatever number we may think proper to continue them,) may be obtained in this Method must be considered as an ample compensation for the great number of Algebraick operations, and for the subtlety and remoteness of the reasonings made use of in investigating the said Series.

This fourth Method of investigating the terms of the Series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum*, which is equal
to

to the quantity $\overline{1+x}^{\frac{-1}{n}}$, or to the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$, has been drawn-up (as I have already acknowledged,) in exact imitation of the investigation that is given of the infinite Series that is equal to $\overline{1+x}^{\frac{1}{n}}$ in the second volume of the *Scriptores Logarithmici*, pages 225, 226, 227, &c. - - 241, and which is either the same with, or derived from, the investigation that had been given of that series in a more concise manner by the late learned Mr. John Landen of Peterborough: so that the reader of these Tracts is in some degree indebted to Mr. Landen for the present investigation, as well as for that relating to $\overline{1+x}^{\frac{1}{n}}$ in the said former volume. But neither Mr. Landen, nor any other writer of Algebra whose works have fallen under my observation, has ever, as far as I can recollect, given an express investigation of the Binomial Theorem in this case of it, or when the index of the power of $1+x$ is $-\frac{1}{n}$, or the quantity to be expressed in an infinite Series of simple terms is $\overline{1+x}^{\frac{-1}{n}}$, or the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$, or $\frac{1}{\sqrt[n]{1+x}}$. Yet, as this famous Theorem has now been published during more than an hundred years, it seems to be high time that it should be demonstrated in all it's cases.

Of the Contents of the Second Tract in this Volume.

The second Tract in this Volume is a Latin Tract intitled *Analysis Fluxionum*, written by the learned Dr. William Hales, Doctor of Divinity in the University of Dublin, and formerly Fellow of Trinity College, and Professor of Oriental Languages in that university, but now rector of the parish of Killesandra in the province of Ulster and diocese of Meath. The design of this Discourse is to vindicate Sir Isaac Newton's Doctrine of Fluxions from the objections that have been made to it's accuracy and perspicuity by Dr. Berkeley, the famous Bishop of Cloyne in Ireland, and other able writers about the year 1735, and by Mr. John

Landen of Peterborough in the year 1758, and some French mathematicians of later times. And the main ground of this vindication consists in an endeavour to shew that *the Doctrine of the Prime and Ultimate Ratios of variable Quantities*, (which Sir Isaac Newton has laid down in the Introduction to his great work called *The Principia*, and has made the foundation of his Method of Fluxions,) is, in truth, the same thing with the Doctrine of *the Limits of the Ratios of variable Quantities*, but expressed in other and fewer words. And, if this be true, it certainly will follow that Sir Isaac's Doctrine of Prime and Ultimate Ratios must be accurate, because the Doctrine of the Limits of Ratios certainly is so; and then the Method of Fluxions (which is built upon the Doctrine of Prime and Ultimate Ratios,) must be accurate likewise. But the identity of *the Doctrine of Prime and Ultimate Ratios* with *the Doctrine of the Limits of Ratios* is not very apparent: and, though, in a few passages of his works, Sir Isaac seems to consider them as the same, yet, in general, he uses the language of the Doctrine of Indivisibles, or infinitely small quantities, which he calls *quantitates quàm minimas*, or *quantitates nascentes vel evanescentes*, as having no magnitude, and yet having certain proportions to each other; which gave a fair handle to Bishop Berkeley to observe that, according to Sir Isaac Newton's manner of speaking of them, Fluxions must be considered as *the ghosts of departed quantities*. Sir Isaac Newton's object in introducing his Doctrine of Prime and Ultimate Ratios, and his Method of Fluxions, which he built upon it, was to avoid the very long and tedious demonstrations *ex absurdo*, (similar to that of Euclid in the 12th book of his Elements, Prop. 2, in which he proves that the areas of two different circles are to each other in the same proportion as the squares of their diameters,) which he must have resorted to in treating of the areas of curvilinear figures, their tangents, and radiusses of curvature, and other properties, if he had adhered to the full and accurate Method of writing on those subjects that was employed by Euclid and Archimedes and Apollonius, and the other antient geometricians, in explaining the like subjects. And he certainly did obtain the advantage of brevity in his demonstrations by so doing; but not without a great loss of perspicuity: as is often the case with writers who are very studious of brevity, according to the observation of Horace, *Brevis esse laboro, obscurus fio*. As a proof of this want of perspicuity in laying-down

down the principles of his Method of Fluxions, I shall here transcribe a most important passage from the first page of his Introduction to his learned Treatise on the *Quadrature of Curves*, in which he defines what he means by *Fluxions* and *Fluents*, and states the method he makes use of to discover the proportion of one Fluxion to another. This passage is in these words.

Considerando igitur quod quantitates æqualibus temporibus crescentes et crescendo genitæ, pro velocitate majori vel minori quâ crescunt et generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum, vel incrementorum, quibus generantur: et has motuum, vel incrementorum, velocitates nominando fluxiones, et quantitates genitas nominando fluentes, incidi paulatim annis 1665 et 1666 in Methodum fluxionum quâ hîc usus sum in Quadraturâ Curvarum.

Fluxiones sunt quàm proximè ut fluentium augmenta æqualibus temporis particulis quàm minimis genitæ; et, ut accuratè loquar, sunt in primâ ratione augmentorum nascentium: exponi autem possunt per lineas quasunque quæ sunt ipsis proportionales.

Upon the words "*ut accuratè loquar*" in this passage I have, in my copy of Newton's *Quadrature of Curves*, written the following note, which contains all that I have to say upon the subject. *Hoc non est accuratè loqui. Nam nulla est prima ratio augmentorum nascentium; ut rectè observavit vir doctissimus et acutissimus, Georgius Berkeley, Episcopus Cloyniensis in Hiberniâ. Sed, si augmenta illa habent aliquam magnitudinem finitam, quantumvis parvam, ratio eorum ad se invicem non erit eorum prima ratio, quoniam augmenta ista potuissent esse etiâ minora quàm sunt: Si v. r. augmenta illa nullam omninò habeant magnitudinem, nullam poterunt habere rationem inter se. Quantitatum enim non-existentium nullæ sunt proprietates. Sed, cum auctor accuratè loqui voluit, debuerat substituisse, pro ratione Primâ augmentorum nascentium, rationem quæ est Limes, vel terminus, à quo incipiunt rationes augmentorum parvorum, sed tamèn finitorum, quantitatum genitarum. Hæc*
substi-

substitutione factâ, auctoris argumentum rectè procedit. Vide Scripta Benjamin Robins de hac re.

Est verò tota hæc Fluxionum Doctrina valdè subtilis et à simplicitate Geometriæ quodammodò aliena, ut quæ contemplationem motûs (quæ ad aliam pertinet Scientiam, scilicet, Mechanicam, Geometriâ difficiliorem,) in eam sine necessitate introduxit. Dico sine necessitate, quoniam nihil est quod per hanc doctrinam Fluxionum, in motûs contemplatione fundatam, inveniri potest, quod non et sine hac doctrinâ potuisset inveniri per doctrinam rationum quæ sunt Limites, sive termini, rationum quas habent inter se augmenta valdè parva, seu partes valdè parvæ, sed finitæ, ex quarum appositione quantitates generari possunt, seu in quas quantitates genitæ dividi et resolvi possunt; idque eadem brevitate quâ per hanc Fluxionum doctrinam, majri verò cum simplicitate et conceptûs facilitate. Vide Scriptores Logarithmicos, tomum primum, paginam 95, in quâ Keplerus refert Judicium doctorum quorundam Geometrarum in Germaniâ circâ A.D. 1621 de methodo quâ Neperus explicaverat inventionem suam perutilem Logarithmorum, quam “in figmento motûs cujusdam Geometrici niti querebantur, cujus lubricitas et fluxibilitas inepta esset in quâ solidus ille stylus rationis demonstrationumque firmum poneret vestigium.” Idem ferè, ut mihi videtur, merito dici potest de hac inventionem Fluxionum à Newtono traditâ.

Indeed the very disputes that have arisen amongst Mathematicians about the grounds of Sir Isaac Newton's Method of Fluxions, and likewise about Mr. Leibnitz's Method of infinitely small Differences, are a proof that neither of those Methods has been set forth by it's Inventor with perspicuity; or such men as Bishop Berkeley and Mr. Landen, and many other men of great genius and learning in the mathematicks, could not have objected to them. For none of them have objected to the demonstration of the second proposition of the 12th book of Euclid's Elements, nor to Archimedes's Quadrature of the Parabola.

Dr. Pemberton, who conversed much with Sir Isaac Newton in his old age, and superintended the publication of the third edition of his *Principia*, informs us (in a passage that is cited at length by Dr. Hales in his first Appendix to his *Analysis Fluxionum*, in pages 162 and 163 of the present volume,) “that

Sir

Sir Isaac particularly recommended Mr. Huygens's style and manner of writing, and thought him the most elegant of any mathematical writer of modern times, and the truest imitator of the antients; and that he always professed himself a great admirer of the taste of the antients and their mode of demonstration, and even censured himself for not following them yet more closely than he had done; and spoke with regret of his mistake at the beginning of his mathematical studies, in applying himself to Des Cartes and other Algebraick writers, before he had considered the Elements of Euclid with that attention which so excellent a writer deserved." This is an interesting anecdote, and does great honour to Sir Isaac Newton's memory, as it is a proof both of the soundness of his judgement and of his great candour and modesty; and we have only to regret that he did not entertain these opinions in the earlier part of his life, and give us an edition of his Method of Fluxions, or, rather, of the Geometry of Curve Lines, without any mention of velocity, and grounded solely on the Limits of the Ratios of the small, but finite, parts, into which curve lines and curvilinear areas and solids may be conceived to be divided, and likewise an edition of his *Principia*, or, at least, of some of the most important parts of it, written in the clear and accurate manner of Mr. Huygens's admirable Treatise, intitled *Horologium Oscillatorium, or De Motu Penduli*, which was published in the year 1673, or only fourteen years before the first edition of the *Principia*, and seemed to be the proper model to be followed by all succeeding writers on the subject of rational Mechanicks, or the doctrine of the motion of bodies.

This Discourse of Dr. Hales, called *Analysis Fluxionum*, begins in page 91, and ends in page 156. And Dr. Hales has added to it two Appendixes; of which the first is entituled *De Analysis antiquâ*, and takes up only five pages, to wit, pages 159, 160, 161, 162, and 163; and the second is intitled *De athere vibratorio—De Modo Sentiendi—et De Ente Supremo*, and extends through 38 pages, beginning in page 167, and ending in page 204. The third, or last, part of this second Appendix, which is intitled *De Ente Supremo*, begins in page 196 and ends in page 204, and contains many curious and important passages from very antient authors, tending to shew, "That the belief of one Supreme, Almighty,

Almighty, Being, the Cre-ator and Governour of the World, had prevailed amongst the different nations of mankind in the most antient times, before the introduction of Polytheism, with which they were afterwards so generally over-run.

Dr. Hales, a considerable time after the foregoing *Analysis Fluxionum*, and the two Appendixes to it, had been printed-off, sent me from Ireland some corrections and additions to it, which are printed at the end of the present volume, in pages 847, 848, 849, &c. - - - 859. And amongst these additions will be found a very large extract from the pamphlet published in the year 1735 by Mr. Benjamin Robins, intituled *A Discourse concerning the nature and certainty of Sir Isaac Newton's Methods of Fluxions, and of Prime and Ultimate Ratios*, which seems to have been the ablest tract that has ever been published on the subject.

Of the Contents of the third Tract in the present Volume.

The third Tract in this Volume is a Letter sent me by James Glenie, Esq. of the Province of New Brunswick in North America, containing a Demonstration of Sir Isaac Newton's Binomial and Residual Theorems in the cases of the quantities $\overline{1+x}^m$, $\overline{1+x}^{\frac{m}{n}}$, or $\overline{1+x}^{\frac{p}{q}}$, and $\overline{1-x}^m$, $\overline{1-x}^{\frac{m}{n}}$, or $\overline{1-x}^{\frac{p}{q}}$, derived from the nature of ratios and Euclid's definition of the proportionality of four quantities in the 5th book of his Elements, and extended to cases in which the letters m and n , or p and q , shall denote not only whole numbers, but any surd numbers, such as $\sqrt{2}$ or $\sqrt{3}$, or $\sqrt[3]{2}$, or $\sqrt[3]{3}$, or any other quantities whatsoever that are incommensurable to 1. This letter is contained in pages 207, 208, 209, &c. - - - 216.

Of

Of the Contents of the fourth Tract in the present Volume.

The fourth Tract in the present Volume is *Dr. Halley's Discourse on Compound Interest*, which was first published in the year 1705, in the Introduction to Mr. Henry Sherwin's Mathematical Tables, and was afterwards published again in the following editions of those Tables. The present Edition is reprinted from the Third Edition of those Tables published by Mr. William Gardiner in the year 1741. It is a very learned tract, and gives good rules for the solution of all the questions that can be put relating to Compound Interest. But it is very concisely, and, in my opinion, obscurely written, like others of Dr. Halley's Tracts on subjects of Algebra: and the rules he has given for the Solution of the three most difficult questions that can be put upon this subject are given without any demonstration, or investigation. Yet the investigations of them are very abstruse and difficult to find; insomuch that for many years, (I believe, for 56 years, or from the year 1705 to the year 1761;) no investigations of them appear to have been published in any books of Arithmetick or Algebra, but calculators seem to have adopted them and made use of them, without any knowledge of the manner of obtaining them, and merely upon the authority of Dr. Halley's declaration of their truth. There were, however, a few persons of superiour sagacity and deeply skilled in Algebra, who were able to penetrate the veil of mystery in which their author had thought proper to involve them; and one of these persons was the famous Algebräist Mr. Abraham De Moivre. For, after Dr. Halley's death in the year 1742, there was found amongst his papers a Letter to him from Mr. De Moivre, in which he tells him, "that he has hit upon a right Method of demonstrating his Theorem for finding the rate of Interest on Annuities," and then describes very fully and clearly his Method of proceeding for this purpose, which produces at length the same Algebräick expression of the value of the rate of Interest, (which was the quantity sought,) as had been given by Dr. Halley's rule. This Letter has no date to it, but was probably written soon after the first publication of Dr. Halley's Discourse on Compound Interest (which was in the year 1705 in

the Introduction to the first Edition of Sherwin's Tables of Logarithms,) when that Discourse, from the accuracy which was found to belong to the abstruse and undemonstrated Algebräick expressions given in it, must naturally have attracted the attention of Mathematicians in general, and particularly of so excellent an Algebräist as Mr. De Moivre, and have excited them to seek for Methods by which they might be regularly investigated. But, whatever may have been the time at which Dr. Halley received this Letter from Mr. De Moivre, he never published it to the world, any more than any investigations of these abstruse expressions that had been invented by himself; and, even after his death, this Letter seems never to have been published till the year 1761, when it was inserted in a periodical publication that was begun in that year, under the title of *The Mathematical Magazine, and Philosophical Repository*, and sold by J. Wilkie, at the Bible in St. Paul's church-yard, but was discontinued, for want of encouragement, after the publication of only five numbers. Nor does it appear that before this publication of Mr. De Moivre's Letter in the year 1761 any other investigations of these abstruse expressions of Dr. Halley had been published by any other person.

This Letter of Mr. De Moivre to Dr. Halley, with some remarks on it, I have published in a subsequent part of the present volume, in pages 520, 521, 522, &c. - - - 530.

These abstruse Algebräick expressions, given us by Dr. Halley, seemed therefore to be, even at the present time, fit objects of the attention of Mathematicians, and worthy of a fuller investigation than they had yet received: and with this view my learned friend, Professor Vilant, of the University of Saint Andrew's in Scotland, expressed a wish that I would take them into consideration, and endeavour either to investigate them anew in a fuller and clearer manner than had hitherto been done, or set forth the investigations that might be already given of them in such a manner as to render them more intelligible and satisfactory than they had hitherto appeared, and that I would insert the
result

result of my endeavours on this subject in this volume of the *Scriptores Logarithmici*, the doctrine of Compound Interest being a branch of Mathematicks that is nearly connected with Logarithms, and cannot well be managed without their continual assistance. In consequence of this suggestion I began to read with attention this crabbed Discourse of Dr. Halley, and to endeavour to investigate those three undemonstrated expressions. But I found this a very difficult task, and therefore consulted my learned friend, Mr. Morgan, the Actuary of the Equitable Society for Assurances on Lives, in Chatham Square, (who is deeply skilled in this branch of Mathematicks,) upon the subject, and desired to know if he could furnish me with a Method of investigating these expressions. And he, in the most ready and obliging manner, furnished me with investigations of them, which, I apprehend, he had found-out by his own exertions, and without the assistance of any other person: for, when I afterwards communicated to him Mr. De Moivre's Letter to Dr. Halley on this subject, (of which Professor Vilant had sent me a copy in manuscript, taken from the *Mathematical Magazine and Philosophical Repository* above-mentioned, which had been published in the year 1761,) he informed me that he had never met with that Letter before.

These investigations of Mr. Morgan I now applied myself to study with attention, and, when I had thoroughly understood them and made myself master of them, I endeavoured to express them anew in a fuller and plainer manner than he had done, by inserting every step of the reasonings employed in them separately and distinctly, and setting down at length all the Algebraick operations that were necessary to be performed in them, so as to render them as easy to be understood as the nature of the subject, (which was very subtle and abstruse,) and the number of processes and Algebraick operations of which they consisted, (which was very great,) would admit; and I resolved to insert them, when thus expanded, in the present Volume, under the title of *Notes* to this Discourse of Dr. Halley: and, on account of the great length of these Notes, (which much exceeded that of the Discourse itself to which they related,) I thought it best to place them at the end of it, instead of placing

them (as short notes usually are placed) in the same pages as the Discourse itself, and at the bottom of the pages in which the passages to which they relate occur. Of these investigations, thus furnished me by Mr. Morgan, and of the other matters contained in those Notes, I will now give some account.

Of the First Note on Dr. Halley's Discourse on Compound Interest.

The first of these three abstruse Algebraïck expressions, given us by Dr. Halley without a demonstration, is the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, in which the Letter b stands for the fraction $\frac{6}{t+1}$, and the Letter y stands for the binomial quantity $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$, or the excess of that power of the fraction $\frac{z}{at}$ of which the fraction $\frac{2}{t-1}$ is the index, above 1; which expression Dr. Halley declares to be nearly equal to the value of r in the binomial equation $\frac{z}{a} \times r - r^t \times \frac{z}{a} = 1$; and in this equation (in which r is supposed to be the only unknown quantity,) r stands for the *rate of the Interest* of money, (or the value of one pound, together with the Interest arising from the loan of it for a year, or some other given portion of time,) and a stands for the number of pounds in an annuity, or yearly rent or payment, that has been granted by one person to another for a certain number of years, and t stands for the number of years for which the said annuity has been granted, or during which it is agreed that it shall continue to be paid, and z stands for the number of pounds in the sum of money that ought to be paid to the grantee of the said annuity at the end of the said term of t years in one payment by the grantor of the said annuity, if the grantee of it has, at the desire of the grantor, forborne to require the payment of it during the whole of the said term of t years, which sum of money Dr. Halley calls the *amount* of the said annuity in the said term of t years. The investigation of this expression

sion $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, of the value of r in this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is the principal subject of the first Note, which is a very long one, and extends through 52 pages, beginning in page 231 and ending in page 282. But it contains some other matter relating to the resolution of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, besides the investigation of the said expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, given by Dr. Halley for the value of it's root r . It's principal contents are as follows.

In the first place I have stated at length in the beginning of this Note, (Art. 1 and 2, pages 231 and 232,) the whole passage of Dr. Halley's Discourse that relates to the resolution of the binomial equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of that expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$ given by him for the value of it's root r , and by means of a certain correction (also given by him,) of the said expression, which may be used, if thought necessary in any case, in order to obtain the value of the root r to a still greater degree of exactness. This passage I have stated in Dr. Halley's own words; and then, (in Art. 3, pages 232 and 233,) I have set forth the said correction of the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$ in a fuller and clearer manner than it is described by Dr. Halley, and have observed that it turns-out to be nothing more than a process of Mr. Raphson's excellent Method of approximating to the value of the root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ grounded on the near value of r obtained by Dr. Halley's first expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, as a first and known near value of it; though Dr. Halley makes no mention of Mr. Raphson on the occasion. And then, (in Art. 4, pages 233 and 234,) I have given the investigation of Dr. Halley's said correction of the first expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$ by means of Mr. Raphson's Method of Approximation.

I then

I then return (in Art. 5, pages 234, 235, 236, &c,) to the more difficult business of investigating the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$ itself, and I set forth in a very full and distinct manner the first principles of the doctrine of Compound Interest, which is grounded on the properties of a Series of quantities in continued Geometrical proportion, and on the 12th proposition of the 5th book of Euclid's Elements; and I prove, from those principles, that, if r be put for the value of one pound, together with the interest due upon it for one year, *the amount* of an annuity of one pound *per annum* at the end of t years, or the sum of money that will be due to the annuitant at that time, if the annuitant has forborne to receive his annuity during the whole of the said term of t years, will be $= \frac{r^t - 1}{r - 1}$ pounds, and consequently that the amount of an annuity of a pounds *per annum* at the end of the said t years will be $= a$ times $\frac{r^t - 1}{r - 1}$ pounds, or $\frac{ar^t - a}{r - 1}$ pounds, which amount Dr. Halley denotes by the Letter z . Therefore z is $= \frac{ar^t - a}{r - 1}$, and consequently $\frac{z}{a}$ is $= \frac{r^t - 1}{r - 1}$, and (multiplying both sides into $r - 1$,) we shall have $\frac{z}{a} \times r - \frac{z}{a} = r^t - 1$, and (adding $\frac{z}{a}$ to both sides,) we shall have $\frac{z}{a} \times r = r^t - 1 + \frac{z}{a}$, or $\frac{z}{a} \times r = r^t + \frac{z}{a} - 1$. But z (which is the amount of the annuity a after it has been forborne for t years,) is evidently greater than a , or the said annuity for one year. Therefore the fraction $\frac{z}{a}$ must be greater than 1, and consequently the trinomial quantity $r^t + \frac{z}{a} - 1$ must be greater than the single quantity r^t . Therefore r^t may be subtracted from $r^t + \frac{z}{a} - 1$, and consequently from, it's equal, $\frac{z}{a} \times r$. Let it be so subtracted. And we shall then have $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; which is the equation for the resolution of which Dr.

Halley

Halley has given us the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, which we must now endeavour to investigate.

Concerning this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ it is easy to observe that one of it's roots will be 1. For, if r is equal to 1, we shall have $r^t (= 1^t) = 1$, and $\frac{z}{a} \times r (= \frac{z}{a} \times 1) = \frac{z}{a}$; and consequently $\frac{z}{a} \times r - r^t$, or the first side of the equation, will, in this case, be equal to $\frac{z}{a} - 1$, which is the second side of the equation; or, in other words, 1 will be a root of this equation. But this root of the said equation cannot be the true value of r , or the rate of Interest which is required to be found by Dr. Halley's Problem, because that rate must (by the very definition of the rate of Interest,) be greater than 1. We must therefore conclude that 1 will be the lesser of the two roots of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and that it's greater root will be the rate of Interest which we are required to find.

Now, since r is greater than 1, let v be put for it's unknown excess above 1, so that r will be $= 1 + v$. And let $1 + v$ be substituted instead of r in the proposed equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. And we shall then have $\frac{z}{a} \times \overline{1 + v} - \overline{1 + v}^t = \frac{z}{a} - 1$.

But $\frac{z}{a} \times \overline{1 + v}$ is $= \frac{z}{a} + \frac{zv}{a}$, and (by Sir Isaac Newton's Binomial Theorem,) $\overline{1 + v}^t$ is = the Series $1 + tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c$ continued to $t + 1$ terms. Therefore $\frac{z}{a} + \frac{zv}{a} -$ the
said

said Series will be $= \frac{z}{a} - 1$; and (adding the said Series to both sides,) we shall have $\frac{z}{a} + \frac{zv}{a} = \frac{z}{a} - 1 +$ the said Series $1 + tv + t + \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c = \frac{z}{a} +$ the Series $tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c$; and (by subtracting $\frac{z}{a}$ from both sides,) we shall have $\frac{zv}{a} =$ the Series $tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c$; and (by dividing all the terms by v ,) we shall have $\frac{z}{a} =$ the Series $t + t \times \frac{t-1}{2} \times v + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$; and (by dividing all the terms by t ,) we shall have $\frac{z}{at} =$ the Series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$; by the resolution of which equation we should obtain the value of v , and consequently that of $1 + v$, or r . But the resolution of this equation is a matter of considerable difficulty. The Method taken by Mr. Morgan for this purpose (and which was probably the Method taken by Dr. Halley, though he did not think proper to disclose it,) is as follows.

Mr. De Moivre has given us a Theorem, of the same nature as Sir Isaac Newton's Binomial Theorem, by which we may raise a trinomial quantity, or a quadrinomial quantity, or a quinquinomial quantity, or any higher multinomial quantity, to any integral power whatsoever, or any power of which the Index

dex m is a whole number. And, as Sir Isaac Newton's Binomial Theorem, (which was first discovered to be true in the case of Integral powers, so as to enable us to find a Series of terms that shall be equal to the quantity $\sqrt[m]{1+x}$, or the m th power of the binomial quantity $1+x$, when the Index m is a whole number,) has been since extended by Sir Isaac Newton's sagacious and happy conjectures, to the fractional powers of a binomial quantity, so as to enable us to find a Series of terms that shall be equal to the quantity $\sqrt[n]{1+x}$, or the m th power of the n th root of the binomial quantity $1+x$, when m and n denote any two whole numbers whatsoever; so Mr. Morgan supposes that Mr. De Moivre's Theorem concerning the m th power of a multinomial quantity will be true likewise concerning the $\frac{m}{n}$ th power, or the m th power of the n th root, of such a multinomial quantity, when m and n denote any two whole numbers whatsoever. And by the help of this supposition he proceeds in the following manner to resolve the foregoing equation $\frac{z}{at} = \text{the Series } 1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c.$ He first raises the known quantity $\frac{z}{at}$ to the power of which the fraction $\frac{2}{t-1}$, (or the reciprocal of the fraction $\frac{t-1}{2}$ which is the co-efficient of v in the second term of the said Series,) is the Index; which may be done without much difficulty by the help of Logarithms: and then he raises the Series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ to the same power, of which the said fraction $\frac{2}{t-1}$ is the Index, by means of Mr. De Moivre's Multinomial Theorem, so extended to the case of fractional powers; by which operation it appears that the five first terms of the Series that is equal to the $\frac{2}{t-1}$ th power of the former Series are $1 + v + \frac{t-1}{12} \times vv + 0 \times v^3$

$\frac{-t^3+4t^2-t-6}{1440} \times v^4$, of which the fourth term $0 \times v^3$ is equal to 0, which greatly contributes to make these five terms approach nearly to the true value of the whole Series. And thus he obtains the following new equation, to wit,

$$\left(\frac{z}{at}\right)^{\frac{2}{t-1}} = 1 + v + \frac{t-1}{12} \times vv + 0 \times v^3 - \frac{-t^3+4t^2-t-6}{1440} \times v^4 \&c,$$
 or (omitting the fourth term $0 \times v^3$, which is equal to 0,) the equation

$$\left(\frac{z}{at}\right)^{\frac{2}{t-1}} = 1 + v + \frac{t+1}{12} \times vv * \frac{-t^3+4t^2-t-6}{1440} \times v^4 \&c,$$
 and (by subtracting 1 from both sides,) the equation

$$\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1 = v + \frac{t+1}{12} \times vv * \frac{-t^3+4t^2-t-6}{1440} \times v^4 \&c,$$
 or (placing the known quantity $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$ on the right-hand side of the mark of equality,) the equation

$$v + \frac{t+1}{12} \times vv * \frac{-t^3+4t^2-t-6}{1440} \times v^4 \&c = \left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1,$$
 or (neglecting the term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ and all the following terms included under the mark &c, which will involve in them v^5, v^6, v^7 , and the following higher powers of v , as being very small in comparison of the two first terms v and $\frac{t+1}{12} \times vv$,) the quadratic equation

$$v + \frac{t+1}{12} \times vv = \left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1.$$
 Now let y be put for the known quantity $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$; and we shall have

$$v + \frac{t+1}{12} \times vv = y.$$
 Therefore, if we multiply all the terms of this equation into the fraction $\frac{12}{t+1}$, in order to free vv from it's coefficient $\frac{t+1}{12}$, we shall have the equation

$$\frac{12}{t+1} \times v + vv = \frac{12}{t+1} \times y;$$
 and if we put $b = \frac{6}{t+1}$, we shall have $2b = \frac{12}{t+1}$, and consequently $2b \times v + vv = 2b \times y$, or $vv + 2bv = 2by$. Therefore (adding bb to both sides,) we shall have

$$vv + 2bv + bb = bb + 2by,$$
 and (extracting the square-roots of both sides,)

$$v + b = \sqrt{bb + 2by},$$
 and consequently

$$v = \sqrt{bb + 2by} - b,$$

$-b$, and $1 + v = 1 + \sqrt{bb + 2by} - b$, or $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - 1$; that is, r will be $= 1 + \sqrt{bb + 2by} - b$, or $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - 1$, which is Dr. Halley's expression for the value of r , or the rate of Interest in his fourth Problem, or for the greater root of the binomial equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. Q. E. I.

The operation of raising the multinomial quantity $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ to the $\frac{t}{t-1}$ th power by means of Mr. De Moivre's Multinomial Theorem, I found to be very laborious. That my readers may not have so much trouble with it as I have had, I have set-down all the different steps of it at full length in Art. 13, 14, and 15, pages 241, 242, 243, &c, - - - 253.

In Art. 16, pages 253 and 254, it is shewn that, if t , or the number of years during which the annuity is to continue, is either 4, or any number greater than 4, the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, obtained in the foregoing articles, will always be less than the true value of r , or the rate of Interest sought, or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; because in all those cases the quantity $\frac{t^3+t+6}{1440} \times v^4$, (which is the part of the term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ which has the sign $-$ prefixed to it, and is therefore to be subtracted from the former terms $v + \frac{t+1}{2} \times vv$), will be greater than the quantity $\frac{4t^3}{1440} \times v^4$, which is the part of the said term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ which has the sign $+$ prefixed to it, and is therefore to be added to the said former terms $v + \frac{t+1}{12} \times vv$. And it may be further observed that, if t , or the number of years during which the annuity is to continue, is 3, the said term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$

will be equal to 0, as well as the former term of the Series which involved v^3 . For in this case the said term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ will be = $\frac{-3^3+4 \times 3^2-3-6}{1440} \times v^4 (= \frac{-27+4 \times 9-9}{1440} \times v^4 = \frac{-36+36}{1440} \times v^4 = \frac{0}{1440} \times v^4) = 0$. And, if t is = 2, the said term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ will also be = 0. For in this case the said term will be = $\frac{-2^3+4 \times 2^2-2-6}{1440} \times v^4 (= \frac{-8+4 \times 4-8}{1440} \times v^4 = \frac{-16+16}{1440} \times v^4 = \frac{0}{1440} \times v^4) = 0$.

But, when t , or the number of years for which the annuity is granted, is a great number, (as, for example, 70 years,) the co-efficient of the term $\frac{-t^3+4t^2-t-6}{1440} \times v^4$ will be of considerable magnitude. For, if t is = 70, we shall have $t^2 = 4900$, and $t^3 = 343,000$, and $\frac{-t^3+4t^2-t-6}{1440} \times v^4 = \frac{-343,000+4 \times 4900-70-6}{1440} \times v^4 (= \frac{-343,000+19,600-76}{1440} \times v^4 = \frac{-343,076+19,600}{1440} \times v^4 = \frac{-320,476}{1440} \times v^4 = \frac{-160,238}{720} \times v^4 = \frac{-80,119}{360} \times v^4) = -222,552,777 \times v^4$, the co-efficient of which, to wit, 222,552,777, is a number of considerable magnitude, and will so much increase the quantity v^4 that the neglect of the term $-222,552,777 \times v^4$ seems likely to diminish in a considerable degree the exactness of the expression $1 + \sqrt[3]{bb + 2by} - b$ obtained for the value of r without it, and make the said value be considerably less than the truth. But in these cases the correction given by Dr. Halley, or one process of Mr. Raphson's Method of Approximation, will give us the value of r to as great a degree of exactness as we need desire.

Having set-forth Mr. Morgan's investigation of this expression of Dr. Halley as fully and clearly as I could, I proceed in Art. 17, pages 254, 255, and 256, to apply it to the example given us by Dr. Halley, or to the resolution of the binomial equation $17.861,415,7 \times r - r^{12.5} = 16.861,415,7$. And I find that

that the value of r resulting from it is 1.059,990,2; which is wonderfully near 1.06, which is the true value of it. This example therefore is a strong proof of the usefulness of this expression of Dr. Halley in cases in which t , or the number of years during which the annuity is to continue, is not much greater than 12. And probably it will be sufficiently exact when t is not greater than 20. But, when t is greater than 20 years, and especially if it be equal to 50, or 60, or 70, years, it will be advisable to employ Dr. Halley's correction of it, or (which would be equivalent to the said correction,) a single process of Mr. Raphson's Method of Approximation.

Though the expression $1 + \sqrt[3]{bb + 2by} - b$, given us by Dr. Halley for a near value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and of which I have published Mr. Morgan's investigation, is a very useful one, on account of it's near approach to the truth in many cases, yet the difficulty of investigating it by the application of Mr. De Moivre's Multinomial Theorem, and the remoteness and subtlety, and, I may add, the uncertainty, of the principles on which that investigation is grounded (since the extension of Mr. De Moivre's Theorem to the case of fractional powers of a multinomial quantity has never, as far as I know, been demonstrated,) induced me to endeavour to find some other expression for a first near value of r that should be more easily attainable, and be derived from simpler and more natural principles. And the result of my inquiries was the discovery of another expression which is much more easily obtained than the foregoing one of Dr. Halley, and is derived from the Binomial Theorem in it's simplest case, (to wit, that of the integral powers of a binomial quantity,) without any mention of Mr. De Moivre's Multinomial Theorem, and which (though not so exact as Dr. Halley's expression,) is near enough to the truth, in many cases, to be highly useful of itself, or without any correction, and in some other cases (though not in all,) will be an excellent ground-work, or basis, for a further approach to the true value of r by a process of Mr. Raphson's Method of Approximation. This expression of the value of r is $1 + \sqrt{g + ee - e}$; in which g is equal to the fraction $\frac{6z - 3at}{at^3 - 3at^2 + 2at}$, (which involves only the known quantities a , t , and z),

and

and e is $= \frac{3}{2t-4}$. It's investigation is given in Art. 20 and 21, pages 257 and 258, and takes up less than two pages. And in Art. 23, pages 259 and 260, it is applied to the resolution of the same equation $17.861,415,7 r - r^{12.5} = 16.861,415,7$, or (omitting the three last decimal figures,) the equation $17.8614 r - r^{12.5} = 16.8614$, which had before been resolved by the expression of Dr. Halley. And the value of $1 + v$, or r , resulting from this expression $1 + \sqrt{g + ee} - e$, is 1.061, 98, which exceeds 1.06, or the true value of r , by only 0.001,398, which is less than 0.0014. And therefore the Interest upon 100 pounds for one year would, according to this value of r , obtained by means of the expression $1 + \sqrt{g + ee} - e$, be somewhat less than $100 \times 0.614 \text{ £}$, or 6.14 £, or 6 pounds, 2 shillings, and ten pence, instead of being exactly 6 pounds. This difference of ^{s. d.} 2, 10 upon 6 pounds seems to be but trifling; and therefore this expression $1 + \sqrt{g + ee} - e$, or $1 + \overline{g + ee}^{\frac{1}{2}} - e$ (which was obtained so much more easily than the expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$ given by Dr. Halley,) seems to be a pretty good approximation to the true value of r in this example.

But, if 1.0614 should not be thought to be near enough to the true value of r in the equation $17.8614 r - r^{12.5} = 16.8614$, it will, at least, serve for an excellent basis, or ground-work, for a further approach towards it's true value by Mr. Raphson's Method of Approximation, by supposing the excess of 1.0614 above the true value of r to be denoted by the Letter w , and inserting the binomial quantity $1.0614 - w$ instead of r in the equation $17.8614 r - r^{12.5} = 16.8614$, and then resolving the transformed equation thereby obtained in the manner prescribed by Mr. Raphson. And we shall find w to be $= 0.001,362,2$, and consequently $1.0614 - w$ to be $(= 1.0614 - 0.001,362,2) = 1.060,037,8$; that is, the value of r , or the rate of Interest sought, will be 1.060,037,8, instead of being exactly 1.060,000,0, or 1.06; which is a very great degree of exactness. This process

cess of Mr. Raphson's Method of Approximation is given in Art. 24, pages 260 and 261.

Dr. Halley seems to be fond of giving general Algebräick expressions, or *formulas*, for correcting the imperfect values of the unknown quantity resulting from his first expressions. And, in imitation of that practice, I have, in Art. 25, 26, 27, and 28, pages 261, 262, and 263, given a general Algebräick expression for the second near value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, after having found a first near value of it by means of the expression $1 + \sqrt{g + ee} - e$, or $1 + \overline{g + ee}^{\frac{1}{2}} - e$; which general expression is $c - \frac{d}{t \times c^t - 1 - \frac{z}{a}}$, in which c stands for $1 + \overline{g + ee}^{\frac{1}{2}} - e$, or the first near value of r obtained by means of that expression, and d stands for the quantity $\frac{z}{a} - 1 - \sqrt{\frac{z}{a} \times c - c^t}$, or the excess of $\frac{z}{a} - 1$ above $\frac{z}{a} \times c - c^t$. And then, in Art. 29, pages 263 and 264, I have applied the said general Algebräick expression to the finding of the second near value of r , which I had before found by a process of Mr. Raphson's Method of Approximation, to be $= 1.060,037,8$. And the said second near value of r , when found by means of the said general expression, appears (as before,) to be $= 1.060,037,8$. But I must confess I differ from Dr. Halley and some other eminent writers of Algebra, with respect to the expedience of drawing-up such general Algebräick expressions, or *formulas*, to govern our numerical calculations. For I have always found it more easy and convenient to resolve any proposed equation by applying to it distinctly the original reasonings and principles upon which Mr. Raphson's Method is grounded, than to proceed by means of such a general *Canon*, or *Formula*: and, one great advantage of this method of proceeding is, that it is much *safer*, or less likely to lead us into mistakes (by some omission of a number, or a letter, or by a change of a sign $+$, or $-$, or $\times$;) than working by a general *Canon*, or *Formula*;—not to mention the greater

greater satisfaction of mind that we receive when we proceed in a plain and distinct manner (in which we see our way before us, and perceive the reason of every arithmetical operation at the time we perform it,) than when we work in the dark in computing the several members of a tedious and intricate Algebraick general expression.

After having thus resolved the equation $17.861,415,7 r - r^{12.5} = 16.861,415,7$ (which was chosen by Dr. Halley as an example of the application of his expression $1 + \sqrt[1/2]{bb + 2by} - b$,) in two different ways, namely, first, by means of Dr. Halley's said expression $1 + \sqrt[1/2]{bb + 2by} - b$, and, secondly, by the other expression $1 + \sqrt[1/2]{g + ee} - e$;—and having, by the said resolutions of this equation, found that Dr. Halley's expression gives the value of r in that case to a very great degree of exactness, making r equal to 1.059,990,2 instead of 1.060,000,0, or 1.06, which is it's accurate value, and that the expression $1 + \sqrt[1/2]{g + ee} - e$ also gives the said value to a very considerable degree of exactness, (though less than that of Dr. Halley's expression,) making r equal to 1.061,398, which exceeds the true value 1.06 by only 0,001,398;—and having shewn that this value 1.061,398, or 1.0614, may, by a single process of Mr. Raphson's Method of Approximation, be changed into the more exact value 1.060,037,8, which exceeds the true value by only 0.000,037,8;—I proceed to apply both these expressions $1 + \sqrt[1/2]{bb + 2by} - b$ and $1 + \sqrt[1/2]{g + ee} - e$ to the resolution of another equation of the same form as the equation $17.861,415,7 r - r^{12.5} = 16.861,415,7$, but in which t , or the index of r^t , or the number of years during which the annuity is to continue, is much greater than in the former case, and is 70 years; which is a very unfavourable case for both the said expressions.

In forming this second example for a trial of the efficacy of these two expressions $1 + \sqrt[1/2]{bb + 2by} - b$ and $1 + \sqrt[1/2]{g + ee} - e$, I have, first, supposed the interest of money to be known and to be 6 per cent., or r to be = 1.06, and

and have supposed the annuity itself to be the same as in the former example, to wit, $\overset{\text{£.}}{34}$, $\overset{\text{s.}}{8}$, or $\overset{\text{£.}}{34.4}$, *per annum*, but to be granted for a term of 70 years, instead of a term of 12 years and a half; and I have computed the value of the amount z of such an annuity at the end of the said 70 years. And it ap-

pears to be $\overset{\text{£.}}{33,297.055,903.04}$. This computation is made, in Art. 30, pages 265 and 266. And, having thus found that, if the Interest of money

is 6 per cent., the amount of the said annuity of $\overset{\text{£.}}{34.4}$ *per annum* forborne during 70 years will be $\overset{\text{£.}}{= 33,297.055,903.04}$, I then reverse the Problem, and

suppose that z , or the said amount, is known to be $\overset{\text{£.}}{= 33,297.055,903.04}$, and that r , or the rate of Interest, is required to be found from it. And this quantity r I compute, first, in Art. 31, pages 266, 267, by Dr. Halley's expression $1 + \sqrt[3]{bb + 2by}^{\frac{1}{2}} - b$, and afterwards by the expression $1 + \sqrt[3]{g + ee}^{\frac{1}{2}} - e$; and by Dr. Halley's expression I find it to be $\overset{\text{£.}}{= 1.058,712.7}$; which is nearer the true value, 1.06, than I had expected to find it, when t was so great a number as 70. For, upon the sum of 100 pounds, the Interest for one year, as

given by this expression, would be $\overset{\text{£.}}{100} \times \overset{\text{£.}}{0.058,712.7}$, or $\overset{\text{£.}}{5.871,27}$, instead of $\overset{\text{£.}}{6}$,

of which it falls short by only $\overset{\text{£.}}{0.128,73}$, or about $\overset{\text{s.}}{2}$, $\overset{\text{d.}}{7}$. We may therefore conclude that this expression $1 + \sqrt[3]{bb + 2by}^{\frac{1}{2}} - b$, given us by Dr. Halley for the value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, will, in all magnitudes of t that are not greater than 70, be a very useful approximation to the true value of r , though more so when t , or the number of years during which the annuity is forborne, is small, than when it is great.

I then, (in Art. 33, pages 267, 268, and 269,) correct this first near value of r , obtained by Dr. Halley's first expression $1 + \sqrt[3]{bb + 2by}^{\frac{1}{2}} - b$, to wit,

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1.058,712,7,

1.058,712,7, in the manner prescribed by Dr. Halley, by adding to it the value of the fraction $\frac{d}{t \times c^t - 1 - \frac{z}{a}}$, in which c stands for 1.058,712,7, or the

first near value of r obtained by the expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, and d stands for the excess of $\frac{z}{a} \times c - c^t$ above $\frac{z}{a} - 1$. This fraction is found to be = 0.001,363,5, which, added to 1.058,712,7, makes 1.060,076,2, which exceeds the true value of r , to wit, 1.06, by only the small difference 0.000,076,2; which, upon the Interest of 100 pounds for one year, would amount to only (100 $\times$ 0.000,076,2, or) 0.007,62 of a pound, or something less than two pence. This seems to be sufficiently near the truth for all useful purposes: and therefore I conclude that this expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, of Dr. Halley, together with his correction of it, the fraction $\frac{d}{t \times c^t - 1 - \frac{z}{a}}$, will

always give us the value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ to a sufficient degree of exactness when t (or the number of years during which the annuity is forborne,) is a great number (such as 70, or 72, or 75, or, perhaps, any number not greater than 80,) as well as when it is a small number. And, if t is a small number, not exceeding 20, the said expression alone, without the correction, will be sufficiently exact.

The value of r in this example is then computed by means of the other expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$, in which g stands for the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$, and e stands for $\frac{3}{2t - 4}$. This computation is made in Art. 34, pages 269; 270; and the result of it is, that r is = 1.107,903. This value of r is so much greater than it's true value, 1.06, that we may justly conclude from it that the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ will be of little or no use in resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ when t , or the number of years during

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ing which the annuity is forborne, is so great a number as 70. But, when t is nearly equal to 12, we have seen above, in Art. 23, that this expression gives the value of r sufficiently exact for most purposes; and I therefore conjecture that it may be safely employed when t is not greater than 20. And, if we correct it in the manner described in Art. 24, 25, &c, - - - 29, by putting $c = 1 + \sqrt{g + ee} - e$, and subtracting $\frac{z}{a} \times c - c^t$ from $\frac{z}{a} - 1$, and putting d for the remainder thence arising, and, lastly, by subtracting from c , or $1 + \sqrt{g + ee} - e$, (which is always greater than the true value of r ,) the fraction $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$, it will probably give us the value of r , or the rate of In-

terest sought, with sufficient exactness, even when t is greater than 20, and of any magnitude not greater than 30. But this can only be ascertained by trying it in other examples in which t is taken of different magnitudes, (such as 20, 30, 40, 50, &c,) between $12\frac{1}{2}$ and 70, in which two magnitudes of t it has been tried in the preceeding articles. But, as to Dr. Halley's expression $1 + \sqrt[3]{bb + 2by} - b$, the trials that have been made above of it's exactness, prove that it is in all cases more exact than the expression $1 + \sqrt{g + ee} - e$, or $1 + \sqrt{g + ee} - e$, and likewise that it is sufficiently exact for most purposes, even when t is so large a number as 70, and that, with Dr. Halley's correction of it, it becomes much more exact than before, and as exact, even in those difficult cases, as any calculator need desire.

As a further trial of the exactness of the two expressions $1 + \sqrt[3]{bb + 2by} - b$, and $1 + \sqrt{g + ee} - e$ in determining the value of r , or the rate of Interest, in questions of the foregoing kind, in which the values of a , t , and z , (or the number of pounds in the annuity, the number of years for which it is granted, and the amount of it, or the sum due to the grantee, or annuitant, at the end of the term for which it has been granted, in consequence of his hav-

ing forborne to receive it,) are known, and r , (or the rate of Interest,) is unknown and to be deduced from them, I have (in Art. 36, 37, and 38, pages 271, 272, and 273,) applied the said expressions to the solution of another question in which the number of years for which the annuity is granted is only 10 years, and in which the Interest of money is much lower than in the former ex-

$\begin{array}{ccc} \text{£. s. d.} & & \text{£.} \end{array}$

ample, being 3, 6, 8 per 100, or 0.033,333,333, &c. £ per 100, instead of 6 per cent. And, upon computing the value of r (which in this case is exactly = 1.033,333,333, &c.) by means of these two expressions, it appears that the said value, when computed by Dr. Halley's expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, will be = 1.033,330,6, and, when computed by the expression $1 + \sqrt{g + ee} - e$, will be = 1.033,485,1. Both these values of r are sufficiently near the true value, 1.033,333,333, &c, to answer all useful purposes; but the former value is the more exact of the two. Therefore we may conclude, that when the term of years for which the annuity is granted is not more than 10 years, either of the two expressions $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$ and $1 + \sqrt{g + ee} - e$ will give us the value of r as exactly as need be desired. And in these cases, perhaps, the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ may deserve to be preferred to the other expression, as being, I think, rather easier to be computed.

But, when t , or the number of years for which the annuity is granted, is 70, or any greater number, it has been shewn before that the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ will give us a value of r very much greater than the truth, and so much greater as to be of little, or no use, without some corrections of it by Mr. Raphson's Method of Approximation. And these corrections would be too numerous in such cases, and approach too slowly towards the true value of r , to make it expedient for us to employ the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ to obtain the first near value of r , from which the subsequent nearer values of it are to be obtained by Mr. Raphson's Method of Approximation. For I have found that, if

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in the former example, (in which t was $= 70$, and the equation to be resolved was $967.937,671,6 \times r - r^{70} = 966.937,671,6$,) we make use of the expression $1 + g + ee^{\frac{1}{2}} - e$ to obtain a first near value of r , (which first near value will be $1.107,903,7$, or nearly 1.108 ,) and afterwards from the said first near value 1.108 (which is much greater than the true value of r in this example, which is 1.06 ,) we make gradual approaches to the true value of r by Mr. Raphson's Method of Approximation, it will require no fewer than six successive processes of approximation to obtain the number $1.060,01$ for the value of r , which number $1.060,01$ may be considered as equal to it's true value 1.06 . These several processes I have gone through, and, for the satisfaction of the intelligent and industrious arithmeticians who may be disposed to inquire curiously into this subject, I have set them all down in this volume in pages 275, 276, 277, &c, - - - 280. And the several successive near values of r (after the first near value $1.107,903,7$, or 1.108 , obtained by the expression $1 + \sqrt{g + ee} - e$) that have been obtained by these several processes, appear to be 1.0933 , 1.0801 , 1.0695 , 1.0628 , 1.0604 , and $1.060,01$. We may therefore conclude that in resolving an example of this kind, where t is equal to, or greater than, 70 years, it would be more judicious to employ Dr. Halley's expression $1 + \sqrt[3]{b + 2by}^{\frac{1}{2}} - b$, together with his correction of it, or (which is equivalent to that correction,) one process of Mr. Raphson's Method of Approximation, than to make use of the expression $1 + \sqrt{g + ee} - e$, and to correct the too great value of r thereby obtained by repeated processes of Mr. Raphson's Method of Approximation.

These are the principal contents of the first Note on Dr. Halley's Discourse on Compound Interest, which begins in page 231, and ends in page 282.

Of

Of the Second Note to Dr. Halley's Discourse on Compound Interest.

The second Note relates to the Solution of a Problem concerning *the present value* of a given annuity, or the price paid for the purchase of it, instead of *the amount* of such an annuity, or the sum of money that ought to be paid to the annuitant, or grantee of the annuity, at the expiration of the term of years for which the annuity was granted, in consequence of his having foreborne to receive it during the whole of the said term, as was the case in the former Problem of Dr. Halley, to which the first Note related. But in both Problems the rate of Interest is the unknown quantity which is required to be found. The present Problem is as follows. “ If a be put for the number of pounds contained in an annuity granted for a certain number of years; and t be put for the number of years for which the said annuity is granted; and z be put for the present value, or the price, of the said annuity, or the number of pounds contained in the sum of money that has been paid for the said annuity by the purchaser of it; and r be put for the rate of Interest of money by which the said price, or present value, z has been regulated; and the three quantities denoted by a , t , and z be known, or given: It is required to find the value of r .”

Now this Problem may be reduced to the following binomial equation, to wit, $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which will have two roots, or (in the language of modern Algebraists,) two real and positive roots, of which the lesser will be $= 1$, and the greater will be that value of r which is fitted to solve the present Problem, or is the rate of Interest that is required to be found in it. And Dr. Halley, in his directions for the solution of this Problem, or for the resolution of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which is derived

from

from it, divides it into two cases, according as t , or the number of years for which the annuity is granted, is greater, or less, than 40 years. In the former of these cases, or when t is greater than 40 years, he tells us that $\frac{z+a}{z}$, or $1 + \frac{a}{z}$, will be nearly equal to r , but somewhat greater than the truth; and that the quantity $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$ will be a second near value of r that will be more accurate than $\frac{z+a}{z}$, or $1 + \frac{a}{z}$. And, lastly, he informs us that, if this second quantity $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$ be denoted by d , and the fraction $\frac{a}{d^t \times d-1}$ be denoted by x , the quantity $1 + \frac{a}{z+x}$ will be a third near value of r , which will be nearer the truth than either of the two former, and will be sufficiently accurate. All these assertions are true; but they are made without any demonstration, or proof, and are not even described with so much care and perspicuity as might have been expected. I have therefore in this Note endeavoured to supply these defects in the following manner.

In Article 1, pages 284 and 285; I have shewn how the foregoing Problem may be reduced to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. And then, in Art. 2, pages 285 and 286, I have shewn that $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, will be a near value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, but will be somewhat greater than its true value. And, having thus obtained the quantity $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, for a first near value of r , or the rate of Interest sought, I proceed in Art. 3, pages 286, 287, and 288, to derive from it, by Mr. Raphson's Method of Approximation, Dr. Halley's second near value of r , to wit, $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$. And then, in Art. 4, pages 288 and 289, I proceed to investigate Dr. Halley's third near value of r , to wit, the expression

$1 + \frac{a}{z+x}$, which is obtained by putting d for the second near value of it, to wit $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$, and x for the value of the fraction $\frac{a}{d^t \times d-1}$.

And, having thus gone through the investigations of Dr. Halley's three expressions of the value of r , to wit, $1 + \frac{a}{z}$, $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$, and $1 + \frac{a}{z+x}$, I proceed in Art. 5, pages 289, 290, and 291, to apply them to the numerical example given us by Dr. Halley, or to the resolution of the numeral equation $1.052,978,0 \times r^{59} - r^{60} = 0.052,978,0$, which results from it, and in which the true value of the greater root r , or the rate of Interest sought is 1.05. And it appears that the first expression of this value, given us by Dr. Halley, to wit, $1 + \frac{a}{z}$, will, in this case, be $= 1.052,978,0$, and that the second expression, to wit, $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$, will, in this case, be $= 1.050,458,3$, and that the third expression, to wit, $1 + \frac{a}{z+x}$, will, in this case, be $= 1.050,096,4$. The first of these near values of r , to wit, $1 + \frac{a}{z}$, or 1.052,978,0, is a very convenient quantity for a first near value of it, and is obtained without much difficulty. The second of them, to wit, the quantity $\frac{z+a}{z} - \left(\frac{z}{z+a}\right)^t \times \frac{a}{z}$, or 1.050,458,3, is derived from the first near value $1 + \frac{a}{z}$ by Mr. Raphson's Method of Approximation, though Dr. Halley makes no mention of Mr. Raphson's name on the occasion. And the third expression, to wit, $1 + \frac{a}{z+x}$, or 1.050,096,4, is derived from the second expression with a good deal of laborious calculation, and rather more, I think, than would have been necessary to the computation of a second process of Mr. Raphson's Method of Approximation, and yet is not quite so exact as the value of r that would have resulted from such second process; for that would have been only

1.050,001,8,

1.050,001,8, (as is shewn in Art. 11, 12, 13, and 14,) which exceeds the true value of r , to wit, 1,05, by a less quantity than Dr. Halley's third near value 1,050,096,4. And hence I think it may be concluded that, when we have a Problem of this kind to be solved, or an equation of this form, $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, to be resolved, and the number t is greater than 40, it would be better to adopt Dr. Halley's first expression of the value of r , to wit, $1 + \frac{a}{z}$, for our first near value of it, and afterwards from that first expression to derive a second and a third near value of r by Mr. Raphson's Method of Approximation, than to take the trouble of computing the second and third expressions, $\frac{z+a}{z} - \left[\frac{z}{z+a} \right]^t \times \frac{a}{z}$ and $1 + \frac{a}{z+x}$, given us by Dr. Halley for that purpose.

This is the substance of the Second Note, which takes-up 19 pages, beginning in page 283, and ending page 301.

Of the Third Note to Dr. Halley's Discourse on Compound Interest.

The Third Note relates to the second case of the same Problem that was the subject of the Second Note, concerning the present value of an annuity granted for a certain term of years; in which second case of the said Problem the number of years is supposed to be less than 40. In this case Dr. Halley tells us that the former rule will avail little, and directs us to find a near value of the rate r in the following manner. Compute the value of the fraction $\frac{at}{z}$, which contains only the known quantities a , t , and z . Then find that power of this fraction $\frac{at}{z}$, which has the fraction $\frac{2}{t+1}$ for it's Index. Thirdly, subtract 1 from this power, and call the remainder y ; and let the fraction $\frac{6}{t-1}$ be called b . Then will $1 + b - \sqrt[6]{bb - 2by}^{\frac{1}{2}}$ be nearly equal to r , the rate sought,

fought, and the more nearly as t , or the number of years for which the annuity is granted, is smaller. And it will be always a little greater than the true value of r . This rule he delivers without a demonstration, or the least hint of the manner in which it may be investigated. But Mr. Morgan, of the Equitable Assurance-Office in Chatham-Square, has furnished me with the following investigation of it.

Let v be put for $r - 1$. And we shall have $1 + v = r$, and $\overline{1 + v}^t = r^t$; and consequently $\frac{a}{v} - \frac{a}{\overline{1 + v}^t \times v}$ will be $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$.

$$\text{But } z \text{ is } = \frac{a}{r-1} - \frac{a}{r^t \times r-1}.$$

Therefore z will also be $= \frac{a}{v} - \frac{a}{\overline{1 + v}^t \times v}$; and consequently zv will be $= a - \frac{a}{\overline{1 + v}^t}$, and $\frac{zv}{a}$ will be $= 1 - \frac{1}{\overline{1 + v}^t}$.

But, by the Binomial Theorem in the case of Integral and negative powers, or of the reciprocals of the Integral powers of a binomial quantity (of which case of the said Theorem a demonstration has been given in the 3d Volume of this Collection of Mathematical Tracts called *Scriptores Logarithmici*, page *124, at the bottom, Art. 42,) the quantity $\overline{1 + v}^{-t}$, or $\frac{1}{\overline{1 + v}^t}$, is = the Series $1 - \frac{t}{1} \times Av + \frac{t+1}{2} \times Bvv - \frac{t+2}{3} \times Cv^3 + \frac{t+3}{4} \times Dv^4 - \&c$ = the Series $1 - \frac{t}{1} \times 1 \times v + \frac{t+1}{2} \times \frac{t}{1} \times 1 \times vv - \frac{t+2}{3} \times \frac{t+1}{2} \times \frac{t}{1} \times 1 \times v^3 + \frac{t+3}{4} \times \frac{t+2}{3} \times \frac{t+1}{2} \times \frac{t}{1} \times 1 \times v^4 - \&c$ = the Series $1 - tv + t \times \frac{t+1}{2} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 - \&c$.

Therefore

Therefore $\frac{zv}{a}$ (which is equal to $1 - \sqrt{1+v}^{-t}$, or $1 - \frac{-1}{1+v}^t$), will be equal to $1 -$ the Series $1 - tv + t \times \frac{t+1}{2} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 - \&c$, and consequently to the Series $tv - t \times \frac{t+1}{2} \times vv + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 + \&c$.

Therefore (dividing both sides of the last equation by v), we shall have $\frac{z}{a}$ = the Series $t - t \times \frac{t+1}{2} \times v + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$, and (dividing all the terms by t), we shall have $\frac{z}{at}$ = the Series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$.

Now let the fraction $\frac{z}{at}$ (which forms the left-hand side of this last equation,) be raised to the power of which the fraction $\frac{-2}{t+1}$ (or the reciprocal of the fraction $\frac{t+1}{2}$, which is the co-efficient of v in the second term, $\frac{t+1}{2} \times v$, of the last series,) is the Index; and let the Series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ (which forms the right-hand side of the same equation,) be raised likewise to the same power of which the Index is $\frac{-2}{t+1}$. Then will the said $\frac{-2}{t+1}$ th power of the said Series be equal to the same power of the said fraction $\frac{z}{at}$.

Now let p be $= \frac{2}{t+1}$.

Then will $\left(\frac{z}{at}\right)^{\frac{-2}{t+1}}$ be $= \left(\frac{z}{at}\right)^{-p}$ ($= \frac{1}{\left(\frac{z}{at}\right)^p} = \frac{1}{\left(\frac{z}{at}\right)^p} = 1 \times \frac{at^p}{z^p} = 1 \times$
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at

$\left(\frac{at}{z}\right)^{\frac{2}{t+1}} = \frac{at}{z}^{\frac{2}{t+1}}$; that is, the $\frac{-2}{t+1}$ th power of the fraction $\frac{z}{at}$ is equal to the $+\frac{2}{t+1}$ th power of the fraction $\frac{at}{z}$, which is the reciprocal of the fraction $\frac{z}{at}$, or is equal to 1 divided by $\frac{z}{at}$. And therefore, in order to obtain the $\frac{-2}{t+1}$ th power of the fraction $\frac{z}{at}$, we need only find the $+\frac{2}{t+1}$ th power of the fraction $\frac{at}{z}$; which may be done by, first, finding the Logarithm of the fraction $\frac{at}{z}$, and then multiplying that Logarithm into the fraction $\frac{2}{t+1}$, (which is the Index of the power of the said fraction $\frac{at}{z}$ which we want to find,) and then seeking in a table of Logarithms the number corresponding to the Logarithm which is equal to the product thence arising. For that number will be equal to $\left(\frac{at}{z}\right)^{+\frac{2}{t+1}}$, and consequently to $\left(\frac{z}{at}\right)^{-\frac{2}{t+1}}$.

But the raising the Series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ to the $\frac{-2}{t+1}$ th power is a business of greater difficulty. And for this purpose we must have recourse to Mr. De Moivre's Multinomial Theorem, (which has been set-forth before in Note 1, page 241,) and we must suppose the said Theorem to be true, not only in the case of Integral powers, but likewise in the more complicated case of fractional powers, and even in the case of powers that are both fractional and negative; though it has never, I believe, been demonstrated in any case but that of Integral powers. However, it is probably true in all these cases; and, if it is, it will enable us to find the $-\frac{2}{t+1}$ th power of the said Series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$, though not without a great deal of troublesome Algebraick calculation, which is set-down at full length

length in Art. 4 and 5, pages 305, 306, 307, 308, and 309. And the result is, that the $\frac{2}{t+1}$ -th power of the said Series is equal to the Series $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c$, or $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \&c$, the co-efficient of v^3 being $= 0$. Therefore the Series $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \&c$, or $1 + v - \sqrt{\frac{t-1}{12}} \times vv \&c$, will be $= \sqrt{\frac{at}{z}} \frac{2}{t+1}$. And consequently $v - \sqrt{\frac{t-1}{12}} \times vv \&c$ will be $= \sqrt{\frac{at}{z}} \frac{2}{t+1} - 1$; and (if we neglect all the terms involving $v^4, v^5, v^6, \&c$, denoted by the mark $\&c$ on the first side of the equation, as being very small in comparison of v and $\frac{t-1}{12} \times vv$), we shall have $v - \sqrt{\frac{t-1}{12}} \times vv = \sqrt{\frac{at}{z}} \frac{2}{t+1} - 1$, which is a quadratick equation.

Now let y be put $= \sqrt{\frac{at}{z}} \frac{2}{t+1} - 1$. And we shall then have $v - \sqrt{\frac{t-1}{12}} \times vv = y$; and (by multiplying both sides into the fraction $\frac{12}{t-1}$), we shall have $\frac{12v}{t-1} - vv = \frac{12}{t-1} \times y$; and (by putting $b = \frac{6}{t-1}$) we shall have $2bv - vv = 2by$; in which equation the binomial quantity $2bv - vv$ (being equal to $2b - v$) $\times v$,) will be less than $\left(\frac{2b}{2}\right)^2$, or b^2 , and therefore may be subtracted from it. Let both sides of this equation $2bv - vv = 2by$ be now subtracted from b^2 , or bb . And we shall then have $bb - 2bv + vv = bb - 2by$. Therefore (extracting the square-roots of both sides,) we shall have $b - v = \sqrt{bb - 2by}$, and consequently $b = \sqrt{bb - 2by} + v$, and $v = b - \sqrt{bb - 2by}$. Therefore r , or $1 + v$, will be $= 1 + b - \sqrt{bb - 2by}$, or $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$.

Q. E. I.

This Investigation of this expression of Dr. Halley, (which was communicated to me by Mr. Morgan,) begins in page 302, and ends in page 310.

After

After this Investigation of the expression $1 + b - \sqrt{bb - 2by}$ ^{1/2} I have applied it in Art. 7, pages 311 and 312, to the example given of this Problem by Dr. Halley, or to the resolution of the equation $1.090,909,0 r^{21} - r^{22} = 0.090,909,0$, in which the true value of r is 1.06814. And it appears that the expression $1 + b - \sqrt{bb - 2by}$ ^{1/2} is in this case $= 1.068,327,5$, which exceeds 1.068,14, or the true value of r , by only 0.000,187,5, which is less than the 5696th part of 1.068,14, or the true value of r . This is as great a degree of exactness as need be desired. For, according to this value of r , the Interest

upon $\begin{smallmatrix} \text{£.} \\ 100 \end{smallmatrix}$ for one year, instead of being exactly $\begin{smallmatrix} \text{£.} & \text{s.} & \text{d.} \\ 6.814, & 0 & 3\frac{1}{4} \end{smallmatrix}$, or 6, 16, $3\frac{1}{4}$, would be $\begin{smallmatrix} \text{£.} & \text{s.} & \text{d.} \\ 6.814 & + & 0.018,75 \end{smallmatrix}$, or $\begin{smallmatrix} \text{£.} & \text{s.} & \text{d.} \\ 6.814 & + & 4\frac{1}{2} \end{smallmatrix}$, or 6, 16, $7\frac{1}{2}$, the excess of which above $\begin{smallmatrix} \text{£.} & \text{s.} & \text{d.} \\ 6.814 & & 3\frac{1}{4} \end{smallmatrix}$

the true Interest, 6, 16, $3\frac{1}{4}$, is too small to be worth attending to. This proves that when t , or the number of years for which the annuity is granted, is not greater than 21 years, Dr. Halley's expression $1 + b - \sqrt{bb - 2by}$ ^{1/2} will give us the value of the rate r to a sufficient degree of exactness. But, if greater exactness were desired, a single process of Mr. Raphson's Method of Approximation, grounded upon the first near value of r obtained by Dr. Halley's expression, to wit, the number 1.068,327,5, would give us a second near value of r that would be $(= 1.068,327,5 - 0.000,193,2) = 1.068,134,3$, which is so near to it's true value, 1.068,14, that it may be considered as perfectly exact. The Investigation of this second near value of r by Mr. Raphson's Method of Approximation is given in Art. 9, pages 313, 314, and 315.

And, as Dr. Halley has only affirmed, but not proved, that the true value of r in this equation $1.190,909,0 r^{21} - r^{22} = 0.090,909,0$ is $= 1.06814$, I have, in Art. 8, pages 312 and 313, shewn that it is so, by shewing that, if it be supposed to be $= 1.06814$, and the value of the expression $\frac{a}{r-1} - \frac{a}{r^t \times r-1}$,
(which

(which is equal to z , or the present value of the annuity a , or $\frac{f}{20}$, for t , or 21, years,) be computed upon that supposition, the said present value z will be $\frac{f}{219.988,075,4}$, or (very nearly) $\frac{f}{220}$, which was the present value supposed in Dr. Halley's example. For, since, when r is $= 1.06814$, z is $= \frac{f}{220}$, it follows, *è converso*, that, when z is $= \frac{f}{220}$, r must be $= 1.06814$.

From the success of Dr. Halley's expression $1 + b - \sqrt{bb - 2by}$ in resolving the example to which he has applied it (and in which the number of years for which the annuity is granted is 21 years,) to a very considerable degree of exactness, and the much greater degree of exactness to which the value of r is carried by a single process of Mr. Raphson's Method of Approximation, it seems reasonable to conjecture, that, by the help of only one such process of Mr. Raphson's Method of Approximation as that exhibited in Art. 9, grounded upon the first near value of r , obtained by means of the expression $1 + b - \sqrt{bb - 2by}$ given us above by Dr. Halley, we shall always be able to find the value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ to a sufficient degree of exactness in all cases in which t , or the number of years for which the annuity is granted, is not greater than 40 years; which are the only cases in which Dr. Halley recommends the use of this expression $1 + b - \sqrt{bb - 2by}$ for the resolution of this equation. But, that this may not be matter of conjecture, but may be known with certainty, I have, in the following articles, proceeded to try the power of the expression $1 + b - \sqrt{bb - 2by}$, together with one process of Mr. Raphson's Method of Approximation grounded upon that expression, in the extreme, or most difficult, case, or when t , or the number of years during which the annuity is to continue, is 40 years. And, to the end that we may know with certainty before-hand, what the exact value of r is in the equation which we propose to resolve in this manner, and consequently may

may be able to discover with certainty how near the two values of r that will be obtained by means of the expression $1 + b - \sqrt{bb - 2by}$ $^{\frac{1}{2}}$ and by means of the subsequent process of Mr. Raphson's Method of Approximation, grounded upon the said expression, will approach to the said exact value, I have, first, supposed r to be some known rate of Interest, and have computed the value of z , or the present value of a given annuity a for the proposed number of 40 years, from it; and having thus found the value of z , I have afterwards reversed the Problem, and supposed z to be equal to the value of it obtained by the former calculation, and have then derived the value of r from the said value of z by means of the expression $1 + b - \sqrt{bb - 2by}$ $^{\frac{1}{2}}$ and of one process of Mr. Raphson's Method of Approximation grounded on it. These calculations are performed in Art. 11, 12, and 13, pages 316, 317, 318, 319, and 320. And it appears from them, that, if a , or the annuity granted, is $\text{£} 20$ *per annum*, and t , or the number of years for which it is granted, is 40 years, and r , or the rate of Interest of money, is $= 1.06$, the present value z of such an annuity will be $= \text{£} 300.926,042,2$, or $\text{£} 300$, 18, 6. This is shewn in Art. 11, pages 316 and 317. And then, in Art. 12, the Problem is reversed, and it is supposed that an annuity of $\text{£} 20$ a year, that is to continue for 40 years, has been sold for the sum of $\text{£} 300$, 18, 6, or $\text{£} 300.926,042,2$; and it is required to determine, from these circumstances, the rate of Interest of money that was allowed to the purchaser of the said annuity when he paid the said sum of $\text{£} 300.926,042,2$ for it.

Here a is $= 20$, as before; and t is $= 40$ years, as before; and r , or the rate of Interest, is the unknown quantity, or the greater of the two roots of the binomial equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, (or $\frac{300.926,042,2 + 20}{300.926,042,2} \times r^{40} - r^{40+1} = \frac{20}{300.926,042,2}$), or $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; which

is to be found by means of the expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} , given us by Dr. Halley, and of a subsequent process of Mr. Raphson's Method of Approximation grounded upon that expression. This expression is computed in Art. 12, pages 317 and 318, and is found to be $= 1.060,910,0$; which is but little greater than 1.06, which is it's true value. For, according to this value of

r , the Interest of 100 pounds for one year would be $(= 100 \times 0.060,910,0)$

$= 6.091,00 = 6, 1, 9\frac{1}{4}$ instead of being exactly 6 pounds. Therefore, even in this extreme case, it appears that this expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} ,

given us by Dr. Halley as a near value of r , is but a little greater than the truth. But, if we make use of the value obtained by this expression as the ground of another approximation to the true value of r in the Method recommended by Mr. Raphson, the second near value of r that will be thereby obtained will be $= 1.060,034,8$, which is as near to it's true value, 1.06, as need to be desired. For, according to this value of r , the Interest of a hun-

dred pounds for one year would be $= 100 \times 0.060,034,8 = 6.003,48 =$

$6, 0, 0\frac{3}{4}$, or 6 pounds and about 3 farthings. We may therefore safely conclude that, when t , or the number of years during which the annuity is to

continue, is not greater than 40 years, the value of r , or the rate of Interest sought, or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, may be

obtained to a very great degree of exactness by means of the expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} , given us by Dr. Halley, together with one process of Mr. Raphson's Method of Approximation grounded on that expression.

The process of Mr. Raphson's Method of Approximation by which the second near value of r , to wit, 1.060,034,8, is obtained, is exhibited in Art. 13, pages 318, 319, and 320.

But though this expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} , given us by Dr. Halley,

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appears,

appears, after this examination of it, to be a very useful one, on account of it's near approach to the true value of r , or the greater root of the equation $\frac{z+a}{z} \times r^i - r^{i+1} = \frac{a}{z}$, in all those cases of that equation in which Dr. Halley recommends the use of it to his readers; yet the difficulty of investigating it by the application of Mr. De Moivre's Multinomial Theorem to it in the case of fractional and negative powers, and the remoteness and subtlety of the principles on which that investigation is grounded, have made me consider it, upon the whole, as unsatisfactory, and have induced me to endeavour to find some other expression for a first near value of r , or the rate of Interest sought, or the greater root of the said equation $\frac{z+a}{z} \times r^i - r^{i+1} = \frac{a}{z}$, that should be derived from simpler and more natural principles. And the result of my endeavours has been the discovery of another expression of the near value of r , which is grounded on a certain Theorem similar to a Theorem given us by Sir Isaac Newton for the reversion of an infinite Series of decreasing quantities, and which is obtained by a train of reasoning much more perspicuous and satisfactory than that by which we obtained the foregoing expression of Dr. Halley, and which is also near enough to the truth to be, in many cases, highly useful of itself alone, or without any correction of the value of r exhibited by it, and to be in all other cases an excellent ground-work, or basis, for a further approach to the true value of r by a process of Mr. Raphson's Method of Approximation. The manner of finding this expression of the near value of r may be described in the following manner.

Sir Isaac Newton, in his second Letter to Mr. Oldenburgh, (Secretary to the Royal Society of London,) dated on the 24th day of October, 1676, and published in the *Commercium Epistolicum* of Mr. John Collins and other persons, in the year 1712, has given us the following Theorem for the reversion of an infinite series of decreasing quantities. *Sit* $z = ay + byy + cy^3 + dy^4 + ey^5 + \&c.$ *Et vicissim erit* $y = \frac{z}{a} - \frac{b}{a^3} \times z^2 + \frac{2bb - ac}{a^5} \times z^3 + \frac{5abc - 5b^3 - aad}{a^7} \times z^4 + \frac{3aacs - 21abbc + 6aabd + 14b^4 - a^3e}{a^9} \times z^5 \&c.$ And I have

have given an investigation of this Theorem in the Third Volume of this Collection of Mathematical Tracts called *Scriptores Logarithmici*, pages 695, 696, 697, &c - - - 700.

And afterwards, in pages 703, 704, 705, &c, - - - 709 I have given an investigation of a second Theorem similar to the foregoing Theorem, and which Sir Isaac Newton seems to have considered as involved in it, which is as follows. "If the infinite Series $ay - by^2 + cy^3 - dy^4 + ey^5 - \&c$, (in which the second and other following terms are marked with the sign $-$ and the sign $+$ alternately,) be equal to z , and the said quantity z be less than 1, so that it's powers $z, z^2, z^3, z^4, z^5, \&c$ form a decreasing progression, the quantity y will be equal to a series of decreasing quantities of which the first five terms will be $\frac{z}{a} + \frac{b}{a^3} \times zz + \frac{2bb - ac}{a^5} \times z^3 + \frac{5b^3 - 5abc + a^2d}{a^7} \times z^4 + \frac{14b^4 + 3a^2c^2 + 6a^2bd - 21ab^2c - a^3e}{a^9} \times z^5$."

And consequently, if a , or the co efficient of y , be $= 1$, we shall have $y = z + b \times zz + \frac{2bb - a}{a^3} \times z^3 + \frac{5b^3 - 5bc + d}{a^5} \times z^4 + \frac{14b^4 + 3c^2 + 6bd - 21b^2c - e}{a^7} \times z^5$.

These Theorems are grounded on the plainest principles of Arithmetick, or the operations of Addition, Subtraction, Multiplication, and Division, without any mention of either the Multinomial Theorem of Mr. De Moivre or Sir Isaac Newton's Binomial Theorem.

Further, these Theorems will be true, not only when $b, c, d, e, \&c$, or the co-efficients of y^2, y^3, y^4, y^5 , and the following powers of y , are decreasing quantities, but also when they are increasing quantities, c being greater than b , d than c , and e than d , and so on, provided the increase of the co-efficients $b, c, d, e, \&c$, is more than counter-balanced by the decrease of y^2, y^3, y^4, y^5 , and the following powers of y , with which they are combined, so as to make the whole terms $by^2, cy^3, dy^4, ey^5, \&c$ form a decreasing progression.

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Now,

Now, to prevent confusion in adapting this Theorem to the resolution of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in which z denotes the present value of the annuity a , let us substitute b instead of z in the said Theorem, and v instead of y . And it will then be as follows. If the series $v - bv^2 + cv^3 - dv^4 + ev^5 - \&c$ is $= b$, the quantity v will be $=$ the Series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{14b^4 + 3c^2 + 6bd - 21b^2c - e} \times b^5 \&c$. We must therefore endeavour to deduce from the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ an equation of the same form as the equation $v - bv^2 + cv^3 - dv^4 + ev^5 - \&c = b$. And this may be done as follows.

It has been shewn above, in Art. 2, pages 302 and 303, that, if $1 + v$ be put $= r$, we shall come to the equation $\frac{z}{at} =$ the Series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 - \&c$; and it is shewn, in Art. 15, pages 321 and 322, that from this last equation may be derived the equation $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c = \frac{2at - 2z}{at^2 + at}$, or, (putting $b, c, d, e, f, \&c$ for the several co-efficients of $vv, v^3, v^4, v^5, v^6, \&c$, and h for the absolute term $\frac{2at - 2z}{at^2 + at}$), the equation $v - b \times vv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = h$, or $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+3}{4} \times b \times v^3 - \sqrt{\frac{t+4}{5}} \times c \times v^4 + \frac{t+5}{6} \times d \times v^5 - \sqrt{\frac{t+6}{7}} \times e \times v^6 + \&c = h$; which is the best and simplest form to which this equation can be reduced.

Now in this last equation the unknown quantity v (being the Interest of one

one pound for one year,) is much less than 1, and the absolute term b (being equal to $\frac{2at - 2z}{at^2 + at} = 1 - \frac{z}{at} \times \frac{2}{t+1}$), is also less than 1, and consequently the powers of both v and b , to wit, $v, v^2, v^3, v^4, v^5, v^6$, &c, and $b, b^2, b^3, b^4, b^5, b^6$, &c, will form decreasing progressions. Therefore the value of v may be found by means of the foregoing Theorem for reverting the series $v - bv^2 + cv^3 - dv^4 + ev^5 - fv^6 + \&c$, and will be equal to the Series $b + b \times b^2 + \frac{2bb - c}{2} \times b^3 + \frac{5b^3 - 5bc + d}{6} \times b^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{24} \times b^5$ &c.

This method of resolving the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ I have applied to the resolution of the two numeral equations $1.090,909,0 r^{21} - r^{22} = 0.090,909,0$ and $1.066,461,5 r^{40} - r^{41} = 0.066,461,5$, which had been before resolved by Dr. Halley's expression $1 + b - \sqrt{bb - 2by} \sqrt{\frac{1}{2}}$. And I have found it to answer very well in point of exactness: but it is attended with a good deal of labour of calculation. The application of it to the equation $1.090,909,0 r^{21} - r^{22} = 0.090,909,0$ is contained in Art. 17, 18, 19, &c, - - - 24, pages 323, 324, 325, &c. - - - 329; and the application of it to the equation $1.066,461,5 r^{40} - r^{41} = 0.066,461,5$ is contained in Art. 25, 26, 27, &c, - - - 33. And in Art. 34, page 337, a short and convenient direction is given for the application of it to the resolution of other equations that come under the general form $\frac{z+a}{r} \times r^t - r^{t+1} = \frac{a}{z}$.

And this concludes the Third Note, on Dr. Halley's Tract on this subject.

Of the Fourth Note to Dr. Halley's Discourse on Compound Interest.

The fourth Note to Dr. Halley's Discourse on Compound Interest relates also to a Problem in which r , or the rate of the Interest of money that has been allowed to the purchaser of an annuity for a certain term of years in addition

dition to a former term of years for which he was before entitled to enjoy the same annuity, is the unknown quantity sought. This Problem is expressed by Dr. Halley in the words following.

A Problem.

“ Let it be required to find the Interest allowed the purchaser, when he pays
“ a sum z for an annuity a wherein he has already a term t , to have it prolonged
“ for a certain time equal to x .”

Example.

“ An annuity of $\frac{£}{20}$ *per annum* that is already granted for a term of years of
“ which 21 are still to come, may, for the sum of 40 pounds paid-down, be
“ prolonged for 10 years more, or to 31 years. What is the rate of Interest
“ required?”

For the solution of this Problem Dr. Halley gives us the following expression, but without any investigation.

“ Put $T = 2t + x + 1$, and $\frac{1}{2} T$ shall be the Index of a root of $\frac{ax}{z}$. Let
“ $\frac{ax}{z} \left| \frac{1}{2} T \right|$ be $= 1 + y$, and $\frac{6T + 6}{xx}$ be $= b$. I say, $r - 1$ is very near
“ to $b - \sqrt{bb - 2by} \left| \frac{1}{2} \right|$.”

As Dr. Halley has not given us any investigation of this expression, though it seems very much to require one, I have, in this fourth Note, endeavoured to supply this defect by stating, in as full and clear a manner as I could, an investigation of it, or, rather, of an expression nearly equal to it, and bearing a great resemblance to it, which has been communicated to me by my learned and ingenious friend Mr. Morgan the Actuary of the Society for Equitable Assurances

ances on Lives, in Chatham-Place near Black-Friars bridge, to whom I have been obliged for the investigations of two former expressions given us by Dr. Halley for the solution of Problems of the like nature, which have been set forth in the first and third of these Notes on the said Discourse of Dr. Halley. Of this Investigation I will here give a short account.

In Art. 1, page 339, it is shewn that, if t be put for the number of years that are yet to come in the first term for which the annuity has been granted, and x be put for the number of years in the additional term for which the annuity is to be prolonged, and a be put for the number of pounds of which the annuity consists, or which are to be paid every year to the purchaser of the annuity, and z be put for the number of pounds contained in the sum of money which is to be paid for the prolongation of the annuity for a second term of x years, and r be put for the rate of the Interest of money allowed the purchaser in this bargain, or the value of one pound, together with it's Interest for one year; it is shewn, upon these suppositions, that z will be $= \frac{a}{r^t \times r - 1} - \frac{a}{r^{t+x} \times r - 1}$. Therefore $\frac{z}{a}$ will be $= \frac{1}{r^t \times r - 1} - \frac{1}{r^{t+x} \times r - 1}$, and $\frac{z}{a} \times r - 1$ will be $= \frac{1}{r^t} - \frac{1}{r^{t+x}}$.

Then, in Art. 2 and 3, pages 340, 341, 342, and 343, it is shewn that, if v be put for the Interest of one pound for a year, and consequently r is $= 1 + v$, and v is $= r - 1$, we shall have $\frac{zv}{a} = \frac{1}{1+v)^t} - \frac{1}{1+v)^{t+x}}$, which (by converting each of the two fractions $\frac{1}{1+v)^t}$ and $\frac{1}{1+v)^{t+x}}$ into an infinite Series by Sir Isaac Newton's Binomial Theorem,) produces a very complicated equation which is set down in page 341, to wit, the equation $\frac{zv}{a} =$ the Series

$$\begin{aligned}
 xv - \frac{\left\{ \begin{array}{l} 2tx + x \\ + x^2 \end{array} \right\}}{2} \times vv & \quad \frac{\left\{ \begin{array}{l} + 3t^2x + 6tx + 2x \\ + 3tx^2 + 3x^2 \\ + x^3 \end{array} \right\}}{6} \times v^2 \\
 - \frac{\left\{ \begin{array}{l} 4t^3x + 6x^3 + 11x^2 + 6x \\ + 6t^2x^2 + 18t^2x + 22tx \\ + 4tx^3 + 18tx^2 \\ + x^4 \end{array} \right\}}{24} \times v^4 + \&c, \text{ and (by dividing}
 \end{aligned}$$

both sides of the equation by xv) the equation $\frac{z}{ax} =$ the Series

$$\begin{aligned}
 1 - \frac{\left\{ \begin{array}{l} 2t + 1 \\ + x \end{array} \right\}}{2} \times v & \quad + \frac{\left\{ \begin{array}{l} + 3t^2 + 6t + 2 \\ + 3tx + 3x \\ + x^2 \end{array} \right\}}{6} \times vv \\
 - \frac{\left\{ \begin{array}{l} + 4t^3 + 6x^2 + 11x + 6 \\ + 6t^2x + 18t^2 + 22t \\ + 4tx^2 + 18tx \\ + x^3 \end{array} \right\}}{24} \times v^3 + \&c; \text{ which equation (by put-}
 \end{aligned}$$

ting $T = 2t + x + 1$, or the numerator of the co-efficient of v .) is converted into the less complicated equation $\frac{z}{ax} =$ the Series $1 - \frac{T}{2} \times v +$

$$\begin{aligned}
 \frac{T \times T + 1 - \sqrt{4t + tx}}{6} \times vv & \\
 - \frac{\left\{ \begin{array}{l} + 4t^3 + 6x^2 + 11x + 6 \\ + 6t^2x + 18t^2 + 22t \\ + 4tx^2 + 18tx \\ + x^3 \end{array} \right\}}{24} \times v^3 + \&c, \text{ or (omitting the last}
 \end{aligned}$$

term,) to the Series $1 - \frac{T}{2} \times v + \frac{T \times T + 1 - \sqrt{4t + tx}}{6} \times vv - \&c.$

Having

Having thus obtained the equation $\frac{z}{ax} = \text{the Series } 1 - \frac{T}{2} \times v + \frac{T \times T + 1 - \sqrt{tt + tx}}{6} \times vv - \&c$, we proceed, in the next place, (in Art. 4, 5, 6, 7 and 8, pages 343, 344, 345, &c, - - 348,) to raise both sides of it to the power of which the fraction $-\frac{2}{T}$ (or the reciprocal of the fraction $-\frac{T}{2}$, which is the coefficient of v in the term $-\frac{T}{2} \times v$ of the said Series,) is the Index, employing, for the purpose of raising the said Series to the said power, Mr. De Moivre's Multinomial Theorem, upon a supposition that it is true in the case of fractional powers, and of powers that are both fractional and negative, as well as in the more simple case of integral and affirmative powers. And we thereby obtain (but not without great difficulty) the equation $\left[\frac{ax}{z}\right] + \frac{2}{T} = \text{the Series } 1 + v - \left[\frac{xx-1}{12T}\right] \times vv \&c$. Therefore (subtracting 1 from both sides,) we shall have $\left[\frac{ax}{z}\right] + \frac{2}{T} - 1 = v - \left[\frac{xx-1}{12T}\right] \times vv \&c$, or $v - \left[\frac{xx-1}{12T}\right] \times vv \&c = \left[\frac{ax}{z}\right] + \frac{2}{T} - 1$; and (putting $y = \left[\frac{ax}{z}\right] + \frac{2}{T} - 1$,) we shall have $v - \left[\frac{xx-1}{12T}\right] \times vv \&c = y$, or (neglecting the terms denoted by the mark &c,) $v - \left[\frac{xx-1}{12T}\right] \times vv = y$; and (multiplying both sides into the fraction $\frac{12T}{xx-1}$,) we shall have $\frac{12T}{xx-1} \times v - vv = \frac{12Ty}{xx-1}$; and (putting $b = \frac{6T}{xx-1}$,) we shall have $2bv - vv = 2by$; and (subtracting both sides from bb ,) we shall have $bb - 2bv + vv = bb - 2by$; and (extracting the square-roots of both sides,) we shall have $b - v = \sqrt{bb - 2by}$, and consequently $b = v + \sqrt{bb - 2by}$, and $v = b - \sqrt{bb - 2by}$, that is, $r - 1$ will be $= b - \sqrt{bb - 2by}$, and r , or $1 + v$, will be $= 1 + b - \sqrt{bb - 2by}$.

Q. E. I.

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But this expression $b - \sqrt{bb - 2by}$, found for the value of v , or $r = 1$, by this Investigation of Mr. Morgan, though it is seemingly the same with Dr. Halley's expression of it, (which is also $b - \sqrt{bb - 2by}$, or consists of the same Letters and signs,) yet is a little different from it. For Dr. Halley makes b equal to the fraction $\frac{6T+6}{xx}$ instead of being equal to $\frac{6T}{xx-1}$, as it is in the foregoing investigation of Mr. Morgan. And Mr. Morgan says, that he does not know whence this difference arises. But he adds, that it is not of much consequence; because both the rules are very accurate, and, of the two, he thinks his own rule (in which b is $= \frac{6T}{xx-1}$), the most accurate. And he is therefore inclined to suspect that Dr. Halley may have made some small mistake in his Investigation, and by this means have obtained a different value for b . But he adds, that, be that as it may, there need not be required a more correct rule than either of the two, as he has found by repeated trials in cases where an inaccuracy was most likely to take place.

After thus exhibiting Mr. Morgan's investigation of the expression $b - \sqrt{bb - 2by}$, in which b is $= \frac{6T}{xx-1}$, I proceed, in Art. 10, pages 342 and 350, to apply it to the Solution of the Problem proposed by Dr. Halley, as an example of the utility of his expression. This Problem is as follows. *An annuity of 20 pounds a year, that is already granted for a term of years of which 21 are still to come, may, for the sum of 40 pounds, be prolonged for ten years more, or to 31 years. What is the rate of Interest required?* Here a is $= 20$,
 $\pounds.$
and z is $= 40$,
 $\pounds.$
and t is $= 21$ years, and x is $= 10$ years. Therefore ax is ($= 20 \times 10$) $= 200$, and $\frac{z}{ax}$ is $= \frac{40}{200}$, and $\frac{ax}{z}$ is ($= \frac{200}{40}$) $= 5$, and T , or $2t + x + 1$, is ($= 2 \times 21 + 10 + 1 = 42 + 10 + 1$) $= 53$, and $\frac{2}{T}$ is
 $= \frac{2}{53}.$

$= \frac{2}{53}$. Therefore $\sqrt[\frac{2}{x}]{T}$ is $= \sqrt[5]{53} =$ (as appears by a Table of Logarithms,)

$1.062,615,6$, and consequently y , or $\sqrt[\frac{2}{x}]{T} - 1$, is $= 0.062,615,6$. And since T is $= 53$, we shall have $6T (= 6 \times 53) = 318$, and $\frac{6T}{xx-1} (= \frac{318}{xx-1} = \frac{318}{100-1} = \frac{318}{99} = \frac{106}{33}) = 3.212,121,2$, that is, b will be $= 3.212,121,2$. Therefore $2by$ will be $(= 2 \times 3.212,121,2 \times 0.062,615,6 = 2 \times 0.201,128,896,210,72) = 0.402,257,792,421,44$, and bb will be $(= 3.212,121,2^2) = 10.317,722,603,489,44$, and $bb - 2by$ will be $(= 10.317,722,603,489,44 - 0.402,257,792,421,44) = 9.915,464,811,068,00$, and consequently $\sqrt{bb - 2by}$ will be $(= \sqrt{9.915,464,811,068,00}) = 3.148,883,1$, and $b - \sqrt{bb - 2by}$ will be $(= 3.212,121,2 - 3.148,883,1) = 0.063,238,1$; that is, v , or $r - 1$, will be $= 0.063,238,1$, and r will be $= 1.063,238,1$. Q. E. I.

This number $1.063,238,1$, which is found for the value of r , by Mr. Morgan's expression, is very nearly equal to, but a little greater than, the value of it found by Dr. Halley's expression, which is $1.063,233$. So that Dr. Halley's and Mr. Morgan's expressions seem to be equally useful in practice. But, as we know the investigation of Mr. Morgan's expression, (which has been given at length in the articles and pages above-mentioned,) and are ignorant of the manner in which Dr. Halley obtained his expression (in which b is made equal to $\frac{6T+6}{xx}$ instead of $\frac{6T}{xx-1}$), I should be inclined, in solving a Problem of this kind, to make use of Mr. Morgan's expression in preference to Dr. Halley's.

The number $1.063,233$, assigned by Dr. Halley as the value of r in the foregoing Problem, is computed in Art. 11, page 350, from Dr. Halley's expression, by making b equal to $\frac{6T+6}{xx}$ instead of $\frac{6T}{xx-1}$.

I then, in Art. 12 and 13, pages 350, 351, and 352, try the exactness of the number 1.063,238,1, found before by means of Mr. Morgan's expression for the value of r , by supposing r to be equal to that number, and computing the sum z which is to be paid for the prolongation of the said annuity of $\text{£. } 20 \text{ per annum}$ for 21 years for an additional term of 10 years upon that supposition. And it appears that the said sum will be $\text{£. } 39.998,792,3$, or $\text{£. s. d. } 39, 19, 10\frac{1}{4}$, instead of being exactly 40, as it was supposed to be in the foregoing Problem. The difference of these two sums $\text{£. } 40$ and $\text{£. s. d. } 39, 19, 10\frac{1}{4}$ is so trifling, that it proves that, at least, in this particular example, or with these values of t and x , or the numbers of years for which the annuity is granted and prolonged, the said expression $1 + b - \sqrt{bb - 2by}$ of the value of r obtained by Mr. Morgan's investigation, is a very useful approximation to the true value of r , or the rate of Interest sought.

In Art. 14, page 353, it is shewn that the general Problem above-mentioned for finding the value of r , when the values of a , z , t , and x are known, may be reduced to the following trinomial equation, to wit, $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$, which in the case of the particular Problem proposed by Dr.

Halley as an example, (in which a is $\text{£. } 20$, and z is $\text{£. } 40$, and t is $= 21$ years, and x is $= 10$ years,) is converted into the equation $\frac{1}{2}r + r^{31} - r^{32} = \frac{1}{2}$, or $0.500,000,0 \times r + r^{31} - r^{32} = 0.500,000,0$. And this numeral equation is resolved, in Art. 15, pages 354, 355, and 356, to a great degree of exactness by Mr. Raphson's Method of Approximation, taking the number found before by Mr. Morgan's expression, to wit, the number 1.063,238,1, for the first near value of r , and substituting $1.063,238,1 - w$ instead of r in

in the said trinomial equation. And the result of this computation is, that w is $= 0.000,002,2$, and consequently that r (or $1.063,238,1 - w$) is ($= 1.063,238,1 - 0,000,002,2$) $= 1.063,235,9$, of which (if no mistake has been made in the calculation,) all the figures will be true.

In Art. 16, pages 356 and 357, it is shewn that 1 is the lesser of the two roots of the equation $0.500,000,0 \times r + r^{31} - r^{32} = 0.500,000,0$, of which the greater root is 1.063,235,9 or the value of r in the Problem from which this equation is derived. And it is observed that our knowledge that 1 is the lesser root of this equation might be employed as a means of obtaining a pretty near value of the greater root of the same equation, by, first, finding the greatest possible value of the trinomial quantity $0.500,000,0 \times r + r^{31} - r^{32}$, (which forms the first, or left-hand, side of the said equation,) by the Doctrine of *Maxima* and *Minima*, and then supposing that greatest value to be an arithmetical mean between the lesser and the greater roots of this equation. But the further consideration of this subject is reserved for a separate tract which follows this Fourth Note immediately. This Fourth Note on Dr. Halley's Discourse on Compound Interest ends in page 357.

Of the Contents of the Fifth Tract in the present Volume.

The Fifth Tract in the present Volume is an Appendix to the preceeding Tract of Dr. Halley on Compound Interest. And the design of it is, to furnish the reader with other methods of solving the three Problems on that subject, in which the rate of the Interest of money allowed in them is the unknown quantity that is required to be found, and for the solution of which Dr. Halley has given three Algebraick expressions for the near value of the said unknown quantity,

quantity, without any demonstrations of them, but which he had probably discovered by the help of Mr. De Moivre's Theorem for raising any trinomial, or quadrinomial, or quinquinomial quantity, or any other higher multinomial quantity whatsoever, to any power of which a whole number denoted by the Letter m is the Index. This Theorem of Mr. De Moivre is demonstrated by him only in this case of the Integral powers of a multinomial quantity, or when the Index m is a whole number; but it is declared by him to be true also in the case of fractional powers of such a quantity, or when the Index of the power to which it is to be raised is any fraction $\frac{m}{n}$, in which m and n are any whole numbers whatsoever: and he has promised his readers to give them, on some future occasion, a demonstration of the Theorem in that case; but, I believe, has never performed this promise. But he has not even declared it to be true in the case of negative powers of such a quantity, either integral or fractional, or in the case of the multinomial quantity $\overline{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}^{-m}$ or of the multinomial quantity $\overline{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}^{-\frac{m}{n}}$, or of the multinomial quantities $\overline{\overline{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}}^{\frac{1}{n}}$ and $\overline{\overline{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}}^{\frac{m}{n}}$. And it would, I imagine, be extremely difficult to form clear demonstrations of this Theorem in all these cases; and, if they were formed, they would be exceedingly intricate and difficult to understand; though it seems probable that the Theorem may be true in all of them, as the Binomial Theorem of Sir Isaac Newton is known to be. But till these cases of Mr. De Moivre's Theorem shall have been fully and clearly demonstrated, no conclusions derived from them can be considered as fully and satisfactorily proved. Now the first of the expressions of Dr. Halley for the value of v , or $r - 1$, here alluded to, (and of which the Investigation given me by Mr. Morgan has been set forth at length in Note I,) to wit, the expression

expression $\sqrt{bb + 2by}^{\frac{1}{2}} - b$, requires us to raise the Series, or multinomial quantity, $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ to the power of which the fraction $\frac{2}{t-1}$ is the Index; which is the case to which Mr. De Moivre has affirmed, but not demonstrated, his Theorem to extend: and therefore this investigation ought not to be considered as perfectly legitimate and satisfactory. And, if it could be admitted as satisfactory, the difficulty of obtaining the $\frac{2}{t-1}$ -th power of the said Series by the application of Mr. De Moivre's Theorem to it (which application of the said Theorem is set-forth in pages 241, 242, 243, &c, - - - 252,) is so great as to make it a very inconvenient and unpleasant method of solving the Problem, or obtaining the value of v , or $r - 1$, and r . And in the investigations of the second and third expressions of Dr. Halley (exhibited in Notes III and IV,) it is required to raise the Series, or multinomial quantity, $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 - \&c$ to the power of which the Index is the negative fraction $\frac{-2}{t+1}$, and to raise the Series, or multinomial quantity, $1 - \frac{T}{2} \times v + \frac{T \times T + 1 - \sqrt{tt + tx}}{6} \times vv - \&c$ to the power of which the negative fraction $-\frac{2}{T}$ is the Index; which are cases in which Mr. De Moivre has not even affirmed that his Theorem is true. And, further, the application of Mr. De Moivre's Theorem to these cases (if we admit that it extends to them,) is even more intricate and difficult than in the investigation of Dr. Halley's first expression, where the Index of the power to which the Series, or multinomial quantity, was to be raised, was the affirmative fraction $\frac{2}{t-1}$. For these reasons I thought it very desirable that other methods of solving

ing these three Problems should be resorted-to, which should be grounded on clearer and simpler principles, and be more satisfactory to the lovers of just and conclusive reasoning on those subjects. And with this view I drew-up this Appendix to Dr. Halley's Discourse.

I have therefore, in Art. 3 and 4, page 361, stated over again Dr. Halley's fourth Problem concerning the amount of a given annuity, for the solution of which he has given us the expression $\overline{bb + 2by}^{\frac{1}{2}} - b$, which has been the subject of the first of the foregoing long Notes; and have shewn how the said Problem may be reduced to the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; in which a stands for the number of pounds in the annuity granted; t for the number of years during which the annuity has been due, but not paid; z for the number of pounds in *the amount* of the said annuity, or in the sum of money which ought to be paid to the annuitant in one payment in consequence of the non-payment of it during the said t years; and r for the value of one pound, together with it's Interest for a year, according to the rate of Interest of money that has been allowed to the annuitant by paying him the said sum z as the amount that is due to him.

This equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is of the same form with the following more general equation, to wit, $Px - x^m = Q$; and therefore the Limits of the magnitudes of the two roots of the former equation may be deduced from the Limits of the magnitudes of the two roots of the latter equation.

Now it is shewn, in Art. 7 and 8, pages 363, 364, and 365, first, that the lesser root of the equation $Px - x^m = Q$ will always be less than $\sqrt[m]{\frac{P}{m}}$; and 2ndly, that the greater root of the same equation will always be greater than

$$\sqrt[m]{\frac{P}{m}}$$

$\frac{P}{m} \sqrt[m-1]{\frac{1}{m}}$, but less than $P \sqrt[m-1]{\frac{1}{m}}$; and, 3dly, that the greatest possible magnitude of the binomial quantity $Px - x^m = Q$ will be $\sqrt[m-1]{m-1} \times \frac{P}{m} \sqrt[m-1]{\frac{m}{m-1}}$, which is that which it has when x is $= \frac{P}{m} \sqrt[m-1]{\frac{1}{m-1}}$. Therefore, if we substitute $\frac{z}{a}$ instead of P , and $\frac{z}{a} - 1$ instead of Q , and r instead of x , and t instead of m , in the equation $Px - x^m = Q$, it will follow that $P \sqrt[m-1]{\frac{1}{m-1}}$ will be $= \frac{z}{a} \sqrt[t-1]{\frac{1}{t-1}}$, and $\frac{P}{m}$ will be $(= \frac{z}{a}) \sqrt[t]{\frac{1}{t}}$ $= \frac{z}{at}$, and $\frac{P}{m} \sqrt[m-1]{\frac{1}{m-1}}$ will be $= \frac{z}{at} \sqrt[t-1]{\frac{1}{t-1}}$, and $\sqrt[m-1]{m-1} \times \frac{P}{m} \sqrt[m-1]{\frac{m}{m-1}}$ will be $= \sqrt[t-1]{t-1} \times \frac{z}{at} \sqrt[t-1]{\frac{t}{t-1}}$, and therefore that the lesser root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ will be less than $\frac{z}{at} \sqrt[t-1]{\frac{1}{t-1}}$, and that the greater root of the same equation will be greater than $\frac{z}{at} \sqrt[t-1]{\frac{1}{t-1}}$, but less than $\frac{z}{a} \sqrt[t-1]{\frac{1}{t-1}}$, and that the greatest possible magnitude of the binomial quantity $\frac{z}{a} \times r - r^t$ will be $\sqrt[t-1]{t-1} \times \frac{z}{at} \sqrt[t-1]{\frac{t}{t-1}}$.

Further, it is shewn in Art. 9, pages 365, 366, and 367, that, if $\frac{P}{m} \sqrt[m-1]{\frac{1}{m-1}}$, or the middle value of x in the binomial quantity $Px - x^m$, or that which it has when the said binomial quantity has attained it's greatest possible magnitude, be denoted by the Letter M , and the excess of this middle value of x above the lesser root of the equation $Px - x^m = Q$ be denoted by the Letter d , the greater root of the said equation will be less than $M + d$. But this

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quantity $M + d$ will often be but a very little greater than the said greater root of the equation $Px - x^m = Q$, and will be sufficiently near to it to be made a convenient ground-work of a process to a nearer value of it by Mr. Raphson's, or some other, Method of Approximation. Whenever therefore, in a binomial equation of this form, $Px - x^m = Q$, we already know the value of it's lesser root, this value may be made subservient to our investigation of the value of it's greater root by finding the value of M , or $\sqrt[m]{\frac{P}{m}}$, and subtracting the lesser root of the equation from M , and calling the remainder, or difference, d , and adding the said difference to M . For the quantity $M + d$, thereby obtained, will be nearly equal to, but something greater than, the said greater root. Now this is the case in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. For the lesser root of this equation is known to be 1. Therefore, if we compute the value of M , (or $\sqrt[t]{\frac{P}{m}}$), or $\sqrt[t]{\frac{z}{at}}$, and put $d = M - 1$, or $\sqrt[t]{\frac{z}{at}} - 1$, we shall have $M + d$, (or $M + M - 1$, or $2M - 1$), or $2 \times \sqrt[t]{\frac{z}{at}} - 1$, for a near value of the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which will be a little greater than the truth. And thus we may obtain a tolerably near value of the greater value of r in this equation, without the subtle and uncertain reasonings and the tedious and intricate Algebraick operations which we were obliged to go through in the investigation of Dr. Halley's expression by the help of Mr. De Moivre's Multinomial Theorem extended to the case of fractional powers.

This value of r , obtained by computing the expression $M + d$, or $2 \times \sqrt[t]{\frac{z}{at}} - 1$, will not indeed be so near the truth as the value of it obtained by

by computing Dr. Halley's expression. But it may easily be improved, or brought nearer the truth, by proceeding in the following manner.

Having computed the value of $M + d$, or $2 \times \left(\frac{z}{at} \right)^{\frac{1}{t-1}} - 1$, let this value be substituted instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, and let the value of that quantity resulting from such substitution be called D . Then, since $M + d$ is greater than the true value of r , let a small portion of it, as, for example, a 200th part of it, be subtracted from $M + d$, and let the remainder $M + d - \left(\frac{M+d}{200} \right)$ be called y . Thirdly, let y be substituted instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, and let the value of the said quantity resulting from such substitution be called E . Then we shall have two sets of quantities, to wit, the three quantities $M + d$, $M + d - \left(\frac{M+d}{200} \right)$, or y , and r , which are nearly equal to each other, and the three quantities D , E , and Q , or $\frac{z}{a} - 1$, which are related in the same manner to the three former quantities $M + d$, y , and r , (being the results of the substitution of the said three former quantities instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$), and which will also be nearly equal each other; and of these six quantities the only one that is unknown is r . We may therefore, by supposing the differences of the three former quantities $M + d$, y , and r , to be to each other in the same proportion as the corresponding differences of the three latter quantities D , E , and Q , or $\frac{z}{a} - 1$, find a new value of the unknown quantity r that will be nearer to it's true value than either $M + d$ or y , and oftentimes as near to the truth as need to be desired. And, if still greater exactness is desired, we may make this new value of r , that has been obtained by this proportion, or by what may be called *The Differential Method of Approximation*, the ground-work of a further approach to it's true value by Mr. Raphson's Method of Approximation; and the value

of r that will be obtained by one process of that Method will be near enough to it's true value to satisfy the most scrupulous calculator. These things are explained, and some other observations made upon the subject, in Art. 10, 11, and 12, pages 368, 369, 370, and 371.

This Method of resolving the equation $\frac{x}{a} \times r - r' = \frac{x}{a} - 1$ is then applied to the resolution of the numeral equation $17.8614 r - r^{12.5} = 16.8614$, which Dr. Halley had resolved by his expression. And the result of this application is, that $M + d$, or the first near value of r , or the greater root of the proposed equation, (which is obtained by means of our knowledge that the lesser root of the said equation is $= 1$,) is 1.063,045,0; and the second near value of r (which is derived from the former near value 1.063,045,0 by the Differential Method of Approximation,) is 1.059,863,0, which is very near it's true value 1.06; and the third near value of it (which is derived from the second near value 1.059,863,0 by one process of Mr. Raphson's Method of Approximation,) is 1.060,009,9, which is so near the true value 1.06, that the Interest of an 100 pounds, computed from this number, would be equal to 6 pounds and only 0.95040 of a farthing, or less than a single farthing, instead of being exactly equal to 6 pounds. This resolution of the equation $17.8614 r - r^{12.5} = 16.8614$ is contained in Art. 13, 14, and 15, pages 372, 373, 374, &c. - - - 377.

If, in this manner of resolving the equation $17.8614 \times r - r' = 16.8614$, by making use of our knowledge that it's lesser root is equal to 1 in order to find the first near value of it's greater root, to wit, $M + d$, or 1.063,045,0, we had not had recourse to the Differential Method of Approximation to find a second near value of it, but, instead of it, had employed Mr. Raphson's Method of Approximation for that purpose, by taking $M + d - \sqrt{\frac{M+d}{200}}$, or 1.057,729,8, for the ground-work or basis, of such approximation, the second
near

near value of r that we should thereby have obtained would have had about the same degree of exactness as 1059,863.0, or the second near value of it obtained by the said Differential Method, and would have been obtained with about the same quantity of arithmetical calculation. So that it is pretty indifferent which of these two Methods of Approximation we employ in order to obtain this second near value. And then a second process of Mr. Raphson's Method of Approximation would have given us a third near value of r , that would have been as near the truth as any one would desire.

The like method of resolving an equation of the same form $\frac{x}{a} \times r - r^t = \frac{x}{a} - 1$, (to wit, by making use of our knowledge that the lesser root of this equation is equal to 1 in order to obtain a first near value of its greater root,) is then applied to the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, which results from a Problem in which t , or the number of years for which the annuity is granted, is 70; which is one of the most difficult cases that can be proposed. Yet even here this method of resolution is found to answer very well. For the true value of r , or the greater root of this equation, is 1.06. And, though $M + d$, or the first near value of it (which is obtained by means of our knowledge that the lesser root of this equation is = 1,) is = 1.077,594,2, which is much too great, yet, by the very easy operation of taking an arithmetical mean between $M + d$ and M , or between 1.077,594,2 and 1.038,797,1, we may find the number 1.058,195,6, which is much nearer than 1.077,594,2 to the true value of r , and is almost as near to it as the number 1.058,712,7, which was found for it by Dr. Halley's expression $1 + \sqrt[bb + 2by]{\frac{1}{2}} - b$; and then, by considering this number 1.058,195,6 as our first near value of r , and making it the foundation of a process of the Differential Method, (by increasing it by $\frac{1.058,195,6}{200}$, or 0.005,290,9; whereby we shall obtain the number 1.063,486,5, and substituting, first, 1.058,195,6 and, secondly,

1.063,486,5

1.063,486,5 in the binomial quantity $967.648,136,7 r - r^{70}$ instead of r , the results of which substitutions are 971.527,000,6 and 954.738,937,0,) we obtain for a second near value of r the number 1.059,733,2, which is nearer than Dr. Halley's number 1.059,712,7, to 1.06, or the true value of it. And, lastly, by making this last near value of r , to wit, 1.059,733,2, the ground-work, or basis, of a single process of Mr. Raphson's Method of Approximation, we shall obtain the number 1.059,997,3; which is so near to 1.06, or the true value of r , that the Interest of a hundred pounds for a year, computed accord-

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ing to this estimation of r , would be equal to 5, 19, and more than $\frac{99}{100}$ of another shilling, instead of being exactly equal to 6 pounds. The difference of these sums is less than half a farthing; and therefore the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$ is now resolved with great exactness. The resolution of this equation by this method is given in Art. 16, 17, 18, &c, - - - 21, pages 377, 378, 379, &c, - - - 384.

But, when we have obtained a pretty good first near value of r by means of the quantities M and $M + d$, there are two other methods of obtaining a more exact value of r besides the Differential Method and Mr. Raphson's Method of Approximation. The first of these methods consists in the resolution of a certain quadratick equation which may be deduced from the equation $\frac{x}{a} \times r - r^f = \frac{x}{a} - 1$, but not without a good deal of computation both algebraïcal and arithmetical, which makes this method, in my opinion, much less convenient than the former method by the help of the Differential and Mr. Raphson's Methods of Approximation: but it gives the value of r to a very great degree of exactness. The second of these methods is grounded on one of Sir Isaac Newton's Theorems for reverting an infinite series. But this method likewise requires a great quantity of very laborious arithmetical calculation, and therefore appears to me less convenient and fit for practice than the

the former method above-described, though it likewise gives the value of r to a great degree of exactness. I will just give a brief statement of these two methods, and of the degree of exactness to which, when applied to the resolution of the above-mentioned numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, they give us the value of r , or the greater root of that equation.

The first of these two methods, which consists in the resolution of a quadratic equation, is as follows.

Let $M + d$, or some quantity less than $M + d$, such as $M + d - \sqrt{\frac{M+d}{200}}$, or $\frac{M+d+M}{2}$, or $M + \frac{d}{2}$, be taken as a first near value of r , or the greater root of the equation $\frac{x}{a} \times r - r^t = \frac{x}{a} - 1$, and denoted by the small Letter b . And let b be substituted instead of r in the binomial quantity $\frac{x}{a} \times r - r^t$. Then, if the quantity $\frac{x}{a} \times b - b^t$ is less than $\frac{x}{a} \times r - r^t$, or, (it's equal,) the absolute term $\frac{x}{a} - 1$, it will follow that b will be greater than r ; and, if the quantity $\frac{x}{a} \times b - b^t$ is greater than $\frac{x}{a} - 1$, it will follow that b will be less than r . Let us, first, suppose that $\frac{x}{a} \times b - b^t$ is less than $\frac{x}{a} - 1$, and consequently that b is greater than r . And then the value of r may be found in the following manner.

Let w be put for $b - r$, or the excess of b above r , and x for the fraction $\frac{w}{b}$, and e for $\frac{x}{a}$, and f for $\frac{x}{a} - 1$. And let the values of the two fractions $\frac{tb^t - eb}{tb^t - tb^t}$ and $\frac{2f + 2b^t - 2eb}{tb^t - tb^t}$ be computed, and the former of them be called G , and the latter of them be called H . And we shall then have $2Gx - xx + \&c = H$, and $x = G - \sqrt{GG - H}$, and w , or $b \times x$, $= bG - b \times \sqrt{GG - H}$ and

and r , or $b - w$, $= b - bG + \sqrt{GG - H}$; that is, r will be nearly equal to $b - bG + b \times \sqrt{GG - H}$. Q. E. I.

This expression of the value of r is investigated in Art. 23, pages 386, 387, and 388.

Secondly, let us suppose that the binomial quantity $\frac{x}{a} \times b - b'$ is greater than $\frac{x}{a} - 1$, and consequently that b is less than r . And then the value of r may be found as follows.

Let w be put for $r - b$, and x be $= \frac{w}{b}$; and let e be $= \frac{x}{a}$, and f be $= \frac{x}{a} - 1$, as before. And let the values of the two fractions $\frac{eb^2 - eb}{e^2b^2 - eb^2}$ and $\frac{2eb - 2b^2 - 2f}{e^2b^2 - eb^2}$ be computed, and the former of them be called G , and the latter of them be called H . And we shall then have $2Gx + xx = H - \&c$, and $x = \sqrt{GG + H} - G$, and $w (= b \times x) = b \times \sqrt{GG + H} - b \times G$, and $r (= b + w) = b + b \times \sqrt{GG + H} - b \times G$; that is, r will be nearly equal to $b + b \times \sqrt{GG + H} - b \times G$. Q. E. I.

This expression of the value of r is investigated in Art. 24, pages 388, 389, 390, and 391.

This last expression is then (in Art. 25, pages 391 and 392,) applied to the resolution of the before-mentioned numeral equation $967.648,136,7 \times r - r^7 = 966.648,136,7$, by taking for b (or the first near value of r , from which the further approximation to it's true value is to begin,) the number 1.058,195,6, which was obtained by taking an arithmetical mean between $M + d$ and M , and which is less than 1.06, or the true value of r . And the result of the computation

computation of this expression $b + b \times \sqrt{GG + H} - b \times G$ is that it is $= 1.059,998,0$; and consequently r is nearly equal to $1.059,998,0$. This value of r is wonderfully near it's true value, 1.06 ; and therefore we may conclude that this method of finding the value of r , by resolving the quadrattick equation $2Gx + xx = H$, is a very exact one.

The other way of approximating further towards the true value of r , or the greater root of the equation $\frac{x}{a} \times r - r^2 = \frac{x}{a} - 1$, after we have obtained a pretty good first near value of it by means of it's two Limits M and $M + d$, is derived from the following Theorem of Sir Isaac Newton for reverting an infinite Series. If y be a quantity less than 1 , so that it's powers $y, y^2, y^3, y^4, y^5, y^6, y^7$, &c form a decreasing progression of terms, and a, b, c, d, e, f, g , &c are numeral co-efficients of the several successive powers of y , to wit, $y, y^2, y^3, y^4, y^5, y^6, y^7$, &c; and, if the Series $ay + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$ is $= z$, and z is also less than 1 , so that it's several successive powers $z, z^2, z^3, z^4, z^5, z^6, z^7$, &c, form likewise a decreasing progression of terms; then will y be equal to an infinite Series of quantities of which the first five terms will be $\frac{z}{a} - \frac{b}{a^3} \times z^2 + \frac{2bb - ac}{a^5} \times z^3 - \frac{5b^3 - 5abc + a^2d}{a^7} \times z^4 + \frac{3a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^9} \times z^5$. And therefore, if a is $= 1$, (as is very often the case,) and the Series $y + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$ is $= z$, the quantity y will be $=$ the Series $z - bz^2 + \frac{2bb - c}{1} \times z^3 - \frac{5b^3 - 5bc + d}{1} \times z^4 + \frac{3c^2 - 21b^2c + 6bd + 14b^4 - e}{1} \times z^5$ &c.

This last Series may be applied to the finding a more accurate value of the greater root of the equation $\frac{x}{a} \times r - r^2 = \frac{x}{a} - 1$ than the first near value of it which had been obtained by means of it's two Limits M and $M + d$. But, to avoid confusion by employing the same Letter z to denote different quantities,

quantities, it will be convenient to substitute another Letter instead of z in the foregoing Theorem. And for this purpose I have chosen the small Letter q . And then the said Theorem will be as follows. If y is less than 1, and q is also less than 1, and the Series $y + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$ is $= q$, quantity y will be $=$ the Series $q - bq^2 + \overline{2bb - c} \times q^3 - \overline{5b^3 - 5bc + d} \times q^4 + \overline{3c^2 - 21b^2c + 6bd + 14b^4 - e} \times q^5 \&c$.

Now let the Letter p be employed, instead of the Letter b , to denote the first near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which has been already obtained by means of it's two Limits M and $M + d$. And let us suppose p to be somewhat less than the true value of r , or the said greater root, as the number 1.058,195,6 (which was the first near value of r so obtained in resolving the numeral equation $967.648,136,7 \times n - r^{70} = 966.648,136,7$) was found to be. And let w be put for the unknown small excess of r , or the said greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, above p , the known near value of it. And, further, to simplify the notation, let P be substituted for $\frac{z}{a}$, the co-efficient of r in the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and Q for the absolute term $\frac{z}{a} - 1$; whereby the said equation will be converted into the equation $Pr - r^t = Q$. And let y be put $= \frac{w}{p}$.

Then will r be $= p + w (= p \times \sqrt[t]{1 + \frac{w}{p}} = p \times \sqrt[t]{1 + y}) = p + py$, and Pr will be $(= P \times \overline{p + py}) = Pp + Ppy$, and r^t will be $= \overline{p + py}^t = p^t \times \overline{1 + y}^t = p^t \times$ the Series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + \&c$. Therefore the binomial

quantity

quantity $Pr - r^t$ will be $= Pp + Ppy - p^t \times$ the Series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + \&c.$
 But $Pr - r^t$ is $= Q$. Therefore the compound quantity $Pp + Ppy - p^t \times$ the Series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + \&c$ will also be $= Q$. And from this equation may be derived, by the reasonings set-forth in Art. 28 and 29, pages 395, 396, and 397, the equation $y + t \times \frac{t-1}{2} \times \frac{p^t}{tp^t - Pp} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp} \times y^4 + \&c = \frac{Pp - p^t - Q}{tp^t - Pp}$; which is an equation of the same form with the equation $y + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c = q$.

Now let q be put for the absolute term $\frac{Pp - p^t - Q}{tp^t - Pp}$ of this equation; and let b be put $= t \times \frac{t-1}{2} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^2 ; and c be put $= t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^3 ; and d be put $= t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^4 ; and e be put $= t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^5 . And the said equation will be thereby converted into the equation $y + by^2 + cy^3 + dy^4 + ey^5 + \&c = q$.

And consequently by the above-mentioned Theorem of Sir Isaac Newton, we shall have $y = q - b \times q^2 + \frac{2bb - d}{1} \times q^3 - \frac{5b^3 - 5bc + d^2}{1} \times q^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{1} \times q^5$ &c.

And, when y is found by computing some of the terms of this Series, we shall have $w = p \times y$, and $r = p + w = p + p \times y$.

Q. E. I.

This method of resolving the equation $\frac{x}{a} \times r - r^2 = \frac{x}{a} - 1$ is extremely laborious; but it gives us in many cases a very accurate value of r . For, in Art. 30, pages 398, 399, and 400, it is applied to the resolution of the numeral equation $967.648,136,7 \times r - r^{20} = 966.648,136,7$, after taking p , (or the first near value of r that is obtained by means of it's two Limits M $M + d$), equal to the number 1.058,195,6. And the result of the computation of only the four first terms of the Series is, that y is $= 0.001,699,4$ and that w , or $p \times y$, is $(= 1.058,195,6 \times 0.001,699,4) = 0.001,798,2$, and consequently that r , or $p + w$, is $(= 1.058,195,6 + 0.001,798,2) = 1.059,993,8$, which is exceedingly near to 1.06, or the true value of r . However, I think the first of these three Methods of resolving this equation $\frac{x}{a} \times r - r^2 = \frac{x}{a} - 1$, (by which, after finding a good first near value of r , or the greater root of it, by means of M and $M + d$ the two Limits of such greater root, we employ either one process of the Differential Method of Approximation and one process of Mr. Raphson's Method, or two successive processes of Mr. Raphson's Method, to find a more exact value of it,) much more convenient than either of the two other Methods of resolving it, and therefore I have, in Art. 31, pages 401, 402, 403, &c. - - - 406, made a full recapitulation of all the reasonings and processes which are to be employed in resolving it by that first method. And this recapitulation is followed by a Scholium, (in Art. 32, pages 406, 407, and 408,) in which I have stated my reasons for thinking that

that either of these three methods of resolving the said equation, and especially the first Method, is preferable to Dr. Halley's Method of resolving it by means of his abstruse Algebraick expression, which is derived from Mr. De Moivre's Multinomial Theorem extended to the case of fractional powers. This Scholium concludes that part of this Appendix to Dr. Halley's Discourse which relates to the resolution of the equation $\frac{x}{a} \times r - r^t = \frac{x}{a} - 1$.

In page 408 I have taken into consideration the second question relating to Compound Interest, for the solution of which Dr. Halley has given us an Algebraick expression derived from Mr. De Moivre's Multinomial Theorem. This is Dr. Halley's fourth Problem relating to the present values of annuities for terms of years, and is expressed by him in these words; "The annuity (a), present value (z), and time (t), being given, to find (r) the rate of Interest." And for the solution of this Problem, when t is less than 40 years, he has given us the following direction. "Let $\frac{t+1}{2}$ be the Index of a root of $\frac{at}{x}$; from which root subtract 1, and call the remainder y . And let $\frac{6}{t-1}$ be called b . I say, that $b - \frac{bb - 2by}{2by}^{\frac{1}{2}}$ is sufficiently near to $r - 1$, and will be still nearer the truth as the number of years is smaller."

This expression has been investigated very fully in the Third of the foregoing Notes on Dr. Halley's Discourse by the help of Mr. De Moivre's Multinomial Theorem, and upon a supposition that the said Theorem is true in the case of powers that are both fractional and negative, which makes the investigation of it exceedingly intricate, obscure, and subtle. In this case therefore I should be inclined to make use of some other method of determining the value of r , or the rate of Interest, that should be derived from some clearer principles than those by which Dr. Halley's said expression

$b - \sqrt{bb - 2by}$ has been investigated in the Third of the foregoing Notes on his Discourse.

Now one such method of finding the value of r in this second case of the foregoing Problem, instead of resorting to the expression $b - \sqrt{bb - 2by}$ given us by Dr. Halley, has been already described in the latter part of the said Third Note on Dr. Halley's Discourse, and is here described again in a summary manner in Art. 35, pages 410 and 411, which is derived from a very useful Theorem given us by Sir Isaac Newton for the reversion of an infinite series. This method enables us to find the value of r to a great degree of exactness; but it requires a great deal of calculation: and therefore I have, in this Appendix, endeavoured to find some other methods of investigating this value, that are less laborious, and consequently more fit for practice. And with this view I shew in the first place, in Art. 36, page 412, how to reduce the foregoing Problem to the binomial equation $\frac{x+a}{x} \times r^x - r^{x+1} = \frac{a}{x}$, of which the quantity sought, or the rate of Interest r , is the greater root. And then, in Art. 37, page 413, it is shewn that the lesser root of this equation is $= 1$, the knowledge of which will afterwards enable us to obtain a tolerably good first near value of it's greater root. But for this purpose it will be convenient, first, to make a few remarks on the number of roots in a binomial equation of the more general form $Px^m - x^{m+1} = Q$, and on the limits of their magnitudes; which remarks we may afterwards transfer to the roots of the equation $\frac{x+a}{x} \times r^x - r^{x+1} = \frac{a}{x}$, which comes under that more general form. This examination of the said general equation $Px^m - x^{m+1} = Q$ is then made in Art. 40, 41, 42, 43, 44, pages 414, 415, 416, &c. - - 419, and it is shewn that the greatest possible magnitude of the lesser root of this equation will be $\frac{m}{m+1} \times P$, and that the least possible magnitude of the greater

greater root of this equation will be the same quantity $\frac{m}{m+1} \times P$, and that the greatest possible magnitude of the said greater root will be P , and that the greatest possible magnitude of the binomial quantity $Px^m - x^{m+1}$ will be $\frac{P}{m+1} \times \frac{mP}{m+1}^m$, or that which it has when x is $= \frac{m}{m+1} \times P$. And hence it follows in the first place, that, if the absolute term, Q , of the equation $Px^m - x^{m+1} = Q$ is greater than $\frac{P}{m+1} \times \frac{mP}{m+1}^m$, the said equation will be impossible; and, 2ndly, that, if the absolute term Q is equal to $\frac{P}{m+1} \times \frac{mP}{m+1}^m$, the said equation will have only one root, and that root will be equal to $\frac{m}{m+1} \times P$; and, 3dly, that, if the absolute term Q is less than $\frac{P}{m+1} \times \frac{mP}{m+1}^m$, the said equation will have two roots, of which the lesser will be less than $\frac{m}{m+1} \times P$, and the greater will be greater than $\frac{m}{m+1} \times P$, but less than P .

These limits of the greater root of the equation $Px^m - x^{m+1} = Q$ are so near to each other (their difference being only $= P - \frac{m}{m+1} \times P = \frac{m+1}{m+1} \times P - \frac{m}{m+1} \times P = \frac{1}{m+1} \times P$), that, if we take an arithmetical mean between them, the said arithmetical mean (which will be $\frac{\frac{m+1}{2} \times P, \text{ or } \frac{2m+1}{2m+2} \times P}{\frac{m+1}{m+1}}$) will be a very good approximation to the true value of the said greater root, which lies between the said limits $\frac{m}{m+1} \times P$ and $(\frac{m+1}{m+1} \times P, \text{ or}) P$. And thus we may easily obtain a good first near value of the greater root of the equation $Px^m - x^{m+1} = Q$, even without a previous knowledge of the lesser root of this equation.

But,

But, if we already know the value of the lesser root of the equation $Px^m - x^{m+1} = Q$, we may thereby obtain a pretty good approximation to the value of the greater root of the said equation by proceeding as follows. Let $\frac{m}{m+1} \times P$ (or the value of x at the instant at which the binomial quantity $Px^m - x^{m+1}$, during the increase of x from 0 to P , attains it's greatest magnitude,) be denoted by the Letter M ; and let d be put for the difference by which M exceeds the lesser root of the equation $Px^m - x^{m+1} = Q$, which is supposed to be known. Then it will be found, that, if we add half the quantity d to the said middle quantity M , the sum $M + \frac{d}{2}$ will often be pretty nearly equal to the greater root of the said equation. I have found by the examples of equations of this form, which I have undertaken to resolve, that $M + \frac{d}{2}$ is much fitter than $M + d$ to be adopted in this conjectural manner for a first near value of x , or the greater root of the equation.

When we have thus taken either $\frac{2m+1}{2m+2} \times P$, or $M + \frac{d}{2}$, for a first near value of x , or the greater root of the equation $Px^m - x^{m+1} = Q$, we must substitute it instead of x in the binomial quantity $Px^m - x^{m+1}$, and compare the value of the said binomial quantity resulting from such substitution with the absolute term Q . And, if the said value is less than Q , the said near value of x , or the greater root of the said equation, will be greater than it's true value; and, if the said value, or result, is greater than the absolute term Q , the said near value of x , or the said greater root, will be less than it's true value. And from the difference between the said result and the absolute term Q , we may form a good judgement of the magnitude of the difference between the said first near value of the said greater root and it's true value. And then we may

may proceed to find a more exact value of the said greater root either by one process of the Differential Method of Approximation, and one Method of Mr. Raphson's Method, or by two processes of Mr. Raphson's Method.

After obtaining these conclusions concerning the general Equation $Px^m - x^{m+1} = Q$, I have, in Art. 45, 46, and 47, pages 420, 421, 422, and 423, transferred them to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, by substituting $\frac{z+a}{z}$ instead of P , and $\frac{a}{z}$ instead of Q , and r instead of x , and t instead of m . And thereby we find M , or $\frac{m}{m+1} \times P$, to be $= \frac{t}{t+1} \times \frac{z+a}{z}$, and $\frac{2m+1}{2m+2} \times P$ to be $= \frac{2t+1}{2t+2} \times \frac{z+a}{z}$, and therefore may conclude that $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ will be a good first near value of the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. And further, since we know that the lesser root of this equation is $= 1$, we shall have $d = M - 1 = \frac{t}{t+1} \times \frac{z+a}{z} - 1$, and consequently $\frac{d}{2} = \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, and $M + \frac{d}{2} (= \frac{t}{t+1} \times \frac{z+a}{z} + \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2} = \frac{3t}{2t+2} \times \frac{z+a}{z} + \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}) = \frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, and therefore may conclude that $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ will also be often a pretty good first near value of the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. One of these two quantities $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ may therefore be chosen for a first near value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and substituted instead of r in the binomial quantity $\frac{z+a}{z} + r^t - r^{t+1}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be less, or greater, than $\frac{a}{z}$, the absolute

term of that equation, and consequently whether the said near value of r so substituted is greater, or less, than it's true value. And then, the said first near value of r may be made use of as the ground-work, or basis, of a process of the Differential Method of Approximation, and afterwards of a process of Mr. Raphson's Method of Approximation, or (omitting the Differential Method,) of two successive processes of Mr. Raphson's Method; by which processes the value of the greater root of the equation $\frac{z+a}{x} \times r^i - r^{i+1} = \frac{a}{x}$ will be obtained to a great degree of exactness.

This method of finding r , or the greater root of the equation $\frac{z+a}{x} \times r^i - r^{i+1} = \frac{a}{x}$, is then applied to the resolution of the numeral equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ which results from the example which Dr. Halley has given us of this Problem, and which he has resolved by his above-mentioned expression $b - (bb - 2by)^{\frac{1}{2}}$.

Now the true value of r , or the greater root of this equation, is 1.068,14, as is declared by Dr. Halley, and has been shewn above in this present volume in the Third Note upon Dr. Halley's Discourse on Compound Interest, in page 312. And the near value of it obtained by Dr. Halley's expression $b - (bb - 2by)^{\frac{1}{2}}$ is 1.068,327, as is shewn in page 312. And in the present method of resolving this equation, the first near value of r , obtained by computing the expression $\frac{2i+1}{2i+2} \times \frac{z+a}{x}$, is = 1.066,115,6, which falls short of the true value 1.068,14 by only 0.002,024,4, or less than it's 527th part; which is a very considerable degree of exactness. The other first near value of r , which is obtained by computing the expression $M + \frac{d}{a}$, or $\frac{3i}{2i+2} \times \frac{z+a}{x} - \frac{1}{2}$, is = 1.061,983,3; which is much less exact than the former near value 1.066,115,6. Therefore 1.066,115,6 is the fitter number to be employed as
a first

a first near value of r , or as the basis of a process of the Differential Method of Approximation. And it is so employed in Art. 50, pages 425, 426, and 427; and the second near value of r resulting from such process is 1.068,171,2; which is true in the first five figures 1.068,1, and is nearer than 1.068,327, or the number obtained by Dr. Halley's expression, to the true value 1.068,14. And, if we make this number 1.068,171 the ground-work of a process of Mr. Raphson's Method of Approximation, we shall obtain the number 1.068,133,2 for a third near value of r , which is exceedingly near the true value 1.068,14, their difference being less than the 158,564th part of the said true value. These calculations are performed in pages 423, 424, 425, &c. - - 429.

Dr. Halley has declared that, when t , or the number of years for which the annuity is granted is greater than 40, he does not recommend the use of his Algebraick expression $b - \sqrt{bb - 2by}$ for the solution of the Problem; but has given us other methods of solving it, which are better adapted to these cases. We may therefore consider the case of an annuity for 40 years as the most difficult case that can be conveniently solved by his expression. I have therefore applied the foregoing method of solution to a Problem in which an annuity is granted for a term of 40 years, with a view to try the efficacy of it in this extreme case. And, in order to judge with certainty of the degree in which the values of the two expressions $\frac{2t+1}{2t+2} \times \frac{x+a}{x}$ and $M + \frac{d}{x}$, or $\frac{3t}{2t+2} \times \frac{x+a}{x} - \frac{1}{2}$, will approach to the true value of r , or the rate of Interest, I have, first, supposed r to be known, and to be = 1.06, and have computed the present value, z , of an annuity of $\overset{£}{20}$ per annum that is to be sold to a purchaser for a term of 40 years, upon that supposition. And it is found that z , or the price to be paid for such an annuity, will be = $\overset{£}{300.926,042,2}$, or

$\begin{matrix} \text{£.} & \text{s.} & \text{d.} \\ 300, & 18, & 6 \end{matrix}$; whence it follows, *è converso*, that, if the sum of $\begin{matrix} \text{£.} \\ 300.926,042,2 \end{matrix}$, or $\begin{matrix} \text{£.} & \text{s.} & \text{d.} \\ 300, & 18, & 6 \end{matrix}$, has been paid for an annuity of $\begin{matrix} \text{£.} \\ 20 \end{matrix}$ for 40 years, the rate of Interest that has been allowed to the purchaser of such annuity, must have been 1.06, or 6 per cent. I then suppose that such an annuity has been sold for the said price of $\begin{matrix} \text{£.} & \text{s.} & \text{d.} \\ 300, & 18, & 6 \end{matrix}$, or $\begin{matrix} \text{£.} \\ 300.926,042,2 \end{matrix}$, but that it is not known what rate of Interest was made the ground of the valuation of it, or was intended to be allowed to the purchaser on that occasion; and that it is required to determine, what that rate of Interest must have been, by reasonings founded on our knowledge of the other circumstances belonging to it, to wit, the quantity of the annuity, the time during which it is to continue, and the price that has been paid for it.

Now, if a be put for the quantity of the annuity, or the number of pounds to be annually paid to the annuitant, and t for the number of years during which the annuity is to continue, and x for the number of pounds in the price that has been paid for it, and r for the rate of Interest allowed the purchaser in requiring the said price for it, (which rate of Interest is supposed to be unknown, and is required to be found from the knowledge of the other three quantities,) the solution of the said Problem may be reduced to the resolution of the binomial equation $\frac{x+a}{x} \times r^t - r^{t+1} = \frac{a}{x}$, and the greater root of this equation will be the value of r that we are to find.

Now, since a is, in this example, $\begin{matrix} \text{£.} \\ 20 \end{matrix}$, and, x is $\begin{matrix} \text{£.} \\ 300.926,042,2 \end{matrix}$, and t is = 40 years, we shall have $\frac{a}{x} (= \frac{\begin{matrix} \text{£.} \\ 20 \end{matrix}}{300.926,042,2}) = 0.066,461,5$, and $\frac{x+a}{x} (= \frac{x}{x} + \frac{a}{x} = 1 + \frac{a}{x} = 1 + 0.066,461,5) = 1.066,461,5$, and $r^t = r^{40}$,

and

and $r^{t+1} = r^{41}$; and consequently the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will, in this case, become $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and the greater root of this equation will be the value of r , or the rate of Interest sought. This value of r must therefore now be investigated by the method described in the foregoing articles.

We shall therefore have, for a first near value of r , the quantity $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$
 £.

$(= \frac{81}{82} \times 1.066,461,5) = 1.053,455,8$; which is less than the true value 1.06 by 0.006,544,2, or less than the 163d part of the said true value, which is near enough to the truth to be a good ground-work of a further approach to the said true value by either the Differential Method of Approximation or Mr. Raphson's Method. And we shall also have, for another first near value of r , the quantity $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ $(= \frac{3 \times 40}{2 \times 40 + 2} \times 1.066,461,5 - \frac{1}{2} = \frac{120}{82} \times 1.066,461,5 - \frac{1}{2} = \frac{60}{41} \times 1.066,461,5 - \frac{1}{2} = \frac{63.987,690,0}{41} - \frac{1}{2} = 1.560,675,3 - 0.500,000,0) = 1.060,675,3$; which is somewhat greater than 1.06, or the true value of r , but approaches much nearer to it than the former number 1.053,455,8, and therefore will be still fitter than that former number to be made the basis of a further approximation to the said true value. I have then made use of both these numbers 1.053,455,8 and 1.060,675,3 conjointly as a ground-work of a process of the Differential Method of Approximation, and have thereby obtained, for a second near value of r , the number 1.059,794,9; which is nearer to the true value 1.06 than even the number 1.060,675,3. And then, with this second near value of r , to wit, 1.059,794,9; as the ground-work of a process of Mr. Raphson's Method of Approximation, I have obtained, for a third near value of it,

it, the number 1.060,037,1; which is as near the true value 1.06, or 1.060,000,0, as need to be desired.

These computations relating to the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$ are contained in Art. 54, 55, 56, &c. - - - 63, pages 430, 431, 432, &c. - - - 436. And with them I conclude that part of this Appendix which relates to the resolution of the equation $\frac{x+a}{x} \times r^t - r^{t+1} = \frac{a}{x}$.

But, before we entirely dismiss the consideration of the equation $\frac{x+a}{x} \times r^t - r^{t+1} = \frac{a}{x}$, it may not be amiss to observe that in some cases the expression $\frac{2t+1}{2t+2} \times \frac{x+a}{x}$ will approach nearer than the expression $\frac{3t}{2t+2} \times \frac{x+a}{x} - \frac{1}{2}$ to the true value of the greater root of the equation $\frac{x+a}{x} \times r^t - r^{t+1} = \frac{a}{x}$, and in other cases less near to it. For, in the numeral equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, (of which the greater root is $= 1.068,14$.) the expression $\frac{2t+1}{2t+2} \times \frac{x+a}{x}$ was found to be $= 1.066,115,6$, which falls short of the true value 1.068,14 by only 0.002,024,4, or less than the 527th part of the said true value; but the other expression $\frac{3t}{2t+2} \times \frac{x+a}{x} - \frac{1}{2}$ was found to be $= 1.061,983,3$, which falls short of the said true value 1.068,14 by 0.006,156,7, or the 173d part of the said true value, which is more than three times the former difference 0.002,024,4. Therefore in resolving the said equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ the number 1.066,115,6 (obtained by means of the first expression $\frac{2t+1}{2t+2} \times \frac{x+a}{x}$), is much fitter than the number 1.061,983,3, (obtained by means of the second expression $\frac{3t}{2t+2} \times \frac{x+a}{x} - \frac{1}{2}$), to be employed as a first near value of the said greater root,

or

or to be made the ground-work, or basis, of a further approach to it's true value, by the Differential Method, or Mr. Raphson's Method, or any other Method, of Approximation. But the reverse of this took place in the case of the other numeral equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, of which the greater root is $= 1.06$. For in that equation the first expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ is $= 1.053,455,8$, which falls short of the true value 1.06 by $0.006,544,2$, or about the 163d part of the said true value; but the second expression $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ is $= 1.060,675,3$, which exceeds the said true value 1.06 by only the small quantity, $0.000,675,3$, which is less than the 1569th part of the said true value. Therefore in resolving this equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, the number $1.060,675,3$, (obtained by means of the second expression $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$), is much fitter than the number $1.053,455,8$, (obtained by means of the first expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$) to be employed as a first near value of the said greater root, or to be made the ground-work, or basis, of a further approach to the said true value by the Differential Method, or Mr. Raphson's Method, or any other Method, of Approximation. Therefore in resolving any numeral equation that comes under the general form $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, it will often be expedient to compute both the expressions $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, and substitute each of them successively instead of r in the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, and to denote the value of the said binomial quantity resulting from the substitution of the first expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ by the Letter D, and the value of it resulting from the substitution of the second expression $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ by the Letter E, and compare these results with $\frac{a}{z}$ the absolute term of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in order to discover which

which of these two results D or E differs least from the absolute term $\frac{a}{z}$. And then, the expression which produced the result which is found to differ least from the absolute term $\frac{a}{z}$ will be the expression which will differ least from the true value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and which will therefore be the fittest to be employed as a first near value of the said greater root, or to be made the ground-work, or basis, of a further approach to the said true value by any of the Methods of Approximation.

I then, in Art. 64, enter upon the consideration of the third and last of the three Problems in Dr. Halley's Discourse on Compound Interest which have given occasion to the foregoing Notes, and of the equation which is deduced from it, and on the resolution of which the solution of the said Problem depends. This Problem relates to the finding of the rate of Interest of money allowed in a bargain made for the prolongation of a given annuity, that has been already granted for a given number of years, for an additional number of years, when the four following quantities, to wit, 1st, the quantity of the annuity purchased, or the number of pounds that are to be paid to the annuitant every year; and, 2ndly, the number of years during which the annuity is to continue by virtue of the first grant; and, 3dly, the number of years during which it is to be prolonged beyond the first term by virtue of the new bargain; and, fourthly, the price that is to be paid for such prolongation of it by the purchaser in consequence of such new bargain; are all given or known. And I first state this Problem in Dr. Halley's own words, in pages 436, and 437, and afterwards give a fuller statement of it in Art. 65, in the same page 437, and then, in Art. 66, pages 437, 438, and 439, reduce the said Problem to the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, by the resolution of which

which we may obtain a solution of the said problem. Our efforts must therefore now be employed to resolve this trinomial equation $\frac{a}{x} \times r^x + r^{x+1} - r^{x+1+1} = \frac{a}{x}$.

With a view to the resolution of this equation I shew, in the first place, in Art. 67, page 439, that one of it's roots will be $= 1$; but that this root cannot be the value of r that will solve the Problem from which this equation was derived, because that value of r , (being the rate of Interest of money, or the value of one pound together with it's Interest for a year,) must necessarily be greater than one pound, or 1. And hence it follows that the other and greater root of this equation will be the quantity that will answer our purpose, and give us the solution of the said Problem. We must therefore endeavour to find the greater root of the said trinomial equation $\frac{a}{x} \times r^x + r^{x+1} - r^{x+1+1} = \frac{a}{x}$.

In order to discover the properties of this equation I, first, consider the more general equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, in which the co-efficient of the lowest power of the unknown quantity and the absolute term of the equation are denoted by the single Letters P and Q respectively, which makes the equation somewhat simpler and easier to manage in the course of our inquiry into it's properties. And then, in Art. 68, 69, and 70, pages 440, 441, 442, &c. - - - 446, I demonstrate the following conclusions concerning this equation $Px^m + x^{m+n} - x^{m+n+1} = Q$; to wit,

1st, That the binomial quantity $x^{m+n} - x^{m+n+1}$ must always be less than P, so long as x is any root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$;

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And,

And, 2ndly, That the greatest possible magnitude of the quantity x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, or rather the Limit of it's magnitude in that equation, is the magnitude which it has when the binomial quantity $x^{n+1} - x^n$ is equal to P , or, in other words, is the root of the binomial equation $x^{n+1} - x^n = P$; and therefore that, in order to find the greatest possible magnitude of the quantity x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, or the Limit of it's magnitude, it will be necessary to resolve the binomial equation $x^{n+1} - x^n = P$, (which is an equation that admits of only one root,) or to find the value of it's root x ; which root I call A ;

And, 3dly, That, while x increases from o to A (which is it's greatest possible magnitude,) the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will, first, increase from o to a certain finite quantity, and then decrease from that finite quantity to o , to which it will be equal when x is equal to A , or the root of the binomial equation $x^{n+1} - x^n = P$; and consequently that, if M be put for the value of x at the instant of time at which, during the increase of x from o to it's greatest magnitude A , the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ attains it's greatest magnitude, the greatest magnitude of the said trinomial quantity will be $P \times M^m + M^{m+n} - M^{m+n+1}$;

And, 4thly, That, if Q , or the absolute term of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ is equal to the said greatest quantity $P \times M^m + M^{m+n} - M^{m+n+1}$, the said trinomial equation will have but one root, to wit, the quantity M ; but that, if Q is less than the said greatest quantity,

quantity, or $P \times M^m + M^{m+n} - M^{m+n+1}$, the said trinomial equation will have two roots, of which the lesser will be less than the quantity M , and the greater will be greater than the quantity M , but less than the quantity A ;

And, 5thly, That the quantity M , or the value of x at the instant at which the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ attains it's greatest possible magnitude, is equal to the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$; and therefore that, in order to find the value of this middle quantity M , which, being substituted instead of x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, will make that quantity be of the greatest possible magnitude, it will be necessary to resolve the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$.

It appears therefore that, if we wish to discover the values of the two quantities A and M , which are the two Limits of the magnitude of the greater value of x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, we must resolve the two binomial equations $x^{n+1} - x^n = P$ and $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$.

In Art. 71, pages 446, and 447, I proceed to resolve the first of those two binomial equations, to wit, the equation $x^{n+1} - x^n = P$, upon a supposition that the absolute term P is less than 1, as it usually is in the equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, when that equation arises from the equation $\frac{a}{x} \times r^x + r^{x+1} - r^{x+1+1} = \frac{a}{x}$ relating to the foregoing Problem concerning the price

price paid for the prolongation of a given annuity for an additional number of years. Upon this supposition "that P is less than 1," it is shewn in Art. 71, page 446, that x must be considerably less than 2, though greater than 1, and consequently that, if v be put for the excess of x above 1, and x be therefore $= 1 + v$, the quantity v will be less than 1, and consequently vv will be less v , and v^3 than vv , and every following power of v will be less than the power immediately preceeding it. And hence, (by substituting $1 + v$ instead of x in the equation $x^{n+1} - x^n = P$ by means of Sir Isaac Newton's binomial Theorem,) we obtain, in page 447, the equation $v + n \times v^2 + \&c = P$; which, if we neglect all the terms denoted by the mark $\&c$ (which will involve in them v^3, v^4, v^5 , and the following powers of v), as being very small in comparifon to nv^2 , will be converted into the quadratick equation $v + nv^2 = P$, of which the root is $= \frac{\sqrt{4nP+1}-1}{2n}$. And therefore x , or $1 + v$, will be nearly equal to, but somewhat less than, $1 + \frac{\sqrt{4nP+1}-1}{2n}$. Therefore A , or the greatest possible magnitude of x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = P$, will be nearly equal to, but somewhat less than, $1 + \frac{\sqrt{4nP+1}-1}{2n}$.

This method of determining the value of x in the equation $x^{n+1} - x^n = P$ is evidently imperfect, on account of the neglect of the several terms in the equation $v + n \times v^2 + \&c = P$ that involve the quantities v^3, v^4, v^5 , &c. But yet I thought it might be sufficiently accurate to give us the value of x exact to two decimal places of figures. But this I have since found it will not always be sufficient to effect; because the multiplication of v^3, v^4, v^5 , &c into their respective co-efficients (which will involve the powers of the index n), in the terms denoted by the mark $\&c$, which have been neglected, will sometimes tend to increase those terms, and thereby to counteract and lessen the diminution

diminution of them arising from the decrease of the powers $v^3, v^4, v^5, \&c$, to such a degree as to make those terms be too great to be safely neglected. It is only therefore when v is known to be not only less than 1, but less than $\frac{1}{n}$, that this method of determining the value of v , and consequently that of $1 + v$ or x , by resolving the quadratick equation $v + n \times vv = P$, can be safely resorted-to: and in general, I am now inclined to think, it will be more advisable to resolve the equation $x^{n+1} - x^n = P$ in another manner, to wit, by making a few conjectures concerning the value of x (which we know to be greater than 1, and a great deal less than 2,) and trying the accuracy of such conjectural values by substituting them instead of x in the binomial quantity $x^{n+1} - x^n$, and comparing the values of the said binomial quantity, resulting from such substitutions, with the true value of the said binomial quantity, or with the absolute term P of the equation $x^{n+1} - x^n = P$, which is equal to the said true value. It may, however, be of use to the reader to have both these methods of proceeding laid before him; to the end that, when he shall have any numeral equation of this form, $Px^m + x^{m+n} - x^{m+n+1} = Q$, arising from a question concerning the purchase of the prolongation of a given annuity, for an additional term of years, proposed to him to be resolved, and he shall think fit, in order to such resolution, or to the finding of the greater root of such equation, to determine previously the values of the quantities A and M (which are the higher and lower Limits of the greater root of such trinomial equation,) by resolving the two binomial equations $x^{n+1} - x^n = P$ and $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$, he may judge for himself, in the particular numeral equation which he is desirous of resolving, by which of these two methods it will be most expedient for him to proceed, in order to resolve the said binomial equation $x^{n+1} - x^n = P$, by which he is to determine the value of the quantity A .

I then,

I then, in Art. 72, page 448, proceed to consider the second binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, the root of which is equal to the quantity M , which is the lower Limit of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$. And here it may be observed, that, since the absolute term of this equation, to wit, the quantity $\frac{m}{m+n+1} \times P$, is less than P , or the absolute term of the former binomial equation $x^{n+1} - x^n = P$, and the co-efficient of x^n in the present equation, to wit, the quantity $\sqrt{\frac{m+n}{m+n+1}}$, is very nearly equal to $\frac{m+n+1}{m+n+1}$, or 1, or the co-efficient of x^n in the former equation, the value of x in this latter equation must be less than it was in the former equation. And consequently, if we put v for the excess of x , or the root of the latter equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, above 1, this quantity v will be less than the former quantity v in the former equation $x^{n+1} - x^n = P$, or $(1+v)^{n+1} - (1+v)^n = P$, and consequently the powers v^3, v^4, v^5 , &c will decrease faster in the present equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, or $(1+v)^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times (1+v)^n = \frac{m}{m+n+1} \times P$, than they did in the former equation $(1+v)^{n+1} - (1+v)^n = P$. And consequently, if we substitute the binomial quantity $1+v$, instead of x , in this equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ (finding the values of $(1+v)^n$ and $(1+v)^{n+1}$ by means of the binomial theorem,) the several terms of the new equation, produced by such substitution, that will involve v^3, v^4, v^5 , and the other following powers of v , may more safely be neglected than they could be in the former equation $(1+v)^{n+1} - (1+v)^n = P$, which was derived in the like manner from the equation $x^{n+1} - x^n = P$, and

and the said new equation may therefore, more safely than in the former case, be resolved as a quadratick equation. And such a resolution of this equation $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$ is exhibited in Art. 73, pages 448, 449, and 450; the result of which is, that, if, in order to simplify the notation, we put $f = \frac{m+n}{m+n+1}$, and $g = \frac{m}{m+n+1} \times P$, and thereby convert the equation $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$ into the equation $x^{n+1} - f x^n = g$, and then substitute $1 + v$ instead of x in this latter equation, we shall come to the following equation, to wit, $1 - f + n + 1 - f n \times v + \frac{nn - f n n + n + f n}{2} \times v^2 +$ several other terms involving $v^3, v^4, v^5, \&c = g$, which (if we neglect all the terms involving $v^3, v^4, v^5, \&c$) will be converted into the quadratick equation $1 - f + n + 1 - f n \times v + \frac{nn - f n n + n + f n}{2} \times v v$. Therefore (adding f to both sides,) we shall have $1 + n + 1 - f n \times v + \frac{nn - f n n + n + f n}{2} \times v v = g + f$, and (subtracting 1 from both sides,) we shall have $n + 1 - f n \times v + \frac{nn - f n n + n + f n}{2} \times v v = g + f - 1$, and (multiplying all the terms by 2,) we shall have $2n + 2 - 2 f n \times v + nn - f n n + n + f n \times v v = 2g + 2f - 2$, and (dividing all the terms by $nn - f n n + n + f n$;) we shall have $\frac{2n+2-2fn}{nn-fnn+n+fn} \times v + v v = \frac{2g+2f-2}{nn-fnn+n+fn}$; which is a quadratick equation properly prepared for resolution. Here again we must simplify the notation by putting $b = n + 1 - f n$, and $i = 2g + 2f - 2$, and $k = nn - f n n + n + f n$. And then we shall have $\frac{2b}{k} \times v + v v = \frac{i}{k}$, and (adding $\frac{b^2}{k^2}$ to both sides,) $\frac{b^2}{k^2} + \frac{2b}{k} \times v + v v (= \frac{i}{k} + \frac{b^2}{k^2}) = \frac{ik + b^2}{k^2}$. Therefore (extracting the square-roots of both sides,) we shall have $\frac{b}{k} + v = \frac{\sqrt{ik + b^2}}{k}$, and $v = \frac{\sqrt{ik + b^2}}{k} - \frac{b}{k}$, or $\frac{\sqrt{ik + b^2} - b}{k}$. Therefore $1 + v$, or x , will be $= 1 + \frac{\sqrt{ik + b^2} - b}{k}$; that is, the

the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ will be nearly equal to $1 + \frac{\sqrt{ik+b^2}-b}{k}$. But this general Algebraick expression of this root is so complicated that it would probably be more easy and convenient in practice to neglect it and to proceed to the resolution of any particular numeral equation of the form $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, that we might have occasion to resolve, by applying to it all the reasonings and operations set-forth in Art. 73, instead of computing the values of b and i and k in order to find the value of the general expression $1 + \frac{\sqrt{ik+b^2}-b}{k}$ just now obtained. But this must be left to the judgement and discretion of the calculator in every particular example.

Nor, perhaps, will it always be expedient to have recourse to this method of obtaining a near value of the root of the equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ by resolving a quadratick equation that has been derived from it by neglecting those terms of the accurate equation $1 + v^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n \times 1 + v^n = \frac{m}{m+n+1} \times P$ which involve v^2, v^3, v^4 , &c, as being very small in comparison of the two terms which involve v and vv : but it may sometimes be more advisable to resolve the said equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ in a different manner, to wit, by making a few conjectures concerning the value of it's root x (which we know to be greater than 1, and much less than 2,) and substituting the said conjectural values of x instead of x in the binomial quantity $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n$, and comparing the values of the said binomial quantity resulting from such substitutions with the true value

value of the said binomial quantity in the said equation, or with the absolute term $\frac{m}{m+n+1} \times P$, which is equal to the said true value. In some cases the former Method of proceeding may be more convenient than the latter, when we know that v is not only less than 1, but than $\frac{1}{n}$, and even much less than $\frac{1}{n}$; and in other cases the latter method may be preferable to the former. And of this matter the calculator must judge for himself in each particular example.

When we have thus obtained near values of the roots of the two binomial equations $x^{n+1} - x^n = P$ and $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, (which roots have been shewn to be equal to the quantities A and M, or to the Limits of the magnitude of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$;) we may derive from them a conjectural value of the greater root of the said trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, that will probably be pretty near the truth, by computing the value of the trinomial quantity $P \times M^m + M^{m+n} - M^{m+n+1}$, or the greatest possible magnitude of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, and comparing it with the absolute term Q, or the value of the said trinomial quantity in the equation $Px^m + x^{m+n} - x^{m+n+1} = Q$. And in this manner we may obtain a pretty good first near value of the greater root of the said trinomial equation independently of the lesser root of the said equation, or without any knowledge of the value of the said lesser root.

But, if we already know the magnitude of the lesser root of the said equation, we may in such case obtain another conjectural value of it's greater root by means of it's said lesser root. For, if d be put for the excess of M, (or

the middle value of x , which, being substituted instead of x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, makes that quantity to be of the greatest magnitude possible,) and we add $\frac{d}{2}$ to M , it will, I believe, often be found, that the sum thence arising, to wit, $M + \frac{d}{2}$, will be pretty nearly equal to the greater root of the said equation. And thus we shall in these cases have two ready methods of obtaining a tolerably good first near value of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$.

And, when we have, by either of the two foregoing methods, obtained a pretty good first near value of x , or the greater root of the said trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, we may derive from it a second near value of the said root, that shall be much nearer to the truth than the said first near value, by a process of either the Differential Method of Approximation, or Mr. Raphson's Method: and, if such second near value of the said greater root should not be thought sufficiently exact, we may easily find a third near value of it, by making the said second near value the ground-work, or basis, of another process of Approximation by the Method of Mr. Raphson.

Having thus examined the more general trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ (which includes under it the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, but is expressed in a more simple notation,) I return in Art. 78, page 451, to the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, for the sake of which I entered upon the foregoing examination of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$; and I apply the conclusions that have been obtained concerning the roots of the said equation $Px^m + x^{m+n} - x^{m+n+1}$

$x^{m+n+1} = Q$ to the roots of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$. And I thereby find that the quantity A, or the greatest possible value of r in the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ will be the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$; and that the quantity M, (or the middle value of r in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, which makes that trinomial quantity be of the greatest possible magnitude,) will be the root of the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$; and that the root of the former of these binomial equations, to wit, the equation $r^{t+1} - r^t = \frac{a}{z}$, will be nearly equal to, but somewhat less than, $1 + \sqrt[2t]{\frac{4ta}{z} + 1 - 1}$; and that the greatest possible magnitude of the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ will be $\frac{a}{z} \times M^x + M^{x+t} - M^{x+t+1}$, which must be compared with $\frac{a}{z}$, the absolute term of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, in order to form a good conjecture concerning the first near value of r , or the greater root of the said equation. These conclusions make the subject of Art. 78, 79, 80, 81, and 82, in pages 451, 452, 453, 454, and 455.

Then, in Art. 83, &c, pages 455, &c an example is given of this manner of resolving a trinomial equation of this form, $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, by applying it to the resolution of the numeral equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, which results from the Problem proposed by Dr. Halley, which is set-forth above in Art. 64; in which a , or the annuity granted,

granted, is $\frac{L}{20}$ a year; and t , or the number of years of the first term for which it was granted, that are unexpired, or still to come, is 21 years; and x , or the number of years for which the annuity is to be prolonged beyond the said 21 years, is 10 years; and z , or the sum of money paid-down as the price of such prolongation, is 40 pounds; and r denotes the rate of Interest of money corresponding to the price z , or 40 pounds, and which is required to be found.

The first step to be taken towards the resolution of this trinomial equation $0.500,000,0 \times r^{10} + r^{21} - r^{31} = 0.500,000,0$ in the manner above-described, is to find the value of A , or the higher Limit of it's greater root, by resolving the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, or $r^{22} - r^{21} = 0.500,000,0$, the root of which is equal to A . And the value of this root, as imperfectly ob-

tained by computing the expression $1 + \sqrt[21]{\frac{4^t a}{z} + 1 - 1}$, is = 1.1323. But I have since found, by resolving the equation $r^{22} - r^{21} = 0.500,000,0$ in a different manner, that the more accurate value of it's root is 1.0869. But the former value of A , to wit, 1.1323, though much greater than the truth, is yet sufficiently exact to be of considerable use in assisting us to find a tolerably good first near value of r , or the greater root of the trinomial equation $0.500,000 \times r^{10} + r^{21} - r^{31} = 0.500,000,0$; which was the only purpose for which it was desirable on the present occasion to inquire into the value of this quantity A , or to resolve the binomial equation $r^{22} - r^{21} = 0.500,000,0$.

The next step to be taken towards the resolution of the trinomial equation $0.500,000,0 \times r^{10} + r^{21} - r^{31} = 0.500,000,0$ in the manner above-described, is to find the value of M , or the lower Limit of the magnitude of it's greater root, by resolving the binomial equation $r^{t+1} - \frac{x+t}{x+t+1} \times r^t =$

$$\frac{n}{x+t+1}$$

$\frac{n}{n+1} \times \frac{n}{n} \text{ (or } r^{22} - \frac{10+21}{10+21+1} \times r^{21} = \frac{10}{10+21+1} \times 0.500,000,0, \text{ or } r^{22}$
 $- \frac{31}{32} \times r^{21} = \frac{10}{32} \times 0.500,000,0,)$ or $r^{22} - 0.968,750,0 \times r^{21} =$
 $0.156,250,0.$ And the root of this equation, if imperfectly obtained by resolv-
 ing a certain quadratick equation that is derived from it by means of the bi-
 nomial theorem in the manner above-described, will be $= 1.0437$; as is
 shewn in Art. 86, pages 457 and 458. But I have since found, by resolving
 this equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$ in a different manner,
 that the more accurate value of it's root is $1.038,883$. But the former value
 of M , to wit, 1.0437 , though greater than the truth, is yet sufficiently exact
 to be of great use in assisting us to find a tolerably good first near value of r ,
 or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{22}$
 $= 0.500,000,0$; for which purpose alone it was desirable for us to inquire into
 the value of M , or to resolve the equation $r^{22} - 0.968,750,0 \times r^{21} =$
 $0.156,250,0.$

And here it may be observed, that the imperfect value of M , or the root
 of the binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, obtained
 by resolving a quadratick equation that is derived from it by means of the bi-
 nomial theorem, to wit, the number 1.0437 , approaches much nearer to it's
 more exact value $1.038,883$, than the imperfect value of A , or the root of the
 former binomial equation $r^{22} - r^{21} = 0.500,000,0$, obtained by resolving a
 like quadratick equation that is derived from it by means of the binomial
 theorem, to wit, the number 1.1323 , approaches to it's more exact value
 1.0869 . For the difference between 1.0437 , or $1.043,700$, the imperfect
 value of M , and $1.038,883$, it's more exact value, is $= 0.004,817$, which is
 less than the 215th part of $1.038,883$, it's more exact value; and the differ-
 ence between 1.1323 , the imperfect value of A , and 1.0869 , it's more exact
 value, is $= 0.0454$, which is nearly equal to the 24th part of 1.0869 , it's
 more exact value; which 24th part is more than ten times the 215th part of
 the

the said more exact value. So that the second binomial equation $r^{32} - 0.968,750,0 \times r^{21} = 0.156,250,0$, (of which the root is equal to M,) may be much more safely resolved in the first, or imperfect, method by resolving a certain quadratick equation derived from it, than the first binomial equation $r^{32} - r^{21} = 0.500,000,0$, (of which the root is equal to A,) can be resolved in the like imperfect manner.

Having obtained the number 1.1323 for a near value of A, and the number 1.0437 for a near value of M, I proceed, in Art. 87, page 459, to apply these numbers to the resolution of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$; and, in order thereto, I substitute 1.0437, instead of r , in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$; and I find the value of the said trinomial quantity, resulting from such substitution, to be $= 0.602,317,9$; which must therefore be considered as the greatest possible magnitude of that trinomial quantity. And then, in Art. 88, pages 460, and 461, I proceed to investigate a first near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ in the following manner.

While the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ decreases from 0.602,317,9 (which is it's greatest magnitude,) to 0, the quantity r will increase from M, or 1.0437, to A, or 1.1323; and, while the said trinomial quantity decreases from 0.602,317,9 to 0.500,000,0 (the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$,) the quantity r will increase from M, or 1.0437, to the value of the greater root of the said trinomial equation. Now let us suppose that these two increments of r are to each other, in nearly the same proportion as the two corresponding decrements of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$; which is, indeed, but a coarse supposition, considerably distant from the truth; but yet it will be found to be sufficiently near it, to enable us to find a tolerably good first near value of the greater root of the trinomial equation which we are endeavouring

vouring to resolve. And we shall then find that the increment $r - 1.0437$ will be to the whole increment $1.1323 - 1.0437$ in, nearly, the same proportion as the decrement $0.602,317,9 - 0.500,000,0$ is to the whole decrement $0.602,317,9 - 0$, or $0.602,317,9$; that is, $r - 1.0437$ will be to 0.0886 , in, nearly, the same proportion as $0.102,317,9$ to $0.602,317,9$. Therefore $r - 1.0437$ will be, nearly, $(= \frac{0.0886 \times 0.102,317,9}{0.602,317,9} = \frac{0.009,065,365,94}{0.602,317,9}) = 0.0150$; and consequently r will be, nearly, $(= 0.0150 + 1.0437) = 1.0587$; that is, 1.0587 will be, probably, not very distant from the value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Q. E. I.

And thus, by having computed the values of A and M , or the higher and lower Limits of the magnitude of the said greater root of the said trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and found the said Limits to be equal to 1.1323 and 1.0437 , (though these computations of these Limits are but imperfect,) we have been enabled to form a probable conjecture concerning the value of the said greater root, and to conclude that it is nearly equal to 1.0587 , independently of the lesser root, or without making use of our previous knowledge that the said lesser root is $= 1$.

And this number, 1.0587 , thus found for the first near value of the greater root of the said trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, is tolerably near to the true value of the said greater root, for a first, or conjectural, value of it. For the true value of the said greater root has been found above (in page 356 of the present volume,) to be $= 1.063,235,9$; which exceeds $1.058,7$ by the small quantity $0.004,535,9$, which is less than the 234th part of the said true value $1.063,235,9$. This is a very considerable degree of exactness for a first near value of r , and such as will make the said first near value 1.0587 be an excellent basis, or ground-work, for a further approach to the true value of r , or the said greater root, either by the

Differential

Differential Method, or Mr. Raphson's Method, or any other Method of Approximation.

But, if we make use of our previous knowledge of the magnitude of the lesser root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{33} = 0.500,000,0$ in order to obtain a probable first near value of the greater root of the said equation, by putting d for the excess of M , or 1.0437 , above the said lesser root (which we know to be $= 1$), and supposing the greater root to be nearly $= M + \frac{d}{2}$, we shall thereby obtain another near value of the said greater root, which will be still nearer to it's true value than 1.0587 . For, since M is $= 1.0437$, and the lesser root of the equation is $= 1$, we shall have $d (= M - 1 = 1.0437 - 1) = 0.0437$, and $\frac{d}{2} (= \frac{0.0437}{2}) = 0.0218$, and consequently $M + \frac{d}{2} (= 1.0437 + 0.0218) = 1.0655$; which is greater than $1.063,235,9$, or the true value of the said greater root by only the small quantity $0.002,264,1$, which is less than the 469th part of $1.063,235,9$, or the true value of the said greater root. This difference from the true value of the said greater root is but half the former difference, $0.004,535,9$, of the number 1.0587 from the said true value. So that this number 1.0655 (obtained by means of the expression $M + \frac{d}{2}$, and derived from our previous knowledge of the magnitude of the lesser root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{33} = 0.500,000,0$), is a better first near value of the greater root of the said equation than the former number 1.0587 , which was obtained from the two quantities A and M , or 1.1323 and 1.0437 , which were the limits of the magnitude of the said greater root. But both these numbers 1.0587 and 1.0655 are near enough to the true value of the said greater root to be fit to be taken for first near values of it, or to be employed as the basis, or ground-work, of a further approach to the said true value, either by the

Differential

Differential Method, or Mr. Raphson's Method, or some other Method, of Approximation.

Accordingly I afterwards proceed, in Art. 90, page 462, to employ the two numbers 1.0587 and 1.0665 (having here made an unaccountable mistake in employing the number 1.0665 instead of the number 1.0655, which had been found for a near value of r in the foregoing Article 89,) as the ground of a process of the Differential Method of Approximation towards a second and more exact value of it. And the second near value of r obtained by this process is 1.062,750,3; which falls short of the true value, 1.063,235,9 by the small difference 0.000,485,6, which is less than the 2189th part of the said true value, 1.063,235,9. This is a very considerable degree of exactness; and this second near value of r would have been much more exact, if I had not made the above-mentioned mistake of employing the number 1.0665, instead of the number 1.0655, in this calculation. This process of the Differential Method, by which we have obtained this second near value of r , to wit, 1.062,750,3, is contained in pages 462, 463, and 464.

And then, in Art. 91, pages 464, 465, and 466, I employ the foregoing second near value of r , to wit, 1.062,750,3, as the ground-work of a process of Mr. Raphson's Method of Approximation, and thereby obtain the number 1.063,243,5 for a third near value of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, which exceeds the true value 1.063,235,9 by the very small quantity 0.000,007,6, which is less than the 139,899th part of the said true value 1.063,235,9. This equation may therefore now be considered as having been resolved to a great degree of exactness. And this concludes the whole of what is contained in this Appendix to Dr. Halley's Discourse on Compound Interest, concerning the general equation

$\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ deduced from this Problem of Dr. Halley, and concerning the numeral equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} =$

0.500,000,0, which results from the particular example which he has given of the said Problem.

The manner in which the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ has been resolved in this Appendix to Dr. Halley's Discourse on Compound Interest, and which has been just described, seems to be the most regular and scientific method that can be taken for that purpose; because it, first, determines the values of the two quantities A and M, which are the Limits of the magnitude of the greater root of the equation (which is the unknown quantity which we are required to find,) and thereby affords us just and safe grounds for forming reasonable conjectures concerning the first near value of the said unknown quantity, which first near value we are afterwards to make use of as a ground-work, or basis, of a further approach to the true value of the said unknown quantity, or greater root, by the Differential Method, or Mr. Raphson's Method, or some other Method, of Approximation. But this investigation of these Limits, A and M, of the magnitude of the said greater root (which required the resolution of the two binomial equations $r^{22} - r^{21} = 0.500,000,0$ and $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$ before we began to form our conjectures concerning the first near value of the said greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, which is the unknown quantity which Dr. Halley's Problem requires us to find,) is not a matter of absolute necessity. For we may, if we please, proceed immediately to the resolution of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ without having previously discovered the values of A and M, or the Limits of the greater root of the said equation, and may begin such resolution by making at once a random conjecture concerning the value of the said greater root upon other grounds of probability that may occur to us. And by this means we should avoid the labour of previously resolving the two binomial equations $r^{22} - r^{21} = 0.500,000,0$ and $r^{22} - 0.968,750 \times r^{21} = 0.156,250,0$, which constituted no small part of the trouble of the foregoing resolution of the principal, or trinomial, equation

tion $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Such a resolution of the said trinomial equation might be performed in the following manner.

A Resolution of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, without previously investigating the values of the two quantities A and M, which are the Limits of the magnitude of the greater root of the said equation, which is the unknown quantity that is required to be found.

This equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ is derived from a Problem proposed by Dr. Halley, in which it is supposed that a person, to whom an annuity of $\text{£} 20$ a year had been already granted for a term of years of which 21 years were unexpired, or still to come, had paid, for the prolongation of the said annuity for an additional term of 10 years, the sum of 40 pounds; and it is required, from the knowledge of these four quantities, to wit, the number of pounds in the annuity, the number of years of the first term that are still to come, the number of years for which the annuity is to be prolonged, and the price paid for such prolongation, to determine the rate of Interest for his money that has been allowed to the said annuitant in requiring him to pay the sum of 40 pounds for the said prolongation of his annuity.

Now, as in Dr. Halley's time, or a little before and after the year 1700, the common rate of the Interest of money was 6 per cent. we may reasonably conjecture that, in the foregoing bargain for the prolongation of the said annuity of $\text{£} 20$ a year for an additional term of 10 years at the price of $\text{£} 40$, that rate of Interest, or a rate of Interest but little different from it, was adopted. And therefore we will suppose the first near value of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ to be equal to 1.06, and will substitute 1.06 instead of r in the trinomial quantity $0.500,000,0$

$\times r^{10} + r^{31} - r^{32}$, in order to discover whether the value of the said trinomial quantity resulting from such substitution, will be greater, or will be less, than 0.500,000,0, or the absolute term of the said equation, and consequently whether 1.06 will be less, or will be greater, than the true value of r , or the greater root of the said equation. This substitution may be made as follows.

The Logarithm of 1.06 is = 0.025,305,9. Therefore the Logarithm of $\overline{1.06}^{10}$ will be ($= 10 \times 0.025,305,9$) = 0.253,059,0; which is the Logarithm of the number 1.790,849,1. Therefore $\overline{1.06}^{10}$ is = 1.790,849,1; and consequently $0.500,000,0 \times \overline{1.06}^{10}$ will be ($= 0.500,000,0 \times 1.790,849,1 = \frac{1.790,849,1}{2}$) = 0.895,424,5.

And since the Logarithm of 1.06 is = 0.025,305,9, the Logarithm of $\overline{1.06}^{31}$ will be ($= 31 \times 0.025,305,9$) = 0.784,482,9; which is the Logarithm of the number 6.088,115,5. Therefore $\overline{1.06}^{31}$ is = 6.088,115,5. And consequently $\overline{1.06}^{32}$ will be ($= \overline{1.06}^{31} \times 1.06 = 6.088,115,5 \times 1.06$) = 6.453,402,4.

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.06}^{10} + \overline{1.06}^{31} - \overline{1.06}^{32}$ will be ($= 0.895,424,5 + 6.088,115,5 - 6.453,402,4 = 6.983,540,0 - 6.453,402,4$) = 0.530,137,6; which is greater than 0.500,000,0, the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore 1.06 must be less than r , or the greater root of that equation.

Q. E. I.

Now let w be put for the unknown excess of r above 1.06, and let $1.06 + w$ be substituted instead of r in the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$; and let the transformed equation resulting from this substitution be resolved as if it were a mere simple equation, according to the directions of Mr. Raphson's Method of Approximation, by neglecting all the terms

terms of it that involve any higher power of the new unknown quantity w than it's simple power, or w itself. This may be done in the manner following.

Since r is $= 1.06 + w$, we shall have $r^{10} = \overline{1.06 + w}^{10}$ ($=$, by the binomial theorem, to the Series $\overline{1.06}^{10} + 10 \times \overline{1.06}^9 \times w + \&c = \overline{1.06}^{10} + 10 \times \frac{\overline{1.06}^{10}}{1.06} \times w + \&c = 1.790,849,1 + 10 \times \frac{1.790,849,1}{1.06} \times w + \&c = 1.790,849,1 + 10 \times 1.689,480,28 \times w + \&c = 1.790,849,1 + 16.894,802,8 \times w + \&c$; and consequently $0.500,000,0 \times r^{10}$ will be ($= 0.500,000,0 \times 1.790,849,1 + 0.500,000,0 \times 16.894,802,8 \times w + \&c = \frac{1.790,849,1}{2} + \frac{16.894,802,8}{2} \times w + \&c$) $= 0.895,424,5 + 8.447,401,4 \times w + \&c$.

And r^{31} will be $= \overline{1.06 + w}^{31}$ ($=$, by the binomial theorem, to the Series $\overline{1.06}^{31} + 31 \times \overline{1.06}^{30} \times w + \&c = \overline{1.06}^{31} + 31 \times \frac{\overline{1.06}^{31}}{1.06} \times w + \&c = 6.088,115,5 + 31 \times \frac{6.088,115,5}{1.06} \times w + \&c = 6.088,115,5 + 31 \times 5.743,505,18 \times w + \&c$) $= 6.088,115,5 + 178.048,660,5 \times w + \&c$.

And r^{32} will be $= \overline{1.06 + w}^{32}$ ($=$, by the binomial theorem, to the Series $\overline{1.06}^{32} + 32 \times \overline{1.06}^{31} \times w + \&c = 6.453,402,4 + 32 \times 6.088,115,5 \times w + \&c$) $= 6.453,402,4 + 194.819,696,0$.

Therefore the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be equal to the following compound quantity, to wit,

$$\left\{ \begin{array}{l} 0.895,424,5 + 8.447,401,4 \times w + \&c \\ + 6.088,115,5 + 178.048,660,5 \times w + \&c \\ - 6.453,402,4 - 194.819,696,0 \times w + \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{l} 6.983,540,0 + 186.496,061,9 \times w + \&c \\ - 6.453,402,4 - 194.819,696,0 \times w - \&c \end{array} \right\} =$$

$$0.530,137,6 - 8.323,634,1 \times w \&c.$$

But

But the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ is $= 0.500,000,0$.

Therefore the compound quantity $0.530,137,6 - 8.323,634,1 \times w$ &c will also be $= 0.500,000,0$. And consequently $0.530,137,6$ will be $= 0.500,000,0 + 8.323,634,1 \times w$, and $8.323,634,1 \times w$ will be $(= 0.530,137,6 - 0.500,000,0) = 0.030,137,6$, and w will be $(= \frac{0.030,137,6}{8.323,634,1}) = 0.0036$. Therefore r , or $1.06 + w$, will be $(= 1.06 + 0.0036) = 1.0636$; that is, the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ will be nearly $= 1.0636$. Q. E. I.

This second near value of r is somewhat greater than the truth. For, if we substitute it instead of r in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, the value of the said trinomial quantity arising from such substitution will be less than $0.500,000,0$, or the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$; as will appear by making the said substitution in the manner following.

The Logarithm of 1.0636 is $= 0.026,778,3$. Therefore the Logarithm of $\overline{1.0636}^{10}$ will be $(= 10 \times 0.026,778,3) = 0.267,783,0$; which is the Logarithm of the number $1.852,605,5$. Therefore $\overline{1.0636}^{10}$ is $= 1.852,605,5$. And consequently $0.500,000,0 \times \overline{1.0636}^{10}$ will be $(= 0.500,000,0 \times 1.852,605,5 = \frac{1.852,605,5}{2}) = 0.926,302,7$.

And, since the Logarithm of 1.0636 is $= 0.026,778,3$, the Logarithm of $\overline{1.0636}^{31}$ will be $(= 31 \times 0.026,778,3) = 0.830,127,3$; which is the Logarithm of the number $6.762,812,5$. Therefore $\overline{1.0636}^{31}$ will be $= 6.762,812,5$. And consequently $\overline{1.0636}^{32}$ will be $(= \overline{1.0636}^{31} \times 1.0636 = 6.762,812,5 \times 1.0636) = 7.192,927,375,00$.

Therefore

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.0636}^{10} + \overline{1.0636}^{31} - \overline{1.0636}^{32}$ will be $(= 0.926,302,7 + 6.762,812,5 - 7.192,927,3 = 7.689,115,2 - 7.192,927,3) = 0.496,187,9$; which is nearly equal to, but somewhat less than, $0.500,000,0$, or the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore 1.0636 will be nearly equal to, but somewhat greater than, the true value of r , or the greater root of that equation.

Q. E. D.

But by another process of Mr. Raphson's Method of Approximation we may obtain a third near value of r , or the greater root of the said trinomial equation, that will be as exact as need to be desired. Such a process will be as follows.

Let w be put for the unknown excess of 1.0636 above the true value of r . And we shall then have $r = 1.0636 - w$, and consequently $r^{10} = \overline{1.0636 - w}^{10}$ ($=$, by the binomial theorem, to the series $\overline{1.0636}^{10} - 10 \times \overline{1.0636}^9 \times w + \&c = \overline{1.0636}^{10} - 10 \times \frac{\overline{1.0636}^{10}}{1.0636} \times w + \&c = 1.852,605,5 - 10 \times \frac{1.852,605,5}{1.0636} \times w + \&c = 1.852,605,5 - 10 \times 1.741,825,40 \times w + \&c) = 1.852,605,5 - 17.418,254,0 \times w + \&c$. And consequently $0.500,000,0 \times r^{10}$ will be $(= 0.500,000,0 \times 1.852,605,5 - 0.500,000,0 \times 17.418,254,0 \times w + \&c = \frac{1.852,605,5}{2} - \frac{17.418,254,0}{2} \times w + \&c) = 0.926,302,7 - 8.709,127,0 \times w + \&c$.

And r^{31} will be $= \overline{1.0636 - w}^{31}$ ($=$, by the binomial theorem, to the Series $\overline{1.0636}^{31} - 31 \times \overline{1.0636}^{30} \times w + \&c = \overline{1.0636}^{31} - 31 \times \frac{\overline{1.0636}^{31}}{1.0636} \times w + \&c = 6.762,812,5 - 31 \times \frac{6.762,812,5}{1.0636} \times w + \&c = 6.762,812,5 - 31 \times 6.358,417,16 \times w + \&c) = 6.762,812,5 - 197.110,931,9 \times w + \&c$.

And

And r^{32} will be $= \overline{1.0636 - w}^{32}$ ($=$, by the binomial theorem, to the Series $\overline{1.0636}^{32} - 32 \times \overline{1.0636}^{31} \times w + \&c = 7.192,927,3 - 32 \times 6.762,812,5 \times w + \&c) = 7.192,927,3 - 216.410,000,0 \times w + \&c$.

Therefore the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be equal to the following compound quantity, to wit,

$$\left\{ \begin{array}{l} 0.926,302,7 - 8.709,127,0 \times w + \&c \\ + 6.762,812,5 - 197.110,931,9 \times w + \&c \\ - 7.192,927,3 + 216.410,000,0 \times w - \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{l} 7.689,115,2 - 205.820,058,9 \times w + \&c \\ - 7.192,927,3 + 216.410,000,0 \times w - \&c \end{array} \right\} =$$

$$0.496,187,9 + 10.589,941,1 \times w \&c.$$

But that trinomial quantity is $= 0.500,000,0$:

Therefore the compound quantity $0.496,187,9 + 10.589,941,1 \times w$ will also be $= 0.500,000,0$. And consequently (subtracting $0.496,187,9$ from both sides,) we shall have $10.589,941,1 \times w = 0.003,812,1$ and $w (= \frac{0.003,812,1}{10.589,941,1}) = 0.000,359$. Therefore r , or $1.0636 - w$, will be ($= 1.063,600 - 0.000,359) = 1.063,241$; that is, the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ will be nearly $= 1.063,241$. Q. E. I.

This third value of r agrees with the true value of it, $1.063,235,9$ (found above in page 356,) in the first five figures 1.0632 , and exceeds the said true value by the very small quantity $0.000,005,1$, which is less than the 204,557th part of the said true value. This equation therefore may now be considered as resolved to a great degree of exactness.

— This latter resolution of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ has been performed with rather less labour of calculation than

than the former resolution of it, in which we previously investigated the values of A and M, or the two Limits of the magnitude of r , or it's greater root, by resolving the two binomial equations $r^{22} - r^{21} = 0.500,000,0$ and $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$. But which of these two Methods of resolving trinomial equations of this form $\frac{a}{x} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{x}$ will be fittest to be adopted in general, I will not undertake to determine; and I believe it will be best determined by the judgement and discretion of the calculator himself in every particular numeral equation of this kind which he may have occasion to resolve.

I have now given an account of the contents of the principal part of this Appendix from the beginning of it in page 359 to the end of page 466; in which I have shewn how the two binomial equations $\frac{x}{a} \times r - r^t = \frac{x}{a} - 1$, and $\frac{x+a}{x} \times r^t - r^{t+1} = \frac{a}{x}$, and the trinomial equation $\frac{a}{x} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{x}$, (for the resolution of which Dr. Halley has given us three very abstruse Algebräick expressions, obtained with great labour and difficulty by means of Mr. De Moivre's Theorem for raising a multinomial quantity to any given power, upon a supposition that the said theorem is true in the case of fractional powers and of powers that are both fractional and negative, as well as in the case of Integral powers, to which alone Mr. De Moivre's demonstration of it extends,) may be resolved by other Methods that are founded on clear and simple principles and are obtained by Algebräick operations of much less intricacy and complication than those which were employed in the investigation of Dr. Halley's expressions. But, as the full explanation of those Methods of resolution unavoidably extended the subject through so great a number as 108 pages, I thought it proper to recapitulate the results of the reasonings used in those pages, and to compress them into as narrow a compass as possible, and to deduce from them such practical directions as seemed to be

necessary to enable the reader to apply the said results with ease and expedition to the resolution of the three equations to which they relate. And this recapitulation extends through 29 pages, beginning in page 467 and ending in page 495.

After this I have solved another Problem relating to Compound Interest, which was pointed-out to me by the learned Mr. William Morgan of Chatham Square, as the only remaining question upon this subject in which the rate of the Interest of money allowed in the purchase of a given annuity was the unknown quantity that was required to be found. This Problem is not mentioned by Dr. Halley in his foregoing Discourse on Compound Interest, but is nearly connected with the last of the three foregoing Problems which relates to the prolongation of an annuity for an additional number of years beyond the term for which it was originally granted. For in this fourth Problem, instead of supposing the annuity of a pounds *per annum*, that has been originally granted for a certain number of years of which t years are still to come, to be prolonged for another term of x years, which is to begin at the expiration of the said first term, we are now to suppose that the said annuity is to be made perpetual at the end of the said first term, in consideration of a sum of z pounds that has been agreed-upon between the grantor of the said annuity and the purchaser of it as the price of such prolongation or perpetuation. And it is required; from the knowledge of a , or the number of pounds contained in the annuity so granted in perpetuity after the expiration of the said first term, or at the end of t years, and of the price z that has been paid for the said prolongation, or perpetuation, to find r , or the rate of Interest, that has been allowed to the purchaser for his money.

This Problem may be considered as the extreme case of the last-mentioned Problem of Dr. Halley concerning the prolongation of a given annuity for a certain number of years beyond the original term for which it was granted, and may be solved by means of an equation that is derived from the equation

$$\frac{a}{r}$$

$\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, which relates to that Problem. The manner of deriving such an equation from the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ is as follows.

Let all the terms of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ be divided by r^x . And we shall then have $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$. Now let x , or the number of years during which the annuity is to be prolonged beyond the original term of which t years are still to come, be not (as in the foregoing numeral equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and the Problem, or Example, from which that numeral equation was deduced,) a short term of 10 years, but some exceedingly long term, as, for instance, a term of ten thousand millions of years. Then it is evident that r^x will be a very great number, and consequently that $\frac{a}{z \times r^x}$ will be a very small quantity in comparison of $\frac{a}{z}$, and of r^t and r^{t+1} , and therefore that the magnitude of r in the equation $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$ will be nearly the same as it would be in the equation $\frac{a}{z} + r^t - r^{t+1} = 0$; in which case we should have $\frac{a}{z} + r^t = r^{t+1}$, and $\frac{a}{z} = r^{t+1} - r^t$, or (if we range the terms of this equation in the usual manner, with the absolute term, or known quantity, on the right hand,) $r^{t+1} - r^t = \frac{a}{z}$. And, if we suppose the number x to be absolutely infinite, or the annuity to be continued for ever, instead of being prolonged for any certain number of years, however great, the quantity $\frac{a}{z \times r^x}$ will not be extremely small, as before, but will be absolutely equal to 0, and the equation $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$ will be con-

verted into the equation $\frac{a}{z} + r^t - r^{t+1} = 0$, and consequently $r^{t+1} - r^t$ will be exactly equal to $\frac{a}{z}$. And therefore the value of r , or the rate of the Interest of money allowed the purchaser of the perpetuity of the annuity a in the Problem just now proposed, and which the said Problem requires us to find, will be the value of r in the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, or the root of the said equation. Q. E. D.

But this Problem may also be reduced to the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ in another manner, without any relation to the trinomial equation $\frac{a}{z} \times r^s + r^{s+t} - r^{s+t+1} = \frac{a}{z}$ resulting from the foregoing Problem of Dr. Halley. This other manner of reducing the present Problem to the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ is described at length in Art. 111, pages 497 and 498. And it may also be reduced to the same equation $r^{t+1} - r^t = \frac{a}{z}$ in a third manner, which is described in Art. 112 and 113, pages 498, 499, and 500. And this third manner of obtaining this equation is more direct than either of the two former, being grounded on the properties of a decreasing series of quantities in geometrical proportion.

Having thus shewn three different methods of reducing this Problem to the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, I proceed in Art. 114, pages 501, 502, and 503, to shew how this equation may be resolved. And then in Art. 115, and 116, pages 503, 504, and 505, I illustrate this Problem by a particular example, which produces the numeral equation $r^{41} - r^{40} = 0.617, 145, 0$; and then, in Art. 117, pages 505, 506, and 507, I resolve the said numeral equation $r^{41} - r^{40} = 0.617, 145, 0$, in the manner described in Art. 114, and I find it's root r to be $= 1.060, 042, 3$; which is very near to it's true value (which

(which is 1.06,) the difference being only 0.000,042,3, which is less than the 25,059th part of the said true value.

The remaining part of this Appendix, in Art. 118, 119, 120, &c - - - 129, pages 507, 508, 509, &c, - - - 516, contains some remarks on the different methods that have been taken in some of the foregoing articles for resolving the binomial equation $r^{i+1} - r^i = \frac{a}{z}$, which, I hope, will be found worthy of the industrious reader's attentive perusal and consideration, but of which it would be difficult to give a summary account in this Preface. And therefore I here conclude my account of the contents of this Appendix to Dr. Halley's Discourse on Compound Interest.

Of the Contents of the Fifth Tract in the present Volume.

The Fifth Tract in the present Volume is a Letter from the late celebrated Algebraist, Mr. Abraham De Moivre, F. R. S. to Dr. Edmund Halley, F. R. S. concerning the investigation of one of the Algebraick expressions given by Dr. Halley in his Discourse on Compound Interest that has been re-printed in pages 219, 220, 221, &c, - - - 230 of the present Volume.

This Letter is printed here in two different manners, to wit, first, in the words and notation used by the writer of it, Mr. De Moivre, in pages 520, 521, and 522, and, secondly, (in pages 522, 523, and 524,) in the notation used by Dr. Halley. And then in pages 524, 525, 526, &c, - - - 530, I have added some remarks on it, that tend to facilitate the understanding of Mr. De Moivre's Investigation of the expression $b - \sqrt{bb - 2by}$ of the value of v , or $r - 1$, which is finally obtained by it, and to illustrate it by a comparison with the Investigation of the same expression which had been given me by Mr. Morgan, and printed above in the present volume in the Third Note

to

to the Discourse of Dr. Halley on Compound Interest. And after these remarks, I have added, (in pages 530, 531, 532, &c. - - - 536,) a conjecture which I have ventured to form concerning the manner in which Dr. Halley had, probably, investigated the expression $1 + \sqrt{bb + 2by} - b$; or $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, which is the first of the three Algebræick expressions which he has given us without demonstrations in the foregoing Discourse on Compound Interest, and which is a near value of r , or the greater root of the binomial equation $\frac{z}{a} \times r - r^2 = \frac{z}{a} - 1$.

Of the Contents of the Sixth Tract in the present Volume.

The Sixth Tract in this Volume is a paper published in the Philosophical Transactions for the year 1770, being number XLIII of the 60th volume of that learned collection. It is a Letter from Mr. John Robertson, the late Librarian of the Royal Society, (who was a learned Mathematician and the author of an excellent Treatise of Navigation,) to James West, Esq. the then President of that learned Society, on the foregoing subject of Compound Interest, containing the Investigations of twenty different cases of it, and, amongst the rest, of those difficult cases in which the rate of Interest allowed in the bargain is the unknown quantity that is required to be found, and for the solution of which Dr. Halley had given those abstruse Algebræick expressions which have been the subject of our consideration in several of the foregoing Tracts in this Volume. But he never makes mention of Dr. Halley as the author of any solutions of these cases, but supposes them to have been solved only by the celebrated Mr. William Jones, an eminent Mathematician in the early part of the eighteenth century, who published a well-known collection of the most remarkable discoveries that had been made in the mathematical sciences under the title of *Synopsis Palmariorum Matheseos* in the year 1706. What he says upon this subject is in these words. *The late William Jones, Esq. F. R. S. among the variety of mathematical matters to which he gave attention, considered*

sidered the business of Compound Interest fully, and did, many years ago, cause to be engraved on a copper-plate more cases in Interest than had been exhibited before that time. Several copies of that Plate were distributed among his friends; to whom it appeared that he had treated this subject in a more extensive manner than had been done by other Mathematicians.

The Theorems, or rules, for the cases of Compound Interest, without their Investigations, were inserted by Mr. Jones in the quarto edition of Logarithms published by Gardiner, and the rules were also communicated to Mr. Dodson, who published them, by Mr. Jones's leave, with examples to illustrate the use of his Anti-logarithmick Tables. But the Investigations of these Theorems not having yet been made publick, it is apprehended that gentlemen curious in these speculations would be pleased to see them. And, should the following Essay be approved by you, (who have been for many years acquainted with the Finances of this Nation, and the importance of a ready knowledge in computing the Interest on the National Loans,) you will be pleased to communicate it to the Royal Society.

From these words of Mr. Robertson one would be apt to think that he had never perused with attention Dr. Halley's Discourse on Compound Interest that has been re-printed in this volume, and more especially that he had not examined, or made use of, Dr. Halley's Algebraick expressions of the value of r , or the rate of interest in the difficult cases above-mentioned; because, if he had done so, he would hardly have omitted to make some mention of Dr. Halley, as the inventor and publisher of those expressions, which are in truth the same with the expressions given by Mr. Jones for the solution of the same cases.

And we may observe further that, since Mr. Robertson, in the passage here cited from his Letter to Mr. West, says "that the investigations of those Theorems had not yet (in the year 1770) been made publick," we must suppose that he had not seen, or did not recollect, the Letter of Mr. De Moivre to Dr. Halley, (containing an Investigation of one of those abstruse expressions
of

of the rate of Interest sought,) which had been published in the year 1761 in the Periodical publication intituled, *The Mathematical Magazine, and Philosophical Repository*, and therefore conceived himself to have been the first inventor, or publisher, of the Investigations of these expressions. But, however this may be, Mr. Robertson has certainly here given us not only proper rules for the solution of twenty different cases of Compound Interest, but also the Investigations of the said rules; which Investigations, in those difficult cases in which the rate of Interest is the unknown quantity that is required to be found, and which produce affected, or multinomial, equations, he has derived, in the same manner as Mr. De Moivre and Mr. Morgan, from Mr. De Moivre's Multinomial Theorem, extended to the cases of fractional powers, and of powers that are both fractional and negative. [But all his Investigations, and more especially those of these difficult cases which are derived from the said Multinomial Theorem, are expressed in so very concise a style and manner that, I am afraid, they will be found very difficult to understand by readers that are not already well acquainted with the subject of Compound Interest.

This Sixth Tract begins in page 539, and ends in page 546.

Of the Contents of the Seventh Tract in the present Volume.

The Seventh Tract in the present Volume consists of the Theorems, or Rules, given by the late William Jones, Esq. F. R. S. for the solutions of the several questions relating to Compound Interest, which were mentioned in Mr. Robertson's Letter to Mr. West; which are extracted from the Introduction to Gardiner's large Tables of Logarithms published in quarto in the year 1742, intituled *The Explication and Use of the Logarithmick Tables*, in pages 8 and 9 of the said Introduction.

This Tract begins in page 549, and ends in page 552.

Of

Of the Contents of the Eighth Tract of the present Volume.

The Eighth Tract of the present Volume consists of the same Theorems, or Rules, of the said Mr. William Jones concerning the several questions relating to Compound Interest, together with examples to illustrate the use of them: extracted from the preliminary Discourse of Mr. James Dodson to his Anti-Logarithmick Canon, or Table of Numbers corresponding to all Logarithms under 100,000, published in one volume, folio, in the year 1742, in pages 52, 53, 54, &c, - - - 65 of the said Preliminary Discourse.

This Tract begins in page 555, and ends in page 571.

Of the Contents of the Ninth Tract in the present Volume.

The Ninth Tract of the present Volume contains several questions with their Solutions, relating to Compound Interest and Annuities; extracted from the first Volume of Mr. James Dodson's Mathematical Repository, published in the year 1748, in pages 298, 299, 300, &c, - - - 312.

This Tract begins in page 575, and ends in page 585.

Of the Contents of the Tenth Tract in the present Volume.

The Tenth Tract in this Volume contains the Xth, XIth, and XIIth, chapters of the very useful and popular work of the late Mr. John Ward, intituled, *The Young Mathematician's Guide*, which are those parts of that work in which he treats of the Interest of money, both simple and compound, and of the manner of solving the several questions relating to it. He treats the subject
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clearly and judiciously, and in resolving the affected, or Multinomial, equations that result from the Problems in which the rate of Interest is the unknown quantity that is required to be found, he does not employ the expressions given by Dr. Halley for that purpose, (and which are derived from Mr. De Moivre's Multinomial Theorem, extended to fractional powers and to powers that are both fractional and negative,) but proceeds in a plain and satisfactory method by conjectures and trials and Mr. Raphson's Method of Approximation. One of the numeral equations of this kind that he has resolved in this manner is $11.491,317 R - R^9 = 10.491,317$, which is resolved in page 624, and of which R , or the greater root, is found to be $= 1.06$. To some few passages in this tract that seemed to require explanation, I have subjoined short Notes at the bottoms of the pages in which those passages occur.

These extracts from this valuable work of Mr. Ward begin in page 589, and end in page 637; and the notes occur in pages 604, 607, 613, 625, and 637.

Of the Contents of the Eleventh Tract in the present Volume.

After the foregoing extracts from Mr. Ward's *Young Mathematician's Guide*, which relate to the Interest of money, I have, by the advice of my learned friend Professor Vilant, of the University of St. Andrew's in Scotland, reprinted the whole of another treatise of the same Mr. Ward, which relates intirely to this subject, and which he has intitled, "*Clavis Usuræ*, or a key to Interest, both simple and compound; containing practical rules, plainly expressed in words at length; whereby all the various cases of Interest, and Annuities, or leases either in possession or reversion, and purchasing freehold estates, &c. may very easily be resolved, both by the pen, and by a small Table of Logarithms hereunto annexed, for all rates of Interest and times of payment; illustrated by a variety of examples. To which is added, Rules to be observed in estimating the value of Annuities, or Leases, and Insurances for
Lives,

Lives, &c: Also The business of Rebate, or Discompt, and the Equation of payments (very useful for Merchants and other Dealers,) is here rectified and truly determined." By John Ward, author of the Young Mathematician's Guide, &c.

This Tract of Mr. Ward I conceive to be the most compleat treatise on the subject of the Interest of money that has ever been published; and therefore I resolved to re-print it in this volume, as a proper accompaniment to the other tracts that have been here printed on the same subject: and this I thought would be the more agreeable to those persons who are desirous of understanding this useful and practical branch of Mathematicks, because this last tract of Mr. Ward called *Clavis Usuræ* was out of print and very difficult to be procured.

To this Tract of Mr. Ward, called *Clavis Usuræ*, I have added (as well as to the foregoing, or tenth, tract reprinted from the same author's works,) a few notes to some passages that seemed to require them; which I have placed (as they are but short,) at the bottoms of the pages in which the passages to which they relate occur. The principal of these notes occurs in pages 662 and 663, and consists of an easy method of extracting, by means of a Table of Logarithms, the square-root, or cube-root, or any higher root, of a fraction, or quantity less than 1, such as the decimal fraction 0.054,756, without having recourse to either negative Logarithms, or arithmetical complements; which are two sorts of quantities that considerably increase the difficulty of applying Logarithms to the extraction of the roots of fractions, or indeed to their multiplication or division, and therefore (in my opinion) had better be discarded from the rules of Logarithmical Arithmetick. The substance of this note had been published in the year 1760 in the Dissertation on Logarithms contained in my Elements of Plane Trigonometry, pages 316, 317, 318, &c, - - - 321.

Of the Contents of the Twelfth Tract in the present Volume.

The Twelfth Tract in the present volume is an Appendix to the foregoing tract of Mr. John Ward, intitled "*Clavis Ufuræ*, or a Key to Interest, both Simple and Compound," in which I have compared Mr. Ward's resolutions of those affected, or multinomial, equations which result from the examples he has given us to some of his Problems (in which the rate of Interest allowed to a purchaser in a bargain concerning an annuity is the unknown quantity that is required to be found,) with other resolutions of the same equations by some of the methods which I have recommended above in the Appendix to Dr. Halley's Discourse on Compound Interest. And these equations I have resolved in this Appendix very fully by two different methods, to wit, 1st, by the Method employed for their resolution by Mr. Ward in the foregoing tract, called *Clavis Ufuræ*, or in the preceeding, or tenth Tract, taken from his "*Young Mathematician's Guide*," and, 2ndly, by some of the Methods recommended in the foregoing Appendix to Dr. Halley's Discourse, that the reader may be the better able to compare these methods with each other, and determine which of them is the more convenient.

The first of these equations is $11.491,317, R - R^2 = 10.491,317$. This equation is resolved in Art. 2, pages 774 and 775, by Mr. Ward's Method, and R , or it's greater root, is found to be $= 1.069,203$, which is considerably greater than it's true value, which is $1.060,000$, or 1.06 . Mr. Ward, indeed, rejects the figures $0.009,203$, and thereby obtains 1.06 for the value of R , which is the exact value of it. But he gives no reason for such rejection; and he probably was induced to reject them by knowing before-hand that 1.06 was the true value of R , the question that produced this equation being only the reverse of the preceeding question which he had been considering, and in which the Interest of money had been given, and had been 6 per cent. or R had been supposed to be $= 1.06$. But, if he had not known this, and had

not

not been thereby led to neglect all the figures of the quotient 0.069,203, except the first figure 0.06, it would have required at least two more processes of Approximation to enable him to conclude with certainty that the true value of R was 1.06. For, if he had used only one such process, he would have found the next near value of R, after 1.069,203, or 1.069, obtained by his method of approximation (which is that of Mr. Raphson,) to be 1.062,671; as is shewn in Art. 3 and 4, pages 775, 776 and 777, where the said process is exhibited at length. And this value 1.062,12 is still considerably greater than the true value of R, which is 1.06.

Then, in Art. 5, pages 777 and 778, it is observed that Mr. Ward might, even by his own method of proceeding, have obtained a very near value of R, by retaining four terms instead of three, of the Series $1 + 9e + 36e^2 + 84e^3 + \&c$, which is $= \overline{1 + e}^3$, or R^3 ; for that way of proceeding would have produced a quadratick equation, by the resolution of which he would have found that e was $= 0.0607$, and consequently that R, or $1 + e$, would be $= 1.0607$. And it seems rather surprizing that this observation did not occur to him. If he had thus found R to be nearly equal to 1.0607, in which the figure 6 is followed by a cypher, he would (as he was certain that 1.0607 was somewhat greater than the true value of R,) have had a good ground for conjecturing that the true value of R was 1.06, and for substituting 1.06 instead of R in the binomial quantity $11.491,317 R - R^3$ in order to discover whether the result would be equal, or not, to the absolute term 10.491,317 of the proposed equation $11.491,317 R - R^3 = 10.491,317$, and consequently whether 1.06 was, or was not, equal to the true value of R; and he would have found, by such substitution, that it was so. And this, I believe, is the very best method that can be taken for the resolution of the numeral equation $11.491,317 R - R^3 = 10.491,317$, or any other numeral equation that comes under the general equation $\frac{A}{n} \times R - R^n = \frac{A}{n} - 1$ given us by Mr. Ward for the solution of one of these Questions concerning Compound Interest,

when

when t , or the Index of the power of R in the second term of the equation is not greater, or not much greater, than 9. But, if t should be much greater than 9, (as, for instance, if it should be equal to 30, or 40, or any greater number,) the near value of R obtained in this manner, (or by retaining the four first terms of the Series which is equal to $\overline{1 + e^t}$, or R^t), would be much less exact than in the foregoing example of the equation $11.491,317 R - R^9 = 10.491,317$. And therefore in such a case it would be expedient to have recourse to some other method of proceeding, different from that of Mr. Ward, (in which the binomial quantity $1 + e$ was substituted instead of R in the proposed equation,) and to begin our approximation to the value of R from some known quantity that is greater than 1, and that approaches nearer than 1 to the true value of R . And, in pursuance of this observation, I then, in Art. 7, pages 780, 781, and 782, proceed to resolve the said equation $11.491,317 R - R^9 = 10.491,317$ by the first of the three Methods described above in the Appendix to Dr. Halley's Discourse on Compound Interest. And the several successive near values of R , obtained in this method of proceeding, are as follows.

The quantity M , or the lower Limit of the magnitude of R , (which is $= \frac{P}{m} \sqrt[m-1]{\frac{1}{Q}}$ in the general equation $PR - R^m = Q$.) will be $= \frac{11.491,317}{9} \sqrt[9]{\frac{1}{10.491,317}}$
 $= \frac{11.491,317}{9} \sqrt[9]{\frac{1}{10.491,317}} = 1.276,813 \sqrt[9]{\frac{1}{10.491,317}} =$ (by Logarithms,) 1.031,06. Therefore d , or $M - 1$, will be $(= 1.031,06 - 1) = 0.031,06$, and $M + d$ will be $(= 1.031,06 + 0.031,06) = 1.062,12$. Therefore 1.062,12 will be our first near value of R , or the greater root of the proposed equation $11.491,317 R - R^9 = 10.491,317$. And we may observe that this first near value of R , to wit, 1.062,12, is nearer to it's true value 1.060,00, or 1.06, than the number 1.062,671 is, though that number was obtained by two successive processes of Mr. Ward's Method of resolution.

Now

Now let y be put $= 1.062,12$, and let the binomial quantity $11.491,317 y - y^2$ be computed, and the result (which will be $= 10.485,051,6$) be called D. This result is less than $10.491,317$, or the absolute term of the equation $11.491,317 R - R^2 = 10.491,317$. Therefore y , or $1.062,12$ will be greater than the true value of R .

From y , or $1.062,12$, subtract $\frac{y}{200}$, or $\frac{1.062,12}{200}$, or $0.005,310,6$; and the remainder will be $= 1.056,809,4$. Let this remainder be substituted instead of R in the binomial quantity $11.491,317 R - R^2$, and let the result (which will be $= 10.499,923,8$) be called E. This result is greater than the absolute term $10.491,317$. Therefore $1.056,809,4$ will be less than the true value of R .

Now, having the three values of the binomial quantity $11.491,317 R - R^2$, which correspond to the three quantities $1.062,12$, R , and $1.056,809,4$, to wit, the three quantities D, $10.491,317$, and E, or $10.485,051,6$, $10.491,317$, and $10.499,923,8$, make the following proportion between their differences; to wit, as $E - D$ is to $10.491,317 - D$, so (very nearly,) will $1.062,12 - 1.056,809,4$ be to $1.062,12 - R$, or as $E - D$ is to $10.491,317 - D$, so (very nearly,) will $0.005,310,6$ be to $1.062,12 - R$, or as $10.499,923,8 - 10.485,051,6$ is to $10.491,317,0 - 10.485,051,6$, so (very nearly,) will $0.005,310,6$ be to $1.062,12 - R$, or as $0.014,872,2$ is to $0.006,265,4$, so (very nearly,) will $0.005,310,6$ be to $1.062,12 - R$. Therefore $1.062,12 - R$ will be, nearly, $= \frac{0.006,265,4 \times 0.005,310,6}{0.014,872,2} = \frac{0.000,033,273,033,24}{0.014,872,2} = 0.002,237,2$; and consequently $1.062,12$ will be $= 0.002,237,2 + R$, and R will be $(= 1.062,120,0 - 0.002,237,2) = 1.059,882,8$; that is, the second near value of R in the proposed equation $11.491,317 R - R^2 = 10.491,317$ will be $= 1.059,882,8$. Q. E. I.

And,

And, if this second near value of R (which is near enough to it's true value 1.06, for all useful purposes, differing from it by only the small quantity 0.000,117,2, which is less than the 9044th part of the said true value,) be made the ground-work, or basis, of a process of further Approximation to the said true value by the Method of Mr. Raphson, we shall obtain for a third near value of it the number 1.059,999,7; which differs from 1.06, or the true value of R , by only the very small quantity 0.0000,00,3, which is less than the 3,533,333d part of the said true value.

This process of Mr. Raphson's Method of Approximation is given at length in pages 781 and 782.

This method of resolving the equation $\frac{A}{u} \times R - R^t = \frac{A}{u} - 1$, or (putting $P = \frac{A}{u}$,) the equation $P \times R - R^t = P - 1$, will always enable us to find a very exact value of R , or the greater root of this equation, even when t , or the Index of the power of R in the second term of the equation, is a great number, (as, for example, 40, or 50, or 60,) as well as when it is a small number, (as 9, or 10, or 12,) as happens in the present example. And of this an Example has been given in the above-mentioned Appendix to Dr. Halley's Discourse on Compound Interest in pages 377, 378, 379, &c. - - - 383, where the equation $967.648,136,7 r - r^{70} = 966.648,136,7$ (in which the Index t is = 70,) is resolved by this method, and it's greater root is found to be nearly = 1.059,997,3, which differs from it's true value, (which is 1.06) by only 0.000,002,7, or less than the 392,592nd part of the said true value 1.06.

The next Example of an affected, or multinomial, equation resulting from a Question concerning Compound Interest, given by Mr. John Ward in the parts of his works that have been here re-printed, occurs in page 626 in the words following.

“ Question

“ Question 4. Suppose one should give $\overset{\text{£. s. d.}}{167, 9, 5}$ for the purchase of a pension, or annuity, of $\overset{\text{£.}}{30}$ *per annum*, to continue seven years. At what rate of Interest per cent. would that purchase be made, allowing compound Interest to the purchaser ?

“ In this Question there is $P = 167.4716$, $u = 30$, and $t = 7$; to find R .
 “ By the fourth Theorem (set down in the preceeding page 625,) expressed in
 “ this equation $\frac{u}{P} = \frac{u}{P} \times R^t + R^t - R^{t+1}$, [or $\frac{u}{P} \times R^t + R^t - R^{t+1}$,
 “ $= \frac{u}{P}$, or $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$,] the said equation being brought
 “ into numbers, and it's root extracted, as in the fourth Question of the last Section (in page 623,) the value of R will be found to be 1.06; and then it will
 “ be $1 : 0.06 :: 100 : 6$, the rate per cent, which was required.”

The numeral equation resulting from the application of the General Theorem expressed by the equation $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$ to the question here proposed, has not been set-down by Mr. Ward: but he has left his reader to derive the said numeral equation from the said General equation $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$, and also to resolve the said numeral equation after it shall have been so derived. And he has supposed that his reader will resolve the said numeral equation by the Method of Approximation which he had given before in page 624 for the resolution of the former equation $11.491,317 R - R^9 = 10.491,317$; which method consisted in substituting $1 + e$ instead of R in the said equation, and retaining only the three first terms of the Series that is equal to $1 + 9e$, and supposing them to be nearly equal to the value of the whole of the said Series, and resolving the new equation resulting from the said substitution in consequence of that supposition. These directions I have accordingly followed in Art. 9, pages 784, 785, and 786; and I have found that the numeral equation that we shall have to resolve will be $1.179,134 \times R^7 - R^8 = 0.179,134$, and that the near value of

$1 + e$, or R , obtained by this resolution of the equation in Mr. Ward's Method, will be 1.078; which is very considerably greater than its true value 1.06. And therefore I am inclined to think that it would be more convenient to resolve this equation by one of the Methods of Approximation given above for this purpose in the Appendix to Dr. Halley's Discourse on Compound Interest.

But, before I have recourse to one of the Methods given in that Appendix for the resolution of equations of this form, I have tried what would be the effect of retaining four terms, instead of three, of the two serieses that are equal to $\overline{1 + e}^7$ and $\overline{1 + e}^8$, and resolving the equation that will result from that method of proceeding. And I have found that it answers our purpose very well. For the near value of R , or $1 + e$, obtained by this Method of proceeding, is 1.061,293, which is much nearer than the former near value of it, 1.078, to its true value 1.06. There is, however, rather more trouble in obtaining the number 1.061,293 than in obtaining the number 1.078, because it is necessary for that purpose to resolve the quadratick equation $e^2 + 0.219,831 \times e = 0.017,231$, whereas the number 1.078 was obtained by resolving the simple equation $3.238,186 \times e = 0.253,938$. But this small difference of labour in the computations is not worth attending to in order to obtain so very good a first near value of R as the number 1.061,293.

The computations necessary for obtaining these two numbers 1.078 and 1.061,293 are set forth at great length in Art. 9 and 10, pages 784, 785, 786, 787, and 788.

In resolving the said numeral equation $1.179,134 R^7 - R^8 = 0.179,134$ I conceive the foregoing Method of proceeding, by retaining four terms of the two serieses that are equal to $\overline{1 + e}^7$ and $\overline{1 + e}^8$, to be the most convenient that can be taken for the purpose. And, if the number 1.061,293, or (neglecting the three last figures,) the number 1.06, was not thought to be near enough

to the truth, we might employ it as a ground-work, or basis, of a further approach to the true value of R by Mr. Raphson's Method of Approximation, and by that means should obtain a much nearer value of it. But, if the Indexes of the powers of R in the equation were 40 and 41, or 50 and 51, instead of being 7 and 8, this Method of resolving the equation $\sqrt[m]{\frac{n}{P} + 1} \times R^t - R^{t+1} = \frac{n}{P}$ by retaining four terms of the two serieses that are equal to R^t and R^{t+1} , or to $\overline{1 + e^t}$ and $\overline{1 + e^{t+1}}$, would be found to be much less exact than in the foregoing equation, because in such a case, the co-efficients of several of the omitted terms would be such high numbers as to make those terms too great to be omitted without making the value of e , obtained by means of their omission, be much greater than it's true value. In such cases therefore I should think it would be convenient to have recourse to some of the Methods given for this purpose in the Appendix to Dr. Halley's Discourse on Compound Interest, and to make use of the Limits of the magnitude of the general binomial equation $Px^m - x^{m+1} = Q$ (under which the equation $\sqrt[m]{\frac{n}{P} + 1} \times R^t - R^{t+1} = \frac{n}{P}$ is comprehended,) which are determined in pages 414, 415, 416, &c, - - - 423. And therefore, to shew how these Limits may be usefully applied to the resolution of an equation of this kind, I then, in Art. 11, page 789, proceed to resolve the foregoing numeral equation over-again in the manner described in those pages.

For this purpose it is necessary to compare the numeral equation 1.179,134 $R^7 - R^8 = 0.179,134$ with the general equation $Px^m - x^{m+1} = Q$, concerning which it has been shewn above in pages 414, 415, &c, that M , or the lower Limit of the magnitude of the greater of it's two roots, will be $= \frac{m}{m+1} \times P$, and that the higher Limit of it's magnitude is P , or $\frac{m+1}{m+1} \times P$, and consequently that the arithmetical mean between these two Limits, or the

quantity $\frac{2m+1}{2m+2} \times P$, will be often a pretty good first near value of the said greater root, and especially when m is a pretty high number, as 40, or 50, which is not the case in the present equation. And it is shewn likewise, that, if the value of the lesser root of the equation is known, and it's difference from M , or $\frac{m}{m+1} \times P$, be called d , the binomial quantity $M + \frac{d}{2}$ will usually be a pretty good first near value of the greater root.

Now in the equation $1.179,134 R^7 - R^8 = 0.179,134$, we shall have $m = 7$, and $P = 1.179,134$, and $Q = 0.179,134$, or $P - 1$. Therefore M , or $\frac{m}{m+1} \times P$, will be $(= \frac{7}{7+1} \times 1.179,134 = \frac{7}{8} \times 1.179,134 = \frac{8.253,931}{8}) = 1.031,742$, and $\frac{2m+1}{2m+2} \times P$ will be $(= \frac{15}{16} \times 1.179,134 = \frac{17.687,000}{16}) = 1.105,438$. Therefore 1.105,438 will be one of the two first near values of R , or the greater root of the proposed equation $1.179,134 R^7 - R^8 = 0.179,134$.

Further, the lesser root of this equation is $= 1$. Therefore d , or the difference between M and the said lesser root, will be $= M - 1 (= 1.031,742 - 1) = 0.031,742$, and $\frac{d}{2}$ will be $(= \frac{0.031,742}{2}) = 0.015,871$, and $M + \frac{d}{2}$ will be $(= 1.031,742 + 0.015,871) = 1.047,613$. Therefore 1.047,613 will also be a tolerably good first near value of R , or the greater root of the equation $1.179,134 R^7 - R^8 = 0.179,134$.

Of these two first near values of R , to wit, 1.105,438 and 1.047,613, the latter is nearest to the true value of R , which is 1.06. But the contrary will often happen when the Index m is a high number, as 40, or 50, or 60; for the limits of the value of R (which are P , or $\frac{m+1}{m+1} \times P$, and $\frac{m}{m+1} \times P$,) will be much nearer to each other in that case than in the present case, in which m is only the small number 7.

I then,

I then, in pages 790, 791, and 792, make use of both these first near values of R , to wit, 1.105,438 and 1.047,613, in a process of the Differential Method of Approximation, and I thereby obtain the number 1.052,35 for a second near value of R ; which is nearer to the truth than either of the two former numbers. And, lastly, in pages 792, and 793, I make use of the number 1.0523, as the ground-work, or basis, of a process of Mr. Raphson's Method of Approximation; and I thereby obtain the number 1.0616 for a third near value of R , which is sufficiently near to 1.06 to make it reasonable to conjecture that the true value of R in this equation will be ≈ 1.06 ; which, upon trial, will be found to make the binomial quantity $1.179,134 R^7 - R^8$ be equal to 0.179,134, and consequently must be equal to the greater root of the equation $1.179,134 R^7 - R^8 = 0.179,134$; and I therefore thought it unnecessary to carry the approximation to the value of R , or the greater root of that equation, any further.

The Third Example of an affected equation resulting from a Question concerning Compound Interest, given by Mr. John Ward in the parts of his works that have been here re printed, occurs in page 718 of the present volume, in the Treatise intitled *Clavis Usuræ, or a Key to Interest, both Simple and Compound*, chapter iv, section iii, and is expressed in the following words;

“ *Example 1. In yearly Rents.*

“ Suppose there were paid $\overset{\text{£.}}{3205}, \overset{\text{s.}}{5}$ for an annuity or a Lease, of $\overset{\text{£.}}{250}$ *per annum*, to continue for 21 years. What rate of Interest per cent &c is allowed to the Purchaser?”

This Question Mr. Ward does not solve in the same manner as he did the two former, to wit, by reducing it to an affected equation, and investigating the value of R , or the greater root of this equation, by supposing it to be equal to $1 + e$, and substituting $1 + e$ instead of R in the said equation, with
an

an omission of all the terms in the serieses which are equal to the powers of R , or $1 + e$, that involve any higher powers of e than it's square, or ee , and then resolving the imperfect, transformed, equation that results from such a substitution; but he solves it by mere conjectural assumptions of different quantities for the value of R , and by computing, by means of such assumed values of R , the values of the annuity which might, at those supposed rates of Interest, be purchased for the given sum of $\overset{\text{£.}}{3205}, \overset{s.}{5}$, till he finds that the annuity corresponding to the last-assumed value of R is the very annuity of $\overset{\text{£.}}{250}$ *per annum* which was given in the Question. In this way of proceeding he, first, assumes 1.06 for the value of R , and finds the annuity that corresponds to it to be $\overset{\text{£.}}{196.1059}$, instead of $\overset{\text{£.}}{250}$, a year, and thence concludes that 1.06 is greater than the true value of R ; and he then assumes 1.05 for the said value, and finds the annuity that corresponds to it to be $\overset{\text{£.}}{250.000}$, or exactly $\overset{\text{£.}}{250}$ a year, or the very annuity given in the Question, and thence concludes that this second assumed number, 1.05, is the true value of R , or that the rate of Interest required is 5 *per cent.*

This way of answering a question of this kind is certainly very clear and satisfactory; but there is some danger of it's becoming tedious by the number of conjectures and trials which it may be necessary to make before we arrive at the true value of R , if these conjectures are not made with great judgement and sagacity, and, one may almost say, with great good luck. I have therefore solved the foregoing question by reducing it to the binomial equation $1.077,997 R^{21} - R^{22} = 0.077,997$, and then resolving the said equation. And this I have done, first, in the manner described in the Appendix to Dr. Halley's Discourse on Compound Interest, and, secondly, in the manner described in Art. 5 and 10 of the present Appendix to Mr. Ward's *Clavis Ufuræ*, by substituting $1 + e$ in it's terms instead of R , and retaining four terms of each of the two serieses that will be equal to R^{21} and R^{22} , or to $\overline{1 + e}^{21}$ and $\overline{1 + e}^{22}$,

$\sqrt{1 + e^{22}}$, and resolving the quadratick equation that will result from that method of Approximation.

In the former of these ways of resolving this equation we shall have P, or 1.077,997, for the greater Limit of the magnitude of the greater root R, and M, or $\frac{m}{m+1} \times P$, (or $\frac{21}{22} \times 1.077,997$), or 1.028,997, for the lesser Limits of it's magnitude, and therefore by taking an arithmetical mean between these two Limits, we shall have the number 1.053,497, or, very nearly, 1.0535, for a pretty good first near value of R, or the greater root of this equation.

Further, the lesser root of this equation $1.077,997 R^{21} - R^{22} = 0.077,997$ is = 1. Therefore d will be $(= M - 1 = 1.028,997 - 1) = 0.028,997$, and $\frac{d}{2}$ will be = 0.014,498, and $M + \frac{d}{2}$ will be $(= 1.028,997 + 0.014,498) = 1.043,495$, or, very nearly, = 1.0435, and will be another pretty good first near value of R, or the greater root of this equation.

Either of these two numbers 1.0535 and 1.0435 might be employed with good success as a first near value of R, or the greater root of the equation $1.077,997 R^{21} - R^{22} = 0.077,997$, or as a convenient ground-work for a further approach to it's true value by either the Differential Method, or Mr. Raphson's Method, or some other Method, of Approximation. But as it seems probable that 1.0535 is somewhat greater, and 1.0435 is somewhat less, than the said true value, I have taken an arithmetical mean between them, to wit, the number 1.0485, for such first near value of R, or the ground-work of a further approach to it's true value, which I have made by the Differential Method of Approximation by taking 1.0495 for the other near value of R, which is to be employed conjointly with 1.0485 for the attainment of a second near value of R, which shall be nearer to it's true value than either 1.0485 or 1.0495. And the second near value thereby obtained is = 1.050,027,3; which is so near to it's true value 1.05, that their difference 0.000,027,3 is less than the 38,888th part of the said true value. This Method of resolving this equation $1.077,997 R^{21} - R^{22} = 0.077,997$ appears to be very convenient and successful.

successful. The operations of this resolution are contained in Art. 13, 14, and 15, pages 795, 796, 797, 798, and 799.

The same equation $1.077,997 R^{21} - R^{22} = 0.077,997$, is then resolved in the second way above-mentioned, or by substituting $1 + e$ in it's terms instead of R , and omitting all the terms of the two serieses that are equal to R^{21} and R^{22} , or to $1 + e^{21}$ and $1 + e^{22}$, except the four first terms of them, and resolving the quadratick equation thereby obtained, to wit, the equation $e^2 + 0.043,482 e = 0.006,003$, the root of which is $= 0.058,730$. Therefore R , or $1 + e$, will be $= 1.058,730$. But this number is much too great, and differs more from 1.05, or the true value of R , than either of the two former near values of R , to wit, 1.0535 and 1.0435, derived from the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$; and therefore this Method of resolving the equation $1.077,997 \times R^{21} - R^{22} = 0.077,997$ is much less convenient than the former method of resolving it.

And from this example we may conclude, that, when m , or t , or the Index of the power of x , or R , in an equation of this form $P \times x^m - x^{m+1} = P - 1$, or $P \times R^t - R^{t+1} = P - 1$, is 21, or any greater number, this manner of resolving it by substituting $1 + e$ in it's terms instead of R , and retaining only the four first terms of the two serieses that are equal to $1 + e^t$ and $1 + e^{t+1}$, will not give us so good a first near value of it as either of the two expressions $\frac{2m+1}{2m+2} \times P$ or $M + \frac{d}{2}$. And I believe the same observation will be true even when m , or t , is less than 21, if it be greater than 11 or 12. But, if it do not exceed 12, and, more especially, if it do not exceed 7, this Method of Approximation will be found to give us a very good first near value of R as we have seen above in Art. 10, pages 786, 787, and 788, where it enabled us to find the number 1.061,293 for a first near value of R , or the greater root of the equation $1.179,134 R^7 - R^8 = 0.179,134$, of which root the true value is 1.06. It is, however, in all cases attended with more labour of calculation than computing the two expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, and finding the arithmetical mean between them.

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This second resolution of this equation $1.077,997 + R^{21} - R^{22} = 0.077,997$ is given in Art. 16, pages 799, 800 and 801.

The fourth Example of a Problem, or Question, concerning Compound Interest that produces an affected equation, given us by Mr. John Ward, occurs in page 719, or the next page to that which contained the last Question which produced the equation $1.077,997 R^{21} - R^{22} = 0.077,997$, which we have been just now resolving. And it produces an equation of the same form with that last equation. It is expressed in the following words;

“ Example II. In Half-yearly Payments.

“ Admit $\overset{\text{£.}}{3244}, \overset{\text{s.}}{5}$ were given for an annuity, or lease, of $\overset{\text{£.}}{250}$ *per annum* to
 “ to be paid half-yearly (viz. $\overset{\text{£.}}{125}$ every half-year) and to continue 21 years.
 “ What rate of Interest *per annum* is allowed the purchaser?”

This Question Mr. Ward solves as he did the last, that is, by mere conjectural assumptions of different quantities for the value of R , or the rate sought, and by making trials of the quantities so assumed by computing, by means of them, the values of the annuity, or, rather, half-yearly pension, which might, at those supposed rates of Interest, be purchased for the given sum of $\overset{\text{£.}}{3244}, \overset{\text{s.}}{5}$, till he finds that the half-yearly pension corresponding to the last-assumed value of R is the very half-yearly pension of $\overset{\text{£.}}{125}$ which was given in the question. In this way of proceeding he, first, assumes 1.06 for the value of R , or of $\overset{\text{£.}}{1}$ together with it's Interest for a year, and finds the half yearly pension that corresponds to it to be $\overset{\text{£.}}{98.9283}$, instead of $\overset{\text{£.}}{125}$; and he thence concludes that the true value of R must be less than 1.06. He then, for a second trial, assumes 1.05 for the value of R , and computes the half-yearly pension

that corresponds to this value of R , and finds it to be exactly $\overset{\text{£}}{125}$, or the very half-yearly pension supposed in the Question, and thence concludes that this second assumed value of R , to wit, 1.05 is it's true value, or that the rate of Interest required by the Question is 5 *per cent*. This conclusion is indisputably just: but, as this conjectural way of solving this question might have proved very tedious, if the second conjecture had not been so very fortunate, I have solved it in another and more regular manner, to wit, by, first, reducing it (in Art. 17, page 803,) to the equation $1.038,529 R^{42} - R^{43} = 0.038,529$, and then, (in Art. 18, pages 803, 804, 805, and 806,) resolving the said equation in the manner described above in the Appendix to Dr. Halley's Discourse on Compound Interest in pages 414, 415, 416, &c, - - - 523. The processes of this resolution are as follows.

In the first place we have, in this case, $P = 1.038,529$, and $x = R$, and $m = 42$, and $m + 1 = 43$, and consequently $\frac{2m+1}{2m+2} \times P (= \frac{84+1}{84+2} \times 1.038,529, = \frac{85}{86} \times 1.038,529) = 1.026,453$; which is therefore a tolerably good first near value of R , or the greater root of the equation $1.038,529 R^{42} - R^{43} = 0.038,529$.

Further, M , or $\frac{m}{m+1} \times P$, or the lesser Limit of the magnitude of the greater root of this equation, will be $(= \frac{42}{42+1} \times 1.038,529 = \frac{42}{43} \times 1.038,529) = 1.014,377$; and the lesser root of this equation is $= 1$. Therefore d , or the excess of M above the lesser root, is $(= M - 1 = 1.014,377 - 1) = 0.014,377$, and $\frac{d}{2}$ is $= 0.007,188$, and $M + \frac{d}{2}$ is $(= 1.014,377 + 0.007,188) = 1.021,565$; which is therefore another tolerably good first near value of R , or the greater root of this equation.

And by taking an arithmetical mean between these two numbers 1.026,453 and 1.021,565, we shall obtain a third number, to wit, 1.024,009, which will be nearer than either of the two former numbers to the true value of R . I therefore

therefore chuse this third number 1.024,009, or (dropping the last figure 9) the number 1.024, for the first near value of R , and make use of it as the ground-work, or basis, of a further approach to the true value of R by a process of Mr. Raphson's Method of Approximation. And for that purpose I substitute it instead of R in the binomial quantity $1.038,529 R^{42} - R^{43}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or will be less, than 0.038,529, or the absolute term of the equation $1.038,529 R^{42} - R^{43} = 0.038,529$, and consequently whether 1.024 will be less, or will be greater, than R , or the greater root of the said equation. And I find that the value of the said binomial quantity resulting from such substitution is $= 0.039,340,173,3$; which is greater than 0.038,529, or the absolute term of the said equation. Therefore 1.024 will be somewhat less than the true value of R , or the greater root of the said equation. Therefore, if we put z for the difference between 1.024 and the true value of R , we shall have $R = 1.024 + z$. This binomial quantity $1.024 + z$ must therefore be substituted instead of R in the equation $1.038,529 R^{42} - R^{43} = 0.038,529$, with an omission of all the terms that involve either z^2 , z^3 , or z^4 , or any higher power of z ; and the transformed equation resulting from such substitution must be resolved. And we thereby shall find z to be $= 0.000,741$. Therefore R , or $1.024 + z$, will be $= 1.024,741$.

But R is the value of $\overset{\text{£.}}{1}$ together with it's Interest for half a year. Therefore 1.024,741 will be the value of $\overset{\text{£.}}{1}$ together with it's Interest for half a year; and consequently $\overline{1.024,741}^2$, or 1.050,094,117.081, will be the value of $\overset{\text{£.}}{1}$ together with it's Interest for a whole year; which number differs so little from 1.05 that it may be considered as equal to it: so that the rate of Interest allowed to the purchaser of this half-yearly pension of $\overset{\text{£.}}{125}$, which is to continue for 42 half-years, or 21 years, is 5 per cent. as Mr. Ward has found it to be by his Method of conjectures and trials.

Q. E. I.

In this example it has appeared that the number 1.024 (which is an arithmetical mean between the two numbers 1.026,453 and 1.021,565, which are derived from the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$,) is a very excellent first near value of R , or the greater root of the equation $1.038,529 R^{42} - R^{43} = 0.038,529$, since it agrees with it's more accurate value 1.024 741 in all the four figures 1.024. And I believe that the like arithmetical mean between the two expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ will often be found to be a very good first near value of R in an equation of the general form $P \times R^m - R^{m+1} = P - 1$.

The fifth Example, given us by Mr. John Ward, of a Problem, or Question, concerning Compound Interest that produces an affected equation, occurs above in page 721, and is expressed in the words following;

“ Example III. Of Quarterly Payments.

“ There is [a sum of] 1600 pounds paid for an annuity of $\text{£} 100$ per annum, [which is] to continue 99 years, and to be paid quarterly. It is required to find what rate of Interest per cent. [per annum] is allowed to the purchaser.”

This Question is solved by Mr. Ward in the same manner as the two last, that is, by mere conjectural assumptions of different quantities, one after another, for the value of R , and by making trials of the quantities so assumed by computing, by means of them, the values of the annuities, or, rather, of the quarterly pensions, which might, at those supposed rates of Interest, be purchased for the given sum of $\text{£} 1600$, till he finds that the quarterly pension corresponding to the last-assumed value of R is very nearly equal to the very quarterly pension of $\text{£} 25$ which was given in the Question. And, by four successive conjectures

jectures and trials of this kind, he concludes that the Interest allowed to the purchaser of this quarterly pension of $\overset{\text{£.}}{25}$ for 99 years, or $(4 \times 99, \text{ or } 396)$ quarters of a year for the sum of $\overset{\text{£.}}{1600}$, will be very nearly $\overset{\text{£.}}{6.3855}$, or $\overset{\text{£.}}{6}, \overset{\text{s.}}{7}, \overset{\text{d.}}{8\frac{1}{2}}$, per $\overset{\text{£.}}{100}$ *per annum*.

This question may be reduced to the following binomial equation of the 397th order, to wit, $1.015,625 R^{396} - R^{397} = 0.15,625$, as is shewn in Art. 20, pages 807 and 808. And this binomial equation is afterwards resolved at great length in the same manner as the two preceeding equations $1.077,997 R^{21} - R^{22} = 0.077,997$ and $1.038,529 R^{42} - R^{43} = 0.038,529$. And the result is, that R, or the greater root of the equation $1.015,625 R^{396} - R^{397} = 0.015,625$, or the value of $\overset{\text{£.}}{1}$ together with it's interest for one quarter of a year, will be $\overset{\text{£.}}{= 1.015,591,4}$, and consequently that the amount of $\overset{\text{£.}}{1}$ together with it's Interest for two quarters of a year, or for half a year, will be $(\overset{\text{£.}}{= 1.015,591,4^2}) = \overset{\text{£.}}{1.031,425,891}$, and that the amount of $\overset{\text{£.}}{1}$ together with it's Interest for two half-years, or for a whole year, will be $(\overset{\text{£.}}{= 1.031,425,8^2}) = \overset{\text{£.}}{1.063,839,180}$, &c, or $\overset{\text{£.}}{1.063,839}$. Therefore the rate of Interest per cent. *per annum* allowed to the purchaser in this bargain will be $\overset{\text{£.}}{63839}$, or $\overset{\text{£.}}{6}, \overset{\text{s.}}{7}, \overset{\text{d.}}{8\frac{1}{2}}$, which differs from the rate of Interest assigned by Mr. Ward, to wit, $\overset{\text{£.}}{6.3855}$, or $\overset{\text{£.}}{6}, \overset{\text{s.}}{7}, \overset{\text{d.}}{8\frac{1}{2}}$, by less than one half-penny.

This Question relates to the annuities of 100 *per annum* for 99 years, payable quarterly at the Exchequer, which were established by Act of Parliament in Queen Anne's reign in the year 1708, and were sold for $\overset{\text{£.}}{1600}$.

The Resolution of this equation $1.015,625 R^{396} - R^{397} = 0.015,625$ is given in Art. 21, 22, 23, &c, - - - 27, pages 808, 809, 810, &c, - - - 816.

These

These are all the Examples, given us by Mr. Ward, of Questions concerning Compound Interest, in which the rate of Interest that has been allowed to the purchaser of an annuity for a given term of years is the unknown quantity that is required to be found. But he gives us one more Question on this subject, in which t , or the number of years during which a given annuity u , that has already been granted for a certain term of years of which T years are unexpired, or still to come, ought to be prolonged beyond the first term of T years, in order to make it worth a certain given sum of money called P , that is to be paid-down immediately as the price of such prolongation, upon a supposition that the annuity is to be paid at the end of every year, and that R , or the rate of Interest of money, or the value of 1 pound together with its Interest for a year, is known, or given;—in which I say, t , or the number of years, during which the said annuity ought, upon all these suppositions, to be prolonged beyond the expiration of the said T years, is the unknown quantity that is required to be found. And he has shewn that the general equation that expresses the relation of these several quantities to each other will be the trinomial equation

$$\frac{u}{P} \times R^t + R^{t+T} - R^{t+T+1} = \frac{u}{P}.$$

And he has also shewn that, if the

annuity u is $\overset{\text{£.}}{175}$ *per annum*, and the sum P is $\overset{\text{£.}}{816}$, $\overset{\text{s.}}{18}$, $\overset{\text{d.}}{9}$, and R , or the rate of Interest of money, is 1.06, and T , or the number of years in the first term that are unexpired, or still to come, is 9 years, the number t of the years during which the annuity ought to be prolonged, in consideration of the pay-

ment of the said sum of $\overset{\text{£.}}{816}$, $\overset{\text{s.}}{18}$, $\overset{\text{d.}}{9}$, will be 11 years. But he has not shewn us how to find the value of R , when all the other quantities u , P , T , and t are known; which would have required either a considerable number of conjectural assumptions of quantities for the value of R , and trials of such quantities by substituting them instead of R in the algebraïck expression of the value of t , till he had found such a number for R as would make the said expression of the value of t be equal to 11 years, which is the value of it given in the Question,

or

or would have required a resolution of the trinomial equation $\frac{u}{P} \times R^t + R^{t+T} - R^{t+T+1} = \frac{u}{P}$, or of the numeral equation which would be derived from this trinomial equation by substituting in it's terms the sum 175 instead of u , and $\overset{f.}{816}$, $\overset{s.}{18}$, $\overset{d.}{9}$ instead of P , and 9 instead of T , and 11 instead of t ; which numeral equation would be $0.214,214 R^{11} + R^{20} - R^{21} = 0.214,214$, as is shewn in page 819. This equation I have therefore here resolved at great length, and with considerable variety, in the manner described above in the Appendix to Dr. Halley's Discourse on Compound Interest. And the result of the last and most exact of the resolutions given of this equation, (in Art. 46, page 842,) is, that R , or the greater root of this equation, or the rate of Interest that is required to be found, will be $= 1.060,004,4$; which is wonderfully near to it's true value 1.06.

The resolution of this equation is very long, various, and laborious, and extends through Art. 29, 30, 31, &c - - - 46, in pages 819, 820, 821, &c - - - 842; but will, I hope, be found very useful by those students of Algebra who shall have the patience to read it all through with care and attention, and to perform, as they go, all the arithmetical operations in it, by contributing in a very considerable degree to make them become familiarly acquainted with these Methods of resolving trinomial equations of these high orders. And with the resolution of this difficult equation, and a Scholium containing some remarks upon it, I conclude this Appendix to Mr. Ward's Treatise on Interest, intitled "*Clavis Usuræ, or a Key to Interest, Simple and Compound.*"

The thirteenth and last Tract in this Volume contains some corrections of Dr. Hales's Treatise intitled *Analysis Fluxionum*, and the two Appendixes to it, that are printed in the former part of this Volume, and likewise some additions to the said Treatise and Appendixes; which corrections and additions were
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sent me by Dr. Hales about a year after the said Treatise and Appendixes had been printed-off. These corrections and additions begin in page 847, and end in page 859, which concludes the volume.

Inner Temple,
January 25, 1804.

FRANCIS MASERES.

A
T A B L E
OF THE
C O N T E N T S
OF THE
F I F T H V O L U M E.

NUMBER I.

AN Investigation of Sir Isaac Newton's Binomial Theorem in the Case of the Reciprocals of the Roots of a Binomial Quantity.
By Francis Maferes, Esq. F. R. S. Curfitor Baron of the Court of Exchequer in England.

In pages 3, 4, 5, &c - - - 86.

NUMBER II.

Analysîs Fluxionum. Auctore Gulielmo Hales, in sacrâ Theologiâ Doctore, Rectore Ecclesiæ de Killesandra, et Nuper Collegii sanctæ Trinitatis in Dubliniâ Socio, ac Linguarum Orientalium in eodem Collegio Professore. Cum duabus Appendicibus.

In pages 91, 92, 93, &c - - - 204.

NUMBER III.

A Letter from James Glenie, Esq. to Francis Maferes, Esq. containing a Demonstration of Sir Isaac Newton's Binomial Theorem.

In pages 207, &c - - - 216. °

NUMBER IV.

A Discourse on Compound Interest; by Dr. Edmund Halley, F. R. S. Astronomer Royal, and Savilian Professor of Geometry in the University of Oxford.

Vol. V.

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ford. First published in the Year 1705 in the Introduction to Mr. Henry Sherwin's *Mathematical Tables*, and afterwards in the following Editions of those Tables. The present Edition is re-printed from the third Edition of those Tables, published by Mr. *William Gardiner* in the Year 1741, and is followed by four very long Notes, containing very full Investigations of some very abstruse Algebraick Expressions given by Dr. Halley in the said Discourse without any Demonstration.

In pages 219, 220, 221, &c - - - 357.

NUMBER V.

An Appendix to the foregoing Discourse of Dr. Halley on Compound Interest; shewing how the Equations in which r , or the Rate of Interest of Money, allowed in Bargains concerning Annuities for Terms of Years, is the unknown Quantity, may be resolved in a clear and satisfactory Manner without having Recourse to the Algebraick Expressions which he has given us for that Purpose.

By Francis Maseres, Esq. F. R. S. Curfitor Baron of the Court of Exchequer in England.

In page 359, 360, 361, &c, - - - 516.

NUMBER VI.

A Letter from the late celebrated Algebraist Mr. Abraham De Moivre, F. R. S. to Dr. Edmund Halley, F. R. S. concerning one of the Algebraick Expressions given by Dr. Halley in his Discourse on Compound Interest, re-printed above in pages 219, 220, &c - - - 230 of this Volume.

In pages 519, 520 - - - *536.

NUMBER VII.

A Letter to James West, Esq. President of the Royal Society: Containing the Investigations of Twenty Cases of Compound Interest, by John Robertson, Librarian to the Royal Society. Being Number XLIII of the 60th Volume of the Philosophical Transactions of the Royal Society of London for the Year 1770, pages 508, 509, 510, &c - - - 517.

In pages 539, 540, &c - - - 546.

NUMBER VIII.

Theorems, or Rules, given by the late William Jones, Esq. F. R. S. for the Solution of the several Questions relating to Compound Interest: Extracted from the Introduction to Gardiner's large Tables of Logarithms, published in Quarto in the Year 1742, intitled *The Explication and Use of the Logarithmick Tables*, in pages 8 and 9 of the said Introduction.

In pages 549, 550, 551, and 552.

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NUMBER IX.

The same Theorems, or Rules, of the said Mr. William Jones; concerning the several Questions relating to Compound Interest; together with Examples to Illustrate the Use of them. Extracted from the Preliminary Discourse of Mr. James Dodson to his Anti-Logarithmick Canon, or Table of Numbers, corresponding to all Logarithms under 100,000, published in one Volume, Folio, in the Year 1742, in page 52, 53, &c. - - - 65 of the said Preliminary Discourse. In pages 555, 556, 557, &c - - - 571

NUMBER X.

Several Questions, with their Solutions, relating to Compound Interest and Annuities: Extracted from the First Volume of Dodson's Mathematical Repository, published in the Year 1748, in pages 298, 299, 300, &c - - - 312. In pages 575, 576, 577, &c - - - 585.

NUMBER XI.

Extracts from Mr. John Ward's Young Mathematician's Guide, from page 233 to page 283; Being the Xth, XIth, and XIIth, Chapters of the Second Part of that useful Work. In pages 589, 590, 591, &c - - - 637.

NUMBER XII.

Clavis Ujuræ; or, a Key to Interest, both Simple and Compound: Containing Practical Rules, plainly expressed in Words at length; whereby all the various Cases of Interest, and Annuities, or Leases, either in Possession, or Reversion, and Purchasing Freehold Estates, &c. may very easily be resolved, both by the Pen and a small Table of Logarithms, hereunto annexed, for all Rates of Interest, and Times of Payment whatsoever; illustrated by Variety of Examples. To which is added, Rules to be observed in estimating the Value of Annuities, or Leases, and Insurances for Lives, &c. Also, the Business of Rebate or Discompt, and the Equation of Payments (very useful for Merchants and other Dealers) is here Rectified and truly Determined. By John Ward, Author of the Young Mathematician's Guide, &c.

In pages 647, 648, 649, &c. - - - 770.

NUMBER

NUMBER XIII.

An Appendix to the foregoing Treatise of Mr. John Ward, intituled, "*Clavis Usuræ, or a Key to Interest, both Simple and Compound:*" Containing Resolutions of some of the Questions proposed in that Work and in his other celebrated Work called *The Young Mathematician's Guide*, by some of the Methods described above in the Appendix to Dr. Halley's Discourse on Compound Interest, in pages 359, 360, &c to page 516 of the present Volume.

By Francis Maseres, Esq. F. R. S. Curfitor Baron of the Court of Exchequer in England.

In pages 773, 774, 775, &c - - - 843

NUMBER XIV.

Corrections and Additions to the Rev. Dr. Hales's Treatise, intituled *Analysis Fluxionum*, and it's two Appendixes, which are printed in the former Part of this Volume, in pages 87, 88, 89, &c. - - - 204; lately received from Dr. Hales.

In pages 847, 848, 849, &c - - - 859

AN
INVESTIGATION
OF
SIR ISAAC NEWTON'S BINOMIAL THEOREM
IN THE CASE OF
THE RECIPROCAL OF THE ROOTS OF A BINOMIAL QUANTITY.

VOL. V.

B

INVESTIGATION

SIR ISAAC NEWTON'S BINOMIAL THEOREM

IN TWO PARTS

THE PROPERTIES OF THE ROOTS OF A BINOMIAL EQUATION

THE RECIPROCAL OF THE ROOTS OF A BINOMIAL QUANTITY.

B 2

$C a^{m-3} b^3 + \frac{m-3}{4} \times D a^{m-4} b^4 + \frac{m-4}{5} \times E a^{m-5} b^5 + \&c$ to the term $\frac{m - \overline{m-1}}{m} \times$ the foregoing co-efficient $\times b^m$, or $\frac{1}{m} \times$ the foregoing co-effi-

cient $\times b^m$, or $1 \times b^m$, or b^m . For it is shewn in art. 8, page 157 of the said 2nd volume that the co-efficient of the last term of this series must always be 1. And in the 3d volume of these Tracts, pages 25, 26, 27, &c - - to 73, another demonstration of this Theorem is given from the doctrine of Permutations and Combinations laid down by that deep and sagacious mathematician of the latter part of the last century, Mr. James Bernoulli, in his excellent treatise *De Arte Conjectandi*. And in the xviiiith tract of the second volume of this collection, intituled, "*A Discourse concerning the Binomial Theorem in the case of fractional powers, or powers of which the indexes are fractions,*" pages 194, 195, 196, &c

- - - 344, this theorem is extended 1st, to the quantity $\overline{1+x}^{\frac{1}{n}}$, and 2ndly, to the quantity $\overline{1+x}^{\frac{m}{n}}$, in which the letters m and n denote any whole numbers whatsoever. And in the iiid tract of the third volume of this collection, intituled "*An Appendix to the foregoing Translation of the three first chapters of Mr. James Bernoulli's Treatise De Arte Conjectandi, or on the art of forming probable conjectures concerning events that depend on chance,*" &c, pages 100*, 101*, &c - -

133*, this theorem is extended to the case of the quantity $\overline{1+x}^{-n}$, or $\frac{1}{\overline{1+x}^n}$, or to the integral and negative powers of the binomial quantity $1+x$, or to the reciprocals of the quantity $\overline{1+x}^n$, or of the integral powers of the said binomial quantity. But it has not yet been shewn to be true in the cases of the

quantities $\overline{1+x}^{\frac{-1}{n}}$ and $\overline{1+x}^{\frac{-m}{n}}$, or $\frac{1}{\overline{1+x}^{\frac{1}{n}}}$ and $\frac{1}{\overline{1+x}^{\frac{m}{n}}}$, though it is

well known that it extends to those cases as well as all the former. These cases I shall therefore now proceed to consider, and to demonstrate that in both of them this celebrated theorem will always be true as well as in all the former

cases. And first we will consider the quantity $\overline{1+x}^{\frac{-1}{n}}$, or $\frac{1}{\overline{1+x}^{\frac{1}{n}}}$, as being

less complicated and difficult than the other quantity $\overline{1+x}^{\frac{-m}{n}}$, or $\frac{1}{\overline{1+x}^{\frac{m}{n}}}$.

Of the Binomial Theorem in the case of the Quantity $\sqrt[n]{1+x}$, or $\frac{1}{\sqrt[n]{1+x}}$.

Art. 2. The binomial theorem in the case of integral and affirmative powers, (which is the simplest and easiest case of it,) is as follows. The quantity $\sqrt[n]{1+x}$ is = the series $1 + \frac{m}{1}Ax + \frac{m-1}{2}Bx^2 + \frac{m-2}{3}Cx^3 + \frac{m-3}{4}Dx^4 + \frac{m-4}{5}Ex^5 + \frac{m-5}{6}Fx^6 + \frac{m-6}{7}Gx^7 + \&c$. Now let $\frac{-1}{n}$ be substituted instead of m ; and this theorem will be converted into the following theorem, to wit, $\sqrt[n]{1+x}^{-1}$ = the series $1 - \frac{1}{n} \times Ax - \frac{1}{n} - 1 \times Bx^2 - \frac{1}{n} - 2 \times Cx^3 - \frac{1}{n} - 3 \times Dx^4 - \frac{1}{n} - 4 \times Ex^5 - \frac{1}{n} - 5 \times Fx^6 - \frac{1}{n} - 6 \times Gx^7 - \&c$ = the series $1 - \frac{1}{n} \times Ax - \frac{1-n}{2} \times Bx^2 - \frac{1-2n}{3} \times Cx^3 - \frac{1-3n}{4} \times Dx^4 - \frac{1-4n}{5} \times Ex^5 - \frac{1-5n}{6} \times Fx^6 - \frac{1-6n}{7} \times Gx^7 - \&c$ = the series $1 - \frac{1}{n} \times Ax - \frac{1-n}{2n} \times Bx^2 - \frac{1-2n}{3n} \times Cx^3 - \frac{1-3n}{4n} \times Dx^4 - \frac{1-4n}{5n} \times Ex^5 - \frac{1-5n}{6n} \times Fx^6 - \frac{1-6n}{7n} \times Gx^7 - \&c$ = the series $1 - \frac{1}{n} \times Ax - \sqrt{\frac{n+1}{2n}} \times Bx^2 - \sqrt{\frac{2n+1}{3n}} \times Cx^3 - \sqrt{\frac{3n+1}{4n}} \times Dx^4 - \sqrt{\frac{4n+1}{5n}} \times Ex^5 - \sqrt{\frac{5n+1}{6n}} \times Fx^6 - \sqrt{\frac{6n+1}{7n}} \times Gx^7 - \&c$ = the series $1 - \frac{1}{n} \times 1 \times x - \sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^2 - \sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^3 - \sqrt{\frac{3n+1}{4n}} \times -\sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^4 - \sqrt{\frac{4n+1}{5n}} \times -\sqrt{\frac{3n+1}{4n}} \times -\sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^5 - \sqrt{\frac{5n+1}{6n}} \times -\sqrt{\frac{4n+1}{5n}} \times -\sqrt{\frac{3n+1}{4n}} \times -\sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^6 - \sqrt{\frac{6n+1}{7n}} \times -\sqrt{\frac{5n+1}{6n}} \times -\sqrt{\frac{4n+1}{5n}} \times -\sqrt{\frac{3n+1}{4n}} \times -\sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times \frac{-1}{n} \times 1 \times x^7 - \&c$

$\sqrt{\frac{3n+1}{4n}} \times -\sqrt{\frac{2n+1}{3n}} \times -\sqrt{\frac{n+1}{2n}} \times -\frac{1}{n} \times 1 \times x^7 - \&c =$ the series $1 - \frac{1}{n}$
 $\times x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times$
 $\frac{2n+1}{3n} \times \frac{3n+1}{4n} \times x^4 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times x^5 + \frac{1}{n}$
 $\times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times \frac{5n+1}{6n} \times x^6 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$
 $\times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times \frac{5n+1}{6n} \times \frac{6n+1}{7n} \times x^7 + \&c.$ Therefore, if the bino-
 mial theorem is true in the case of $1+x\sqrt{\frac{-1}{n}}$, or $\frac{1}{1+x\sqrt{\frac{-1}{n}}}$, the said quantity

$1+x\sqrt{\frac{-1}{n}}$, or $\frac{1}{1+x\sqrt{\frac{-1}{n}}}$, will be equal to the infinite series $1 - \frac{1}{n} \times x + \frac{1}{n} \times$
 $\frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times$
 $x^4 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times x^5 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$
 $\times \frac{3n+1}{4n} \times \frac{4n+1}{5n} \times \frac{5n+1}{6n} \times x^6 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n}$
 $\times \frac{5n+1}{6n} \times \frac{6n+1}{7n} \times x^7 + \&c$ *ad infinitum*, or, (if we put A for $\frac{1}{n}$, and B
 for $\frac{1}{n}$, or the co-efficient of x , and C for $\frac{1}{n} \times \frac{n+1}{2n}$, or the co-efficient of x^2 ,
 and D for $\frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$, or the co-efficient of x^3 , and E, F, G, H, &c
 for the co-efficients of $x^4, x^5, x^6, x^7, \&c$.) to the infinite series $1 - \frac{1}{n} Ax +$
 $\frac{n+1}{2n} \times Bx^2 - \sqrt{\frac{2n+1}{3n}} \times Cx^3 + \frac{3n+1}{4n} \times Dx^4 - \sqrt{\frac{4n+1}{5n}} \times Ex^5 + \frac{5n+1}{6n}$
 $\times Fx^6 - \sqrt{\frac{6n+1}{7n}} \times Gx^7 + \&c$ *ad infinitum*. We must therefore endeavour
 to shew that this last proposition is true, or that the quantity $1+x\sqrt{\frac{-1}{n}}$, or
 $\frac{1}{1+x\sqrt{\frac{-1}{n}}}$, is equal to the series $1 - \frac{1}{n} \times Ax + \frac{n+1}{2n} \times Bx^2 - \sqrt{\frac{2n+1}{3n}} \times$
 $Cx^3 + \frac{3n+1}{4n} \times Dx^4 - \sqrt{\frac{4n+1}{5n}} \times Ex^5 + \frac{5n+1}{6n} \times Fx^6 - \sqrt{\frac{6n+1}{7n}} \times Gx^7$
 $+ \&c$ *ad infinitum*. Now this may be shewn by the following train of rea-
 soning.

Art. 3. In the 1st place, since the quantity $1 + x$ is greater than 1, it follows that the quantity $\sqrt[n]{1+x}$, which is the n th root of $1 + x$, or the first of $n - 1$ mean proportionals between 1 and $1 + x$, must also be greater than 1; that is, the denominator of the fraction $\frac{1}{\sqrt[n]{1+x}}$ will be greater than its numerator.

Therefore the said fraction $\frac{1}{\sqrt[n]{1+x}}$ will be less than 1; that is, the quantity

$\sqrt[n]{1+x}^{-1}$, or $\frac{1}{\sqrt[n]{1+x}}$, (to which we are endeavouring to find an infinite series

of decreasing quantities that shall be equal,) will be less than 1.

In the next place, since $\sqrt[n]{1+x}$ (the denominator of the said fraction $\frac{1}{\sqrt[n]{1+x}}$), is known to be equal to the infinite series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n}x^2 +$

$\times \frac{n-1}{2n} \times \frac{n-1}{2n} \times \frac{1}{n}x^3 - \&c$ *ad infinitum*, (as is shewn in the second volume of this collection of Tracts, page 237, art. 47), it follows that the fraction $\frac{1}{\sqrt[n]{1+x}}$ will be equal to the fraction

$$\frac{1}{1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n}x^3 - \&c}$$

or to the quotient of the division of 1 by the infinite series

$$1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n}x^3 - \&c$$

ad infinitum. We must therefore endeavour to find the terms of the said quotient.

Thirdly, since 1 is the dividend in the said operation of division, and 1 is likewise the first term of the divisor, it is evident, from the nature of the operation of division, that the first term of the quotient will likewise be 1. And,

further, since the second and other following terms of the divisor $1 + \frac{1}{n}x -$

$\frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n}x^3 - \&c$ involve in them the fe-

veral powers of x in their natural order, without any interruption, to wit, $x, xx, x^3, x^4, x^5, x^6, x^7, \&c$, it is evident, from the nature of the operation of division, that the second and other following terms of the quotient of this division will

likewise

likewise involve in them the several powers of x in their natural order, without any interruption, and consequently that the said quotient will be a series of quantities of which the first term will be 1, and the second and other following terms will consist of certain numeral co-efficients multiplied into the several powers of x in their natural order, without any interruption, to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7$, &c *ad infinitum*; so that, if we denote the said numeral co-efficients (which are hitherto unknown,) by the capital letters B, C, D, E, F, G, H, &c, the said quotient will be equal to the series 1, Bx, Cx², Dx³, Ex⁴, Fx⁵, Gx⁶, Hx⁷, &c *ad infinitum*, of which the second term Bx and all the following terms are to be connected with the first term 1 either by subtraction or addition, and consequently are to have either the sign — or the sign + prefixed to them.

And, fourthly, since it has been shewn that the quantity $1 + x^{\frac{-1}{n}}$, or the fraction $\frac{1}{1 + x^{\frac{1}{n}}}$, (to which the said quotient, or infinite series 1, Bx, Cx², Dx³,

Ex⁴, Fx⁵, Gx⁶, Hx⁷, &c *ad infinitum*, is equal) is less than 1, or the first term of the said infinite series, it follows that the second term Bx of the said series must be subtracted from the first term 1, and consequently marked with the sign —; because, if it were added to the first term 1, it might happen, when x was extremely small, that Bx alone would be greater than the sum of all the following terms Cx², Dx³, Ex⁴, Fx⁵, Gx⁶, Hx⁷, &c of the said series put together, and consequently that, even if all the said following terms were marked with the sign —, or subtracted from the first term 1, the whole series 1, Bx, Cx², Dx³, Ex⁴, Fx⁵, Gx⁶, Hx⁷, &c, or $1 + Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (even if it were $1 + Bx - Cx^2 - Dx^3 - Ex^4 - Fx^5 - Gx^6 - Hx^7 - \&c$ *ad infinitum*), would be greater than it's first term 1; which would be contrary to what has been shewn to be true.

And therefore we may now conclude that the quantity $1 + x^{\frac{-1}{n}}$, or the fraction $\frac{1}{1 + x^{\frac{1}{n}}}$, will be equal to an infinite series of terms, of which the first

term will be 1, and the second and other following terms will involve in them the several integral powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7$, &c, in their natural order, without any interruption, combined with, or multiplied into, certain numeral co-efficients, which are hitherto unknown, and which we shall denote by the capital letters B, C, D, E, F, G, H, &c; and that Bx, or the second term of the said series, must be subtracted from the first term 1, and consequently marked with the sign —; and that the following terms of the said series, to wit, Cx², Dx³, Ex⁴, Fx⁵, Gx⁶, Hx⁷, &c, will be either added to the first term 1, and consequently marked with the sign +, or subtracted from it, and consequently marked with the sign —; though which of them are to be marked with the former sign, and which of them with the latter, is hitherto undetermined.

undetermined. Or, in other words, we may conclude that the quantity $\frac{1}{1+x\sqrt[n]{n}}$, or the fraction $\frac{1}{\frac{1}{1+x\sqrt[n]{n}}}$, will be equal to an infinite series of terms

of the following form, to wit, $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$ &c *ad infinitum*. We must now therefore endeavour to investigate the values of all the terms of this series, except the first, or known, term 1, and to determine the signs + and —, which are to be prefixed to the several terms $Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9$, &c *ad infinitum*, after the second term Bx , which, we already know, is to be marked with the sign —. Now this may be done in three or four different ways, which I shall now proceed to lay before the reader.

*An Investigation of the Series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c *ad infinitum*, which is equal to the Quantity $\frac{1}{1+x\sqrt[n]{n}}$, or the Fraction $\frac{1}{\frac{1}{1+x\sqrt[n]{n}}}$, by means of an Operation of Algebraick Division.*

Art. 4. The most obvious and natural method of finding the several terms of this series, which is equal to the fraction $\frac{1}{\frac{1}{1+x\sqrt[n]{n}}}$, or to the quotient of the division of the numerator, 1, of the said fraction, by it's denominator $\frac{1}{1+x\sqrt[n]{n}}$, is to divide the said numerator 1 by the infinite series which we already know to be equal to the said denominator $\frac{1}{1+x\sqrt[n]{n}}$, to wit, the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \&c$ *ad infinitum*. For the quotient of this division will be the series sought.

Now this division may be performed, so far as to obtain the five first terms of the quotient, or of the said series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, in the manner following:

From this operation of division it appears that the five first terms of the quotient, or of the series sought, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4$ &c. We may therefore now conclude that the five first terms of the infinite series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c.$ (which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$), are $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \frac{2n+1}{3n}Cx^3 + \frac{3n+1}{4n}Dx^4$. Q. E. I.

Art. 5. The regularity of the co-efficients of the powers of x in the four terms Bx, Cx^2, Dx^3 , and Ex^4 , that have been obtained by means of the foregoing operation of division, affords us a probable ground for expecting that the following terms of the series, to wit, $Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, &c.$ will be equal to $\frac{4n+1}{5n}Ex^5, \frac{5n+1}{6n}Fx^6, \frac{6n+1}{7n}Gx^7, \frac{7n+1}{8n}Hx^8, \frac{8n+1}{9n}Ix^9, &c.$ respectively, or that the several generating fractions of the co-efficients of the sixth, seventh, eighth, and other following terms, (by the multiplication into which each of the said co-efficients is derived from the co-efficient next before it,) will be formed by continually adding the index n to both the numerator and the denominator of the generating fraction next before it. But this probability is far short of a demonstration that they will be equal to these quantities, or will be formed according to this law of continuation. And therefore, in order to obtain perfect satisfaction on this head, it will be necessary to have recourse to some other method of investigating these terms by which the law of their continuation may be discovered.

Art. 6. But, as to the signs $+$ and $-$ that are to be prefixed to the following terms $Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, &c.$ of the series $1 - Bx + Cx^2 - Dx^3 + Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, &c.$ which have not yet been investigated, we may deduce from the nature of the foregoing operation of Algebraick division, if we consider the several processes of it with sufficient attention, a proof that those signs will be prefixed alternately to the said following terms, in the same manner as they have been to the four terms Bx, Cx^2, Dx^3 , and Ex^4 , or $-Bx, +Cx^2, -Dx^3$, and $+Ex^4$, that have been already investigated by means of the said division; and consequently that the said series will be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + &c.$ with

with the signs + and — following each other continually, to whatever number of terms the said series may be extended. This may be shewn in the manner following :

A Demonstration of the alternate Succession of the Signs — and + in the several Terms $Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, \&c$ ad infinitum of the Quotient of the foregoing Algebraick Division, without continuing the said Division farther than to the Investigation of the first five Terms of the said Quotient, that have been already investigated, to wit, the Terms $1 - Bx + Cx^2 - Dx^3 + Ex^4$, or

$$1 - \frac{1}{n} Ax + \frac{n+1}{2n} Bxx - \frac{2n+1}{3n} Cx^3 + \frac{3n+1}{4n} Dx^4.$$

Art. 7. The first step in the foregoing operation of division is to divide the dividend 1 by 1, the first term of the divisor. This gives 1 for the quotient of this particular division, and for the first term of the quotient of the general division of the dividend 1 by the whole divisor.

The second step of this operation of division is to multiply the whole divisor by 1, the said first term of the quotient, and to set down the product of this multiplication in a line under the dividend 1, adding a star *, or a cypher, 0, to the said dividend for every term in the said product after the first term 1. This product (being produced by the multiplication of the divisor into 1,) will be exactly equal to the divisor itself. Therefore the signs of the terms of this product will be the same as the signs of the terms of the divisor; that is, the second term of the said product will be marked with the sign +, the third with the sign —, the fourth with the sign +, the fifth with the sign —, and the following terms with the sign + and the sign — alternately.

The third step of this operation of division is to subtract the said product from the dividend 1; which is done by subtracting 1, the first term of the said product, from the dividend 1, and each of the following terms of the said product from the star *, or cypher, 0, placed immediately above it. The first of these subtractions, that of 1 from 1, leaves *, or 0, for the remainder; and the second and other following subtractions are performed by setting down the several terms that are subtracted in the same horizontal line with the first remainder, *, or 0, but with contrary signs to those they had before. Therefore the signs of the terms of this general remainder, after the first term * or 0, will be —, +, —, +, —, +, —, +, &c *ad infinitum*, being the contrary signs to those of the second, third, fourth, fifth, and other following terms of the said product,

or of the divisor. This general remainder is *, or 0, $-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx$
 $-\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 - \&c$ *ad infinitum.*

finium. And this alternate succession of the signs — and + must take place in all the following terms of this remainder, to whatever number of terms the said remainder may be continued, as well as in the four terms here set down, because the contrary signs + and — succeed each other continually in the like manner in the terms of the said product, or in the terms of the divisor, to whatever number the terms of the said divisor may be continued.

Art. 8. Now, if we examine the several subsequent steps, or processes, of this operation of division, we shall find that the first term of every new remainder that we shall obtain in the course of it, will be marked with the same sign + or — as the corresponding term, or term involving the same power of x , in

the above-mentioned first remainder $-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^4 - \&c.$ For, since the second term of the divisor $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \&c$ is marked with the sign +, it follows

that, whenever a new term of the quotient is obtained, and the divisor is multiplied into the said new term, the second term of the product thereby obtained must be marked with the same sign as the said new term of the quotient, whether the said sign be + or —; because, by the rules of Algebraick multiplication, when a quantity marked with the sign + is multiplied into a quantity marked with the sign +, the product is marked with the sign +; and when a quantity marked with the sign — is multiplied into a quantity marked with the sign —, the product is marked with the sign —. And the first term of the said product of the multiplication of the divisor into the new term of the quotient must always be marked with the same sign as the said new term of the quotient, because it is the product of the multiplication of the said new term of the quotient into 1, the first term of the divisor. Therefore both the first and the second term of the said product will be marked with the same sign as the said new term of the quotient, and consequently with the same sign the one as the other. But the signs of the terms of the preceding remainder, from which the said product is to be subtracted, are always alternately + and —, or — and +; and the sign of the first term of the said product is always the same with the sign of the first term of the preceding remainder, from which it is to be subtracted, and is likewise equal to the said first term. Therefore the sign of the first term of the said product must be contrary to the sign of the second term of the said preceding remainder. And consequently the sign of the second term of the said product, (which is always the same as the sign of the first term of the said product,) will also be contrary to the sign of the second term of the said preceding remainder; from which second term it is to be subtracted. But, when a quantity, marked with either of the two signs + or —, is to be subtracted from another, the method of performing such subtraction is, first, to change its sign, and

and then to add it, with its sign changed, to the other quantity from which it is to be subtracted. Therefore, when the quantity subtracted is marked with a sign contrary to that of the quantity from which it is to be subtracted (as is the case with the said second term of the said product,) the remainder of the subtraction will have the same sign prefixed to it as was prefixed to the quantity from which the subtraction was made. Therefore the remainder arising from the subtraction of the second term of the said product from the second term of the former remainder will be marked with the same sign $+$ or $-$ as the said second term of the former remainder. But the remainder arising from the subtraction of the first term of the said product from the first term of the preceeding remainder is always equal to 0, because the said first terms are always necessarily equal. Therefore the remainder arising from the subtraction of the second term of the said product from the second term of the preceeding remainder is the first term of the new remainder arising from the subtraction of the whole of the said product from the whole of the said preceeding remainder. Therefore the first term of the said new remainder will be marked with the same sign $+$ or $-$ as the second term of the preceeding remainder, or as the corresponding term, or term involving the same power of x , in the preceeding remainder. And, as these reasonings may be applied to every new remainder obtained in the whole course of this operation of division, to whatever number of remainders, and corresponding new terms in the quotient, the said operation may be continued, it follows that the first term of every new remainder will have the same sign as the corresponding term, or term involving the same power of x , in the first general remainder $-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 - \&c$ *ad infinitum*, to whatever number of such remainders the said operation may be continued; because the first term of every such remainder may be traced through the next preceeding remainder, and the remainder next before that, and the third, and other following, preceeding remainders, up to that first general remainder, in the manner that has been here explained.

But the several terms of the said first general remainder $-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 - \&c$ *ad infinitum* are marked with the signs $-$ and $+$ alternately.

Therefore the first terms of all the following remainders, (which first terms will involve in them the square and cube and other following powers of x ,) will be marked with the same signs, respectively, as the corresponding terms, or terms involving the same powers of x , or the square, cube, and other following powers of x , to wit, the terms $+\frac{1}{n} \times \frac{n-1}{2n}xx, -\frac{1}{n} \times \frac{n-1}{2n} + \frac{2n-1}{3n}x^3, +\frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4, - \&c$, in the said first general remainder

mainder $-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 - \&c \text{ ad infinitum}$; that is, the said first terms of all the following remainders, involving $x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, \&c \text{ ad infinitum}$, will be marked with the sign $+$ and the sign $-$ alternately. Q. E. D.

Art. 9. But the signs of the second and third and other following terms of the quotient of the said operation of division must always be the same with the signs of the first terms of the said first and other following remainders, and must also be equal to the said first terms respectively; because they are the quotients that arise by dividing the said first terms of the said first and other following remainders by 1, the first term of the divisor; which division is in truth only a nominal, and not a real, division, and leaves the several dividends in the same condition as before they were so, nominally, divided. Therefore the signs of the second and third and other following terms of the said quotient will be $-, +, -, +, -, +, -, +, -, +, -, +, \&c$, in alternate succession, to whatever number of terms the said quotient may be continued. Q. E. D.

Art. 10. The reasonings used above, in art. 8, to shew that the first term of every new remainder obtained in the course of the foregoing operation of Algebraick division will be marked with the same sign, $+$ or $-$, as the corresponding term, or term involving the same power of x , in the first general remainder

$$-\frac{1}{n}x + \frac{1}{n} \times \frac{n-1}{2n}xx - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 - \&c \text{ ad infinitum},$$

will, perhaps, be more easily understood if we represent the said operation of division in a somewhat abridged notation, by substituting the small letters $b, c, d, e, f, g, h, \&c$ for the co-efficients of $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$ in the second and other following terms of the

divisor $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \&c \text{ ad infinitum}$. In this way of expressing the terms of the said divisor the foregoing operation of division will be as follows:

The Division of 1 by the Infinite Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 - lx^{10} + \&c$ ad infinitum.

The Divisor.

$$x^7 + bx^6 - cx^5 + dx^4 - ex^3 - fx^2 + gx + h)$$

The Dividend.

$$\begin{array}{ccccccccc} \text{I} & * & * & * & * & * & * & * & \\ + & bx & - & cx^2 & + & dx^3 & - & ex^4 & + & fx^5 & - & gx^6 & + & hx^7 \\ \text{I} & & & & & & & & & & & & & \\ - & bx & + & cx^2 & - & dx^3 & + & ex^4 & - & fx^5 & + & gx^6 & - & hx^7 \\ - & bx & - & bx^2 & + & bx^3 & - & bx^4 & + & bx^5 & - & bx^6 & + & bx^7 \end{array}$$

$$\begin{array}{cccccccc}
 * & + & c & \left. \begin{array}{c} \\ \end{array} \right\} x^2 & - & d & \left. \begin{array}{c} \\ \end{array} \right\} x^3 & + & e & \left. \begin{array}{c} \\ \end{array} \right\} x^4 & - & f & \left. \begin{array}{c} \\ \end{array} \right\} x^5 & + & g & \left. \begin{array}{c} \\ \end{array} \right\} x^6 & - & \delta c \\
 & + & b^2 & \left. \begin{array}{c} \\ \end{array} \right\} & & b_c & \left. \begin{array}{c} \\ \end{array} \right\} & & b_d & \left. \begin{array}{c} \\ \end{array} \right\} & & b_c & \left. \begin{array}{c} \\ \end{array} \right\} & & b_f & \left. \begin{array}{c} \\ \end{array} \right\} & & \delta c \\
 & + & c & \left. \begin{array}{c} \\ \end{array} \right\} x^2 & & b_c & \left. \begin{array}{c} \\ \end{array} \right\} x^3 & & c^2 & \left. \begin{array}{c} \\ \end{array} \right\} x^4 & & b_c & \left. \begin{array}{c} \\ \end{array} \right\} x^5 & & c d & \left. \begin{array}{c} \\ \end{array} \right\} x^6 & & \delta c \\
 & + & b^2 & \left. \begin{array}{c} \\ \end{array} \right\} & & b^3 & \left. \begin{array}{c} \\ \end{array} \right\} & & b^2 c & \left. \begin{array}{c} \\ \end{array} \right\} & & b^2 c & \left. \begin{array}{c} \\ \end{array} \right\} & & b^2 d & \left. \begin{array}{c} \\ \end{array} \right\} & & \delta c
 \end{array}$$

$$\begin{array}{ccccccc} \begin{array}{c} -d \\ -\frac{2bc}{b^3} \end{array} & \left\{ \begin{array}{c} x^3 \\ x^3 \\ x^3 \end{array} \right\} & \begin{array}{c} +e \\ +bd \\ +c^2 \\ +b^2c \end{array} & \left\{ \begin{array}{c} x^4 \\ x^4 \\ x^4 \\ x^4 \end{array} \right\} & \begin{array}{c} -f \\ -bc \\ -cd \\ -b^2d \end{array} & \left\{ \begin{array}{c} x^5 \\ x^5 \\ x^5 \\ x^5 \end{array} \right\} & \begin{array}{c} +g \\ +bf \\ +ce \\ +b^2e \end{array} \\ \begin{array}{c} -d \\ -\frac{2bc}{b^3} \end{array} & \left\{ \begin{array}{c} x^3 \\ x^3 \\ x^3 \end{array} \right\} & \begin{array}{c} -d \\ -\frac{2bc}{b^3} \end{array} & \left\{ \begin{array}{c} x^3 \\ x^3 \\ x^3 \end{array} \right\} & \begin{array}{c} +cd \\ +\frac{2bc^2}{b^3c} \end{array} & \left\{ \begin{array}{c} x^5 \\ x^5 \\ x^5 \end{array} \right\} & \begin{array}{c} -d^2 \\ -\frac{2bcd}{b^3d} \end{array} \end{array}$$

$$\begin{array}{c} \left. \begin{array}{c} + \frac{e}{2} \\ + \frac{2bd}{c^2} \\ + \frac{3b^2c}{b^3c} \\ + \frac{b^4}{b^3c} \end{array} \right\} x^4 \\ \left. \begin{array}{c} - \frac{f}{bc} \\ - \frac{2cd}{b^2d} \\ - \frac{2bc^2}{b^3c} \end{array} \right\} x^5 \\ \left. \begin{array}{c} + \frac{g}{2} \\ + \frac{df}{ce} \\ + \frac{b^2e}{d^2} \\ + \frac{2bcd}{b^3d} \end{array} \right\} x^6 - 8c \end{array}$$

The Quotient.

$$\left\{ \begin{array}{l} 1 - bx + c \\ + b^2 \\ + b^3 \\ + c^2 \\ + 2bd \\ + e \end{array} \right\} x^4 \quad \left\{ \begin{array}{l} d \\ - 2bc \\ - b^3 \end{array} \right\} x^3 \quad \left\{ \begin{array}{l} 8 \\ 2bf \\ 2ce \\ 3b^2e \\ d^2 \\ 6bcd \\ 4b^3d \\ c^3 \\ 6b^2e^2 \\ 5b^4e \end{array} \right\} x^2 \quad \left\{ \begin{array}{l} f \\ - 2be \\ - 2cd \\ - 3b^2d \\ - 3bc^2 \\ - 4b^3c \\ - b^5 \end{array} \right\} x^5 \quad \left\{ \begin{array}{l} x^6 - 8cx \\ + x^5 \\ + x^4 \\ + x^3 \\ + x^2 \\ + x \\ + 1 \end{array} \right\}$$

$$\begin{array}{c}
 \begin{array}{c}
 + \\
 + \\
 + \\
 + \\
 +
 \end{array}
 \left. \begin{array}{c}
 2bd \\
 c^2 \\
 3b^2c \\
 b^4
 \end{array} \right\} x^4 \\
 \\
 \begin{array}{c}
 + \\
 + \\
 + \\
 + \\
 +
 \end{array}
 \left. \begin{array}{c}
 be \\
 2b^2d \\
 b^2c^2 \\
 3b^3c \\
 b^5
 \end{array} \right\} x^5 \\
 \\
 \begin{array}{c}
 - \\
 - \\
 - \\
 - \\
 -
 \end{array}
 \left. \begin{array}{c}
 ce \\
 abcd \\
 c^3 \\
 3b^2c^2 \\
 b^4c
 \end{array} \right\} x^6 + & \&c
 \end{array}
 \quad
 \begin{array}{c}
 * \\
 \hline
 \begin{array}{c}
 - \\
 - \\
 - \\
 - \\
 - \\
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 - \\
 -
 \end{array}
 \left. \begin{array}{c}
 f \\
 2be \\
 acd \\
 3b^2d \\
 3bc^2 \\
 4b^3c \\
 b^5
 \end{array} \right\} x^5 \\
 \\
 \begin{array}{c}
 + \\
 + \\
 + \\
 + \\
 + \\
 + \\
 + \\
 +
 \end{array}
 \left. \begin{array}{c}
 gbf \\
 ace \\
 b^2e \\
 d^2 \\
 abcd \\
 b^2d \\
 c^3 \\
 3b^2c^2 \\
 b^4c
 \end{array} \right\} x^6 - & \&c
 \end{array}
 \quad
 \begin{array}{c}
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 - \\
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 - \\
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 -
 \end{array}
 \left. \begin{array}{c}
 bf \\
 2b^2c \\
 2bcd \\
 3b^3d \\
 3b^2c^2 \\
 4b^4c \\
 b^6
 \end{array} \right\} x^6 + & \&c
 \end{array}
 \quad
 \begin{array}{c}
 \begin{array}{c}
 + \\
 + \\
 + \\
 + \\
 + \\
 + \\
 + \\
 +
 \end{array}
 \left. \begin{array}{c}
 gbf \\
 ace \\
 3b^2c \\
 d^2 \\
 6bcd \\
 4b^3d \\
 c^3 \\
 6b^2c^2 \\
 5b^4c \\
 b^6
 \end{array} \right\} x^6 - & \&c
 \end{array}
 \end{array}$$

Art. 11. In this manner of exhibiting the said operation of division it is apparent at first sight that the first term of every new remainder, obtained by the subtraction of the last product from the next preceeding remainder, will be marked with the same sign + or —, as the second term of the said next preceeding remainder, (which involves in it the same power of x as the said first term of the new remainder,) and consequently with the same sign as the corresponding term, or term involving the same power of x , in the first general remainder — $bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c$, from which all the following remainders are derived. But the signs to be prefixed to the second, third, fourth, and other following terms of the quotient are the same with the signs prefixed to the first terms of the said first, second, third, and other following remainders, respectively; because the said second, third, fourth, and other following terms of the quotient are derived from the said first terms of the said first, second, third, and other following remainders, respectively, by dividing them by 1, (the first term of the divisor) which division makes no change either in the magnitude of the said first terms of the said remainders or in the signs to be prefixed to them. Therefore the signs to be prefixed to the second, third, fourth, and other following terms of the quotient will be the same as those which are prefixed to the corresponding terms, or terms involving the same powers of x , in the said first general remainder — $bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c$ *ad infinitum*, respectively, and consequently will be —, +, —, +, —, +, —, +, —, +, &c, in alternate succession, *ad infinitum*, or to whatsoever number of terms the said quotient may be continued.

Q. E. D.

Art. 12. In the last exhibition (in art. 10,) of the foregoing operation of Algebraick division the co-efficients of x^2 , x^3 , x^4 , x^5 , x^6 , &c in the third, and fourth, and fifth, and other following terms of the quotient, and in the corresponding products and remainders obtained in the course of the said operation, become continually more and more complicated and take up a greater quantity of space in the exhibition of them, as the number of the said successive remainders, and of the terms of the said quotient derived from the first terms of the said remainders, continually increases. Now this tediousness and complication in the notation of the said terms of the quotient, and of the products and remainders corresponding to them, may be avoided without any prejudice to the reasonings that have been used concerning the said terms of the quotient and the said products and remainders corresponding to them, by substituting the single capital letters B, C, D, E, F, G, H, &c instead of the co-efficients of x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c in the second and other following terms of the said quotient, and in the products and remainders corresponding to them. And then the foregoing operation of division will be as follows :

The

The Division of 1 by the Infinite Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - ix^8 + kx^9 - lx^{10} + \&c$, with the Co-efficients of the several Powers of x in the second and other following Terms of the Quotient expressed by the single capital Letters B, C, D, E, F, G, H, &c.

The Divisor.

$$1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + \&c$$

The Dividend.

$$\begin{array}{r} 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + \&c \\ * - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ \hline * - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ + bBx^2 - cBx^3 + dBx^4 - eBx^5 + fBx^6 - \&c \\ \hline * + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ + bBx^2 - cBx^3 + dBx^4 - eBx^5 + fBx^6 - \&c \end{array}$$

D 2

The Quotient.

$$\left\{ \begin{array}{l} 1 - Bx + Cx^2 - Dx^3 \\ + Ex^4 - Fx^5 + Gx^6 \\ - \&c \end{array} \right.$$

Or (putting C for $c + bB$)

$$\begin{array}{r} Cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ - cBx^3 + dBx^4 - eBx^5 + fBx^6 - \&c \end{array}$$

The Product to be subtracted.

$$\begin{array}{r} Cx^2 - bCx^3 - cCx^4 + dCx^5 - eCx^6 + \&c \\ \hline * - dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ - cBx^3 + dBx^4 - eBx^5 + fBx^6 - \&c \\ - bCx^3 + cCx^4 - dCx^5 + eCx^6 - \&c \end{array}$$

Or

Or (putting D for $d + eB + bC$,)

$$\begin{aligned} - Dx^3 + ex^4 - fx^5 + gx^6 - \&c \\ + dBx^4 - eBx^5 + fBx^6 - \&c \\ + cCx^4 - dCx^5 + eCx^6 - \&c \end{aligned}$$

The Product to be subtracted.

$$- Dx^3 - bDx^4 + cDx^5 - dDx^6 + \&c$$

$$\begin{aligned} * \quad & + ex^4 - fx^5 + gx^6 - \&c \\ & + dBx^4 - eBx^5 + fBx^6 - \&c \\ & + cCx^4 - dCx^5 + eCx^6 - \&c \\ & + bDx^4 - cDx^5 + dDx^6 - \&c \end{aligned}$$

Or (putting E for $e + dB + cC + bD$)

$$\begin{aligned} + Ex^4 - fx^5 + gx^6 - \&c \\ - eBx^5 + fBx^6 - \&c \\ - dCx^5 + eCx^6 - \&c \\ - cDx^5 + dDx^6 - \&c \end{aligned}$$

The Product to be subtracted.

$$+ Ex^4 + bEx^5 - cEx^6 + \&c$$

$$\begin{aligned} * \quad & - fx^5 + gx^6 - \&c \\ & - eBx^5 + fBx^6 - \&c \\ & - dCx^5 + eCx^6 - \&c \\ & - cDx^5 + dDx^6 - \&c \\ & - bEx^5 + cEx^6 - \&c \end{aligned}$$

Or (putting F for $f + eB + dC + cD + bE$,

$$\begin{aligned} - Fx^5 + gx^6 - \&c \\ + fBx^6 - \&c \\ + eCx^6 - \&c \\ + dDx^6 - \&c \\ + cEx^6 - \&c \end{aligned}$$

The Product to be subtracted,

$$\begin{array}{r}
 -Fx^5 - bFx^6 + \&c \\
 \hline
 * \\
 + g^6 - \&c \\
 + fBx^6 - \&c \\
 + eCx^6 - \&c \\
 + dDx^6 - \&c \\
 + cEx^6 - \&c \\
 + bFx^6 - \&c
 \end{array}$$

Or (putting G for $g + fB + eC + dD + cE + bF$)

$$+ Gx^6 - \&c$$

The Product to be subtracted.

$$\begin{array}{r}
 + Gx^6 + \&c \\
 \hline
 * - \&c
 \end{array}$$

Art. 13. In this last way of exhibiting the foregoing operation of division it appears, 1st, that B, the co-efficient of x in the quotient $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ (which is equal to the quantity

$\frac{1}{1+x}\frac{1}{n}$, or to the fraction $\frac{1}{1+x}\frac{1}{n}$) is equal to b , or the co-efficient of x in the series $1 + bx - cx^2 +$

$dx^3 - ex^4 + fx^5 - gx^6 + \&c$, which is the divisor in the said operation of division, and which is equal to the quantity $\frac{1}{1+x}\frac{1}{n}$; and, 2ndly, that C, or the co-efficient of x^2 in the said quotient, is equal to $c + bB$; and, 3dly, that D, or the co-efficient of x^3 in the said quotient, is equal to $d + cB + bC$; and, in like manner, that E, F, and G, the co-efficients of x^4 , x^5 , and x^6 in the said quotient, are equal to $e + dB + cC + bD$, $f + eB + dC + cD + bE$, and $g + fB + eC + dD + cE + bF$, respectively. And it is easy to perceive that the following co-efficients H, I, K, L, &c, if the said operation of division were to be carried far

far enough, would be equal to $b + gB + fC + eD + dE + cF + bG$, and $i + bB + gC + fD + eE + dF + cG + bH$, and $k + iB + bC + gD + fE + eF + dG + cH + bI$, and $l + kB + iC + bD + gE + fF + eG + dH + cI + bK$, &c, respectively; in which the first term of the value of every new capital letter H, I, K, L, &c, or of every new co-efficient of a power of x in the said quotient, is equal to a set of terms all added to each other, of which the first term is the co-efficient of the same power of x in the divisor, and the other terms are the products that arise by multiplying the co-efficients of the foregoing terms of the divisor, beginning with b or the co-efficient of its second term bx , to wit, the co-efficients $b, c, d, e, f, g, h, i, k, l$, &c, into the co-efficients of the foregoing terms of the quotient, beginning likewise with B , the co-efficient of the second term Bx , but in an inverted order. And by the help of this observation it would be possible to derive the values of as many of the co-efficients of the powers of x in the terms of the said quotient $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - \&c$ as we pleased, from the co-efficients of the preceeding terms of the said quotient and of the divisor. But the computations necessary for this purpose, by substituting in the expression of the value of every new co-efficient the numerous products of the multiplication of the preceeding co-efficients in both the serieses, (to wit, that which is equal to the quotient and that which is equal to the divisor,) into each other, would be found to become very soon intolerably intricate and laborious. And therefore it will be highly expedient to look-out for some other method of continuing the terms of this series, or deriving them one from another, that may be less difficult to reduce to practice, and more particularly to endeavour to discover the law of continuation of the terms of this series, that is set forth above in art. 2, as that which results from the extension of Sir Isaac Newton's binomial theorem

to the quantity $\sqrt[n]{1+x}$, or $\frac{1}{1+x}^{\frac{1}{n}}$. But, before we enter upon that more

abstruse and difficult part of our present inquiry, I will lay before the reader two other methods of investigating a few of the first terms of the infinite series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, which is

equal to the quantity $\sqrt[n]{1+x}$, or the fraction $\frac{1}{1+x}^{\frac{1}{n}}$; which, though not

better than the foregoing method of finding them by the foregoing Algebræ division of 1, the numerator of the said fraction, by the infinite series $1 +$

$$\frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \&c, \text{ or } 1 + bx - cx^2 + dx^3 - ex^4 + \&c, \text{ which is equal}$$

to

to $\sqrt[n]{1+x}$, it's denominator, nor indeed, quite so simple and so natural as that method, will, nevertheless, serve to illustrate the said method, and to confirm the conclusions deduced from it. The first of these methods is as follows:

Another Method of investigating the second, and third, and other following Terms of the Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the Quantity $\sqrt[n]{1+x}$, or the Fraction $\frac{1}{\sqrt[n]{1+x}}$, and of determining the Signs — and + that are to be prefixed to them.

Art. 14. The quantity $\sqrt[n]{1+x}$ (which is the denominator of the fraction $\frac{1}{\sqrt[n]{1+x}}$, has been shewn in the second volume of this collection of tracts called *Scriptores Logarithmici*, (page 237, art. 47,) to be equal to the infinite series

$$1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}xx + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}x^4 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}x^5 - \&c \text{ ad infinitum.}$$

Therefore, if, for the sake of brevity, we put $b = \frac{1}{n}$, and $c = \frac{1}{n} \times \frac{n-1}{2n}$, or $\frac{n-1}{2n}b$, and $d = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, or $\frac{2n-1}{3n}c$, and $e = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$, or $\frac{3n-1}{4n}d$, and $f = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}$, or $\frac{4n-1}{5n}e$, and $g, b, i, k, \&c$ equal to $\frac{5n-1}{6n}f, \frac{6n-1}{7n}g, \frac{7n-1}{8n}b, \frac{8n-1}{9n}i, \&c$, respectively, we shall have $\sqrt[n]{1+x} =$ the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c \text{ ad infinitum.}$ And consequently the quantity $\sqrt[n]{1+x}$, or the fraction $\frac{1}{\sqrt[n]{1+x}}$, will be equal to the fraction $\frac{1}{1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c \text{ ad infinitum.}}$

But

But the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, is equal to the quantity $\frac{1}{1+x\sqrt[n]{n}}$, or the fraction $\frac{1}{1+x\sqrt[n]{n}}$.

Therefore the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ will be equal to the fraction

$\frac{1}{1+bx-cx^2+dx^3-ex^4+fx^5-gx^6+hx^7-\&c}$ *ad infinitum*. And consequently, if the series $1+bx-cx^2+dx^3-ex^4+fx^5-gx^6+hx^7-\&c$ (which is the denominator of the said fraction,) be multiplied into the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, the product of the said multiplication will be equal to 1, which is the numerator of the said fraction. This multiplication may be performed in the manner following :

*The Multiplication of the Series $1+bx-cx^2+dx^3-ex^4+fx^5-gx^6+hx^7+\&c$ *ad infinitum* into the Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ *ad infinitum*.*

$$\begin{array}{l} 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c \\ 1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c \end{array}$$

$$\begin{array}{l} 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c \\ Bx, bBx^2, cBx^3, dBx^4, eBx^5, fBx^6, gBx^7, \&c \\ Cx^2, bCx^3, cCx^4, dCx^5, eCx^6, fCx^7, \&c \\ Dx^3, bDx^4, cDx^5, dDx^6, eDx^7, \&c \\ Ex^4, bEx^5, cEx^6, dEx^7, \&c \\ Fx^5, bFx^6, cFx^7, \&c \\ Gx^6, bGx^7, \&c \\ Hx^7, \&c \end{array}$$

Let us, for the sake of brevity, call this compound series S. Then will the said compound series S be equal to 1.

Art. 15. In this series none of the terms have the signs + and - prefixed to them, except those in the first horizontal line; because the terms in all the other horizontal

horizontal lines arise from the multiplication of the terms of the multiplicand $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$ into the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, the signs to be prefixed to which are not yet known. But we may here observe, that the signs that ought to be prefixed to the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the said compound series, or product, S , (where the said terms are the first terms of the second, third, fourth, fifth, sixth, seventh, and eighth, and other following, horizontal lines, or rows, of terms,) must be the same with those that ought to be prefixed to the same terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, respectively, in the multiplier $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$; because the said terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the said compound series, or product, S , arise from the multiplication of the first term 1 , of the multiplicand, or series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$ *ad infinitum* into the several terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ of the multiplier, or series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$. And therefore, if we can discover the signs $+$ and $-$ that ought to be prefixed to the several terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the said product, or compound series, S , we shall thereby obtain the signs that ought to be prefixed to the same terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the multiplier, or series $1, Bx, Cx^2, Dx^3, Ex^4,$

$Fx^5, Gx^6, Hx^7, \&c$, which is equal to the quantity $\frac{-1}{1+x}^{\frac{1}{n}}$, or to the fraction $\frac{1}{1+x}^{\frac{1}{n}}$. Now, what these signs ought to be, and what will be the

values of the several co-efficients $B, C, D, E, F, G, H, \&c$, of the powers of x in the several terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the said compound series S , may be gradually discovered by attentively considering the several terms that form the said compound series, and reasoning upon them in the manner following :

The Investigation of the Second Term, Bx , of the Series $1, Bx, Cx^2, Dx^3, Ex^4,$

$Fx^5, Gx^6, Hx^7, \&c$ ad infinitum, which is equal to the Quantity $\frac{-1}{1+x}^{\frac{1}{n}}$,

or to the Fraction $\frac{1}{1+x}^{\frac{1}{n}}$.

Art. 16. Since the whole compound series S is equal to 1 , which is it's first term, it follows that the whole of this series, except the first term 1 , that is, the series $S - 1$, or

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$+ bx$

$$\begin{array}{r}
 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c \\
 \left\{ \begin{array}{l} Bx, bBx^2, cBx^3, dBx^4, eBx^5, fBx^6, gBx^7, \&c \\ Cx^2, bCx^3, cCx^4, dCx^5, eCx^6, fCx^7, \&c \\ Dx^3, bDx^4, cDx^5, dDx^6, eDx^7, \&c \\ Ex^4, bEx^5, cEx^6, dEx^7, \&c \\ Fx^5, bFx^6, cFx^7, \&c \\ Gx^6, bGx^7, \&c \\ Hx^7, \&c \\ \&c \end{array} \right\}
 \end{array}$$

will be equal to 0, and consequently that the sum of those of it's terms which are to be marked with the sign — will be equal to the sum of those which are to be marked with the sign +, from which the former sum is to be subtracted. And this equality between these sums will, it is evident, still continue, if all the terms be multiplied, or divided, by any number whatsoever. Therefore, if all the terms be divided by x , the said equality will still continue, and consequently the new compound series thereby produced will, like the former, be equal to 0. But by this division, this last compound series $S - 1$ will become

$$\begin{array}{r}
 + b - cx + dx^2 - ex^3 + fx^4 - gx^5 + bx^6 - \&c \\
 \left\{ \begin{array}{l} B, bBx, cBx^2, dBx^3, eBx^4, fBx^5, gBx^6, \&c \\ Cx, bCx^2, cCx^3, dCx^4, eCx^5, fCx^6, \&c \\ Dx^2, bDx^3, cDx^4, dDx^5, eDx^6, \&c \\ Ex^3, bEx^4, cEx^5, dEx^6, \&c \\ Fx^4, bFx^5, cFx^6, \&c \\ Gx^5, bGx^6, \&c \\ Hx^6, \&c \\ \&c, \end{array} \right\}
 \end{array}$$

which will be $= \frac{s-1}{x}$. Therefore this last compound series $\frac{s-1}{x}$ will be equal to 0, or the sum of all the terms in it which are to be marked with the sign — will be equal to the sum of all it's other terms which are to be marked with the sign +. And this will always be true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true, when x is equal to 0. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and the last compound series $\frac{s-1}{x}$ will be reduced to the two terms in it's first vertical column, to wit, the terms $+b$ and B . Therefore the two terms $+b$ and B will, together, be equal to 0; and consequently B must be subtracted from b , and therefore must be marked with

with the sign $-$; and it must likewise be equal to b . But the term Bx in the compound series $S - 1$ and in the compound series S must be marked with the same sign as the term B in the compound series $\frac{s-1}{x}$, because the compound series $\frac{s-1}{x}$ is derived from the two former compound series $S - 1$ and S without any operations that can cause a change in the signs $+$ and $-$ which are to be prefixed to it's terms. Therefore the term Bx in the compound series S must have the sign $-$ prefixed to it. And consequently, by art. 15, the term Bx in the multiplying series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ (which is equal to the quantity $1 + x \sqrt[n]{\frac{-1}{n}}$, or to the fraction $\frac{1}{1 + x \sqrt[n]{\frac{1}{n}}}$), must also have the sign $-$ prefixed to it. Therefore the two first terms of the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ will be $1 - Bx$, or (because B is $= b$), $1 - bx$, or (because b is $= \frac{1}{n}$), $1 - \frac{1}{n}x$.

Q. E. I.

The Investigation of the Third Term, Cx^2 , of the Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, ad infinitum, which is equal to the Quantity $1 + x \sqrt[n]{\frac{-1}{n}}$, or to the Fraction $\frac{1}{1 + x \sqrt[n]{\frac{1}{n}}}$.

Art. 17. Since the sign $-$ is to be prefixed to Bx , or the second term of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$ is multiplied above in art. 14, it follows that the second horizontal line of terms in the compound series S , which is produced by the said multiplication, will be $-Bx - bBx^2 + eBx^3 - dBx^4 + eBx^5 - fBx^6 + gBx^7 - \&c$, in which the signs prefixed to the several terms are, respectively, contrary to those which are prefixed to the corresponding terms of the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, (which is multiplied by the said term $-Bx$), agreeably to the rules of Algebraick multiplication. Therefore the compound series $\frac{s-1}{x}$, set down in art. 16, which was obtained by, first, taking away the term 1 from the compound series S , and then dividing the remainder, or the compound series $S - 1$, by x (neither of which operations, it is evident,

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can

can affect the signs + and — which are to be prefixed to it's several terms,) will, when the proper signs are prefixed to the terms in it's second horizontal line, be as follows, to wit,

$$\begin{array}{r}
 + b - cx + dx^2 - ex^3 + fx^4 - gx^5 + bx^6 - \&c \\
 - B - bBx + cBx^2 - dBx^3 + eBx^4 - fBx^5 + gBx^6 - \&c \\
 \left\{ \begin{array}{l} Cx, bCx^2, cCx^3, dCx^4, eCx^5, fCx^6, \&c \\ Dx^2, bDx^3, cDx^4, dDx^5, eDx^6, \&c \\ Ex^3, bEx^4, cEx^5, dEx^6, \&c \\ Fx^4, bFx^5, cFx^6, \&c \\ Gx^5, bGx^6, \&c \\ Hx^6, \&c \\ \&c. \end{array} \right.
 \end{array}$$

But it has been shewn in art. 16 that the compound series $\frac{s-1}{x}$ is equal to 0.

Therefore the compound series just now set down (which is the compound series $\frac{s-1}{x}$ with the proper signs + and — prefixed to the several terms of it's second horizontal line,) will be equal to 0, or the sum of all the terms in it which are to be marked with the sign — will be equal to the sum of all the other terms, which are to be marked with the sign +.

But it has been shewn in art. 16 that $b - B$ is equal to 0.

Therefore, if the terms b and B be left out of this last compound series, the remaining terms of it will still be equal to 0, that is, the compound series

$$\begin{array}{r}
 - cx + dx^2 - ex^3 + fx^4 - gx^5 + bx^6 - \&c \\
 - bBx + cBx^2 - dBx^3 + eBx^4 - fBx^5 + gBx^6 - \&c \\
 \left\{ \begin{array}{l} Cx, bCx^2, cCx^3, dCx^4, eCx^5, fCx^6, \&c \\ Dx^2, bDx^3, cDx^4, dDx^5, eDx^6, \&c \\ Ex^3, bEx^4, cEx^5, dEx^6, \&c \\ Fx^4, bFx^5, cFx^6, \&c \\ Gx^5, bGx^6, \&c \\ Hx^6, \&c \\ \&c, \end{array} \right.
 \end{array}$$

or $\frac{s-1}{x} - b + B$, will be equal to 0, or the sum of all the terms in it that are to be marked with the sign — will be equal to the sum of all the other terms,

terms, which are to be marked with the sign +. And this equality between these sums, it is evident, will continue, if all the terms should be divided by x , by which division the last compound series would be converted into the following compound series, to wit,

$$\left\{ \begin{array}{l} -c + dx - ex^2 + fx^3 - gx^4 + hx^5 - \&c \\ -bB + cBx - dBx^2 + eBx^3 - fBx^4 + gBx^5 - \&c \\ C, \quad bCx, \quad cCx^2, \quad dCx^3, \quad eCx^4, \quad fCx^5, \quad \&c \\ \quad Dx, \quad bDx^2, \quad cDx^3, \quad dDx^4, \quad eDx^5, \quad \&c \\ \quad \quad Ex^2, \quad bEx^3, \quad cEx^4, \quad dEx^5, \quad \&c \\ \quad \quad \quad Fx^3, \quad bFx^4, \quad cFx^5, \quad \&c \\ \quad \quad \quad \quad Gx^4, \quad bGx^5, \quad \&c \\ \quad \quad \quad \quad \quad Hx^5, \quad \&c \\ \quad \quad \quad \quad \quad \quad \&c. \end{array} \right\}$$

Therefore this last compound series will be equal to 0, or the sum of all the terms in it which are to be marked with the sign — will be equal to the sum of all the other terms of it, which are to be marked with the sign +. And this equality will always continue, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will continue also when x is equal to 0. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and the last compound series will be reduced to the three terms in its first vertical column, to wit, the three terms $-c - bB, C$. Therefore these three terms $-c - bB, C$ will, together, be equal to 0. And consequently, as the two first terms c and bB have the sign — prefixed to them, the third term C must have the sign + prefixed to it, and must be equal to the sum of the other two terms c and bB , or $-c$ and $-bB$; because otherwise the three terms $-c, -bB$, and C , or $-c - bB + C$, taken together, could not be equal to 0. But, for the reasons given above in art. 16, the term Cx^2 in the first compound series S (obtained by the multiplication exhibited above in art. 14,) must have the same sign prefixed to it as the term C in this last compound series, which corresponds to the term Cx^2 in the said series S , and is derived from it by arithmetical operations that cannot affect the sign to be prefixed to it. Therefore the term Cx^2 in the said compound series S must have the sign + prefixed to it. And consequently (by art. 15,) the term Cx^2 in the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$ is multiplied in art. 14,) must also have the sign + prefixed to it; that is, the third term Cx^2 of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$,

which is equal to the quantity $\frac{1}{1+x^n}$, or to the fraction $\frac{1}{1+x^n}$, will have

the sign + prefixed to it, or must be added to the first term 1. And consequently

quently the three first terms of that series will be $1 - Bx + Cx^2$, or
 $1 - \frac{1}{n}x + Cx^2$. Q. E. I.

Art. 18. The magnitude of the co-efficient C will be determined by resolving the simple equation $C = c + bB$; which may be done as follows:

Since b is $= \frac{1}{n}$, and c is $= \frac{1}{n} \times \frac{n-1}{2n}$, and B has been shewn to be $= b$, or $\frac{1}{n}$, we shall have $bB = \frac{1}{n} \times \frac{1}{n}$, and $c + Bb = \frac{1}{n} \times \frac{n-1}{2n} + \frac{1}{n} \times \frac{1}{n} (= \frac{1}{n} \times \sqrt{\frac{n-1}{2n} + \frac{1}{n}} = \frac{1}{n} \times \sqrt{\frac{n-1}{2n} + \frac{2}{2n}} = \frac{1}{n} \times \frac{n-1+2}{2n}) = \frac{1}{n} \times \frac{n+1}{2n}$. Therefore Cx^2 will be $= \frac{1}{n} \times \frac{n+1}{2n} \times x^2 = B \times \frac{n+1}{2n} \times x^2 = \frac{n+1}{2n} Bx^2$; and consequently the three first terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to $\sqrt[1+x]{\frac{-1}{n}}$, or to the fraction $\frac{1}{1+x\frac{1}{n}}$, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}x^2$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2$. Q. E. I.

The Investigation of the Fourth Term, Dx^3 , of the Series $1, Bx, Cx^2, Dx^3,$

$Ex^4, Fx^5, Gx^6, Hx^7, \&c$ ad infinitum, which is equal to the Quantity $\sqrt[1+x]{\frac{-1}{n}}$, or to the Fraction $\frac{1}{1+x\frac{1}{n}}$.

Art. 19. Since the sign $+$ is to be prefixed to Cx^2 , the third term of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$ is multiplied above in art. 14, it follows that the third horizontal row of terms in the compound series S , produced by the said multiplication, will be $+ Cx^2 + bCx^3 - cCx^4 + dCx^5 - eCx^6 + fCx^7 - gCx^8 + \&c$, in which the signs prefixed to the several terms are the same, respectively, with those which are prefixed to the corresponding terms of the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$, which is multiplied by the said term Cx^2 , or $+ Cx^2$;

+ Cx^2 ; agreeably to the rules of Algebraick multiplication. Therefore the compound series S , obtained by the said multiplication, will, when the proper signs + and — are prefixed to the terms of it's three first horizontal lines, be as follows, to wit,

$$\left\{ \begin{array}{l} 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c \\ - Bx - bBx^2 + cBx^3 - dBx^4 + eBx^5 - fBx^6 + gBx^7 - \&c \\ + Cx^2 + bCx^3 - cCx^4 + dCx^5 - eCx^6 + fCx^7 - \&c \\ \quad Dx^3, \quad bDx^4, \quad cDx^5, \quad dDx^6, \quad eDx^7, \quad \&c \\ \quad \quad Ex^4, \quad bEx^5, \quad cEx^6, \quad dEx^7, \quad \&c \\ \quad \quad \quad Fx^5, \quad bFx^6, \quad cFx^7, \quad \&c \\ \quad \quad \quad \quad Gx^6, \quad bGx^7, \quad \&c \\ \quad \quad \quad \quad \quad Hx^7, \quad \&c \\ \quad \quad \quad \quad \quad \quad \&c; \end{array} \right.$$

and consequently the compound series derived from this series in art. 17, will, when the proper signs + and — are prefixed to the terms contained in it's three first horizontal lines, be as follows, to wit,

$$\left\{ \begin{array}{l} - c + dx - ex^2 + fx^3 - gx^4 + hx^5 - \&c \\ - bB + cBx - dBx^2 + eBx^3 - fBx^4 + gBx^5 - \&c \\ + C + bCx - cCx^2 + dCx^3 - eCx^4 + fCx^5 - \&c \\ \quad Dx, \quad bDx^2, \quad cDx^3, \quad dDx^4, \quad eDx^5, \quad \&c \\ \quad \quad Ex^2, \quad bEx^3, \quad cEx^4, \quad dEx^5, \quad \&c \\ \quad \quad \quad Fx^3, \quad bFx^4, \quad cFx^5, \quad \&c \\ \quad \quad \quad \quad Gx^4, \quad bGx^5, \quad \&c \\ \quad \quad \quad \quad \quad Hx^5, \quad \&c \\ \quad \quad \quad \quad \quad \quad \&c. \end{array} \right.$$

Therefore this series will be equal to 0, or the sum of the several terms in it that are to be marked with the sign — will be equal to the sum of all the other terms, which are to be marked with the sign +.

But it has been shewn that the three terms $-c - bB + C$ are equal to 0. Therefore, if these three terms are left out of this last compound series, the remaining terms of it will still be equal to 0, or the sum of those of them which are to be marked with the sign — will be equal to the sum of all the other terms, which are to be marked with the sign +. But, when the three terms $-c - bB + C$ are left out of the foregoing compound series, it will become as follows, to wit,

$$+ dx$$

$$\begin{array}{l}
 + dx - ex^2 + fx^3 - gx^4 + bx^5 - \&c \\
 + cBx - dBx^2 + eBx^3 - fBx^4 + gBx^5 - \&c \\
 + bCx - cCx^2 + dCx^3 - eCx^4 + fCx^5 - \&c \\
 D x, \quad b D x^2, \quad c D x^3, \quad d D x^4, \quad e D x^5, \quad \&c \\
 \quad \quad E x^2, \quad b E x^3, \quad c E x^4, \quad d E x^5, \quad \&c \\
 \quad \quad \quad F x^3, \quad b F x^4, \quad c F x^5, \quad \&c \\
 \quad \quad \quad \quad G x^4, \quad b G x^5, \quad \&c \\
 \quad \quad \quad \quad \quad H x^5, \quad \&c \\
 \quad \quad \quad \quad \quad \quad \&c.
 \end{array}$$

Therefore this last compound series will be equal to 0, or the sum of all the terms in it that are to be marked with the sign $-$ will be equal to the sum of all the other terms of it, which are to be marked with the sign $+$. And this equality between these two sets of terms will still continue, when the terms shall be all divided by x . Therefore the compound series arising from such division, that is, the compound series

$$\begin{array}{l}
 d - ex + fx^2 - gx^3 + bx^4 - \&c \\
 + cB - dBx + eBx^2 - fBx^3 + gBx^4 - \&c \\
 + bC - cCx + dCx^2 - eCx^3 + fCx^4 - \&c \\
 D, \quad b D x, \quad c D x^2, \quad d D x^3, \quad e D x^4, \quad \&c \\
 \quad \quad E x, \quad b E x^2, \quad c E x^3, \quad d E x^4, \quad \&c \\
 \quad \quad \quad F x^2, \quad b F x^3, \quad c F x^4, \quad \&c \\
 \quad \quad \quad \quad G x^3, \quad b G x^4, \quad \&c \\
 \quad \quad \quad \quad \quad H x^4, \quad \&c \\
 \quad \quad \quad \quad \quad \quad \&c
 \end{array}$$

will be equal to 0, or the sum of all the terms in it that are to be marked with the sign $-$ will be equal to the sum of all the other terms in it, which are to be marked with the sign $+$. And this will always be true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will be true also when x is equal to 0. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and consequently the last compound series will be reduced to the four terms contained in it's first vertical column, to wit, the four terms $+ d + cB + bC, D$. Therefore these four terms, taken together, must be equal to 0; and consequently, as the three first of them are marked with the sign $+$, or are to be added to each other, the fourth term D must be subtracted from them, and marked with the sign $-$, and likewise must be equal to the sum of the three former terms, or to $+ d + cB + bC$; because otherwise it will be impossible that the said four terms, taken together, should be equal to 0. Therefore the corresponding term, Dx^3 , in the first compound

compound series S, (from which the last compound series has been derived by arithmetical operations that have not caused any changes in the signs of it's terms,) must also be marked with the same sign —; and so (for the reasons given in art. 15,) must the same term Dx^3 in the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - &c$ is multiplied in art. 14, and which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$. And consequently the

first four terms of the said series will be $1 - Bx + Cx^2 - Dx^3$, or $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - Dx^3$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - Dx^3$.

Q. E. I.

Art. 20. To determine the magnitude of the co-efficient D, we must resolve the simple equation $D = d + cB + bC$; which may be done as follows:

Since b is $= \frac{1}{n}$, and c is $= \frac{1}{n} \times \frac{n-1}{2n}$, and d is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and B is $= \frac{1}{n}$, and C is $= \frac{1}{n} \times \frac{n+1}{2n}$, we shall have $D = d + cB + bC = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n} + \frac{1}{n} \times \frac{1}{n} \times \frac{n+1}{2n}$ $(= \frac{n-1 \times 2n-1}{6n^3} + \frac{n-1}{2n^3} + \frac{n+1}{2n^3} = \frac{n-1 \times 2n-1}{6n^3} + \frac{2n}{2n^3} = \frac{n-1 \times 2n-1}{6n^3} + \frac{6n}{6n^3} = \frac{2nn-3n+1}{6n^3} + \frac{6n}{6n^3} = \frac{2nn+3n+1}{6n^3} = \frac{n+1 \times 2n+1}{6n^3} = \frac{n+1 \times 2n+1}{n \times 2n \times 3n}) = \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$. Therefore Dx^3 will be $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3 = \frac{2n+1}{3n} Cx^3$; and consequently the four first terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, (which is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$), will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bxx - \frac{2n+1}{3n}Cx^3$.

Q. E. I.

The Investigation of the Fifth Term, $E x^4$, of the Series $1, B x, C x^2, D x^3, E x^4, F x^5, G x^6, H x^7, \&c$ ad infinitum, which is equal to the Quantity $1 + x^{\frac{-1}{n}}$, or to the Fraction $\frac{1}{1 + x^{\frac{1}{n}}}$.

Art. 21. Since the sign $-$ is to be prefixed to $D x^3$, the fourth term of the series $1, B x, C x^2, D x^3, E x^4, F x^5, G x^6, H x^7, \&c$, by which the series $1 + b x - c x^2 + d x^3 - e x^4 + f x^5 - g x^6 + h x^7 - \&c$ is multiplied above in art. 14, it follows that the fourth horizontal row of terms in the compound series S , which is the product of that multiplication, will be $- D x^3 - b D x^4 + c D x^5 - d D x^6 + e D x^7 - \&c$, in which the signs $+$ and $-$ prefixed to the several terms are respectively contrary to those which are prefixed to the corresponding terms of the series $1 + b x - c x^2 + d x^3 - e x^4 + f x^5 - g x^6 + h x^7 - \&c$, which is multiplied into the said term $- D x^3$; agreeably to the rules of Algebraick multiplication. Therefore the compound series S , obtained by the said multiplication, will, when the proper signs are prefixed to the terms of it's four first horizontal rows, be as follows, to wit,

$$\begin{array}{rcl}
 1 + b x - c x^2 + d x^3 - e x^4 + f x^5 - g x^6 + h x^7 - \&c & & \\
 - B x - b B x^2 + c B x^3 - d B x^4 + e B x^5 - f B x^6 + g B x^7 - \&c & & \\
 + C x^2 + b C x^3 - c C x^4 + d C x^5 - e C x^6 + f C x^7 - \&c & & \\
 - D x^3 - b D x^4 + c D x^5 - d D x^6 + e D x^7 - \&c & & \\
 E x^4, & b E x^5, & c E x^6, & d E x^7, & \&c \\
 F x^5, & b F x^6, & c F x^7, & \&c \\
 G x^6, & b G x^7, & \&c \\
 H x^7, & \&c \\
 \&c; & &
 \end{array}$$

and consequently the last compound series, obtained in art. 19, (which is derived from the compound series S by various operations that do not affect the signs that are to be prefixed to their terms,) will, when the proper signs are prefixed to the terms of it's fourth horizontal row, be as follows, to wit,

+ d

$$\left\{ \begin{array}{l}
 + d - ex + fx^2 - gx^3 + hx^4 - \&c \\
 + eB - dBx + eBx^2 - fBx^3 + gBx^4 - \&c \\
 + bC - cCx + dCx^2 - eCx^3 + fCx^4 - \&c \\
 - D - bDx + cDx^2 - dDx^3 + eDx^4 - \&c \\
 \quad Ex, \quad bEx^2, \quad cEx^3, \quad dEx^4, \quad \&c \\
 \quad \quad Fx^2, \quad bFx^3, \quad cFx^4, \quad \&c \\
 \quad \quad \quad Gx^3, \quad bGx^4, \quad \&c \\
 \quad \quad \quad \quad Hx^4, \quad \&c \\
 \quad \quad \quad \quad \quad \&c.
 \end{array} \right.$$

Therefore this compound series will be equal to 0, or the sum of the several terms in it which are to be marked with the sign — will be equal to the sum of all the other terms in it, which are to be marked with the sign +.

But it has been shewn that the four terms $+ d + eB + bC - D$, which form the first vertical column of terms in this series, are equal to 0. Therefore, if these four terms are left out of this compound series, the remaining terms of it will also be equal to 0, that is, the following compound series, to wit,

$$\left\{ \begin{array}{l}
 - ex + fx^2 - gx^3 + hx^4 - \&c \\
 - dBx + eBx^2 - fBx^3 + gBx^4 - \&c \\
 - cCx + dCx^2 - eCx^3 + fCx^4 - \&c \\
 - bDx + cDx^2 - dDx^3 + eDx^4 - \&c \\
 \quad Ex, \quad bEx^2, \quad cEx^3, \quad dEx^4, \quad \&c \\
 \quad \quad Fx^2, \quad bFx^3, \quad cFx^4, \quad \&c \\
 \quad \quad \quad Gx^3, \quad bGx^4, \quad \&c \\
 \quad \quad \quad \quad Hx^4, \quad \&c \\
 \quad \quad \quad \quad \quad \&c
 \end{array} \right.$$

will be equal to 0, or the sum of the several terms of it that are marked with the sign — will be equal to the sum of all the other terms of it, which are marked with the sign +. And this equality between these two sets of terms will still continue when all the terms shall be divided by x . Therefore the compound series arising from such division, that is, the compound series

$$\left\{ \begin{array}{l}
 -e + fx - gx^2 + bx^3 - \&c \\
 -dB + eBx - fBx^2 + gBx^3 - \&c \\
 -cC + dCx - eCx^2 + fCx^3 - \&c \\
 -bD + cDx + dDx^2 + eDx^3 - \&c \\
 E, \quad bEx, \quad cEx^2, \quad dEx^3, \quad \&c \\
 \quad \quad Fx, \quad bFx^2, \quad cFx^3, \quad \&c \\
 \quad \quad \quad Gx^2, \quad bGx^3, \quad \&c \\
 \quad \quad \quad \quad Hx^3, \quad \&c \\
 \quad \quad \quad \quad \quad \&c,
 \end{array} \right.$$

will be equal to 0, or the sum of the several terms of it that are marked with the sign $-$ will be equal to the sum of all the other terms of it, which are marked with the sign $+$. And this will always be true, of how small a magnitude soever we may suppose x to be taken: and therefore it will be true also when x is equal to 0. But, when x is equal to 0, all the terms in the said compound series which involve x will be equal to 0 likewise, and consequently the whole series will be reduced to the five terms $-e - dB - cC - bD, E$. Therefore these five terms $-e - dB - cC - bD, E$, taken together, will be equal to 0; and consequently, as the four first of them, to wit, e, dB, cC , and bD , are all marked with the sign $-$, the last term E must be marked with the contrary sign $+$, and must likewise be equal to the sum of the said four first terms. Therefore the corresponding term, Ex^4 , in the compound series S , (from which the last compound series is derived by various operations that do not affect the signs that are to be prefixed to it's terms,) must also have the sign $+$ prefixed to it: and so consequently (by art. 15) must the same term Ex^4 in the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$ is

multiplied in art. 14, and which is equal to the quantity $1 + x \sqrt[n]{\frac{-1}{n}}$, or to the fraction $\frac{1}{1+x \sqrt[n]{\frac{-1}{n}}}$. Therefore the five first terms of the said series will be

$$\begin{aligned}
 &1 - Bx + Cxx - Dx^3 + Ex^4, \text{ or } 1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \\
 &\times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + Ex^4, \text{ or } 1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bxx - \sqrt{\frac{2n+1}{3n}}Cx^3 \\
 &+ Ex^4. \quad \text{Q. E. I.}
 \end{aligned}$$

Art. 22. To determine the magnitude of the coefficient E , we must resolve the simple equation $E = e + dB + cC + bD$; which may be done as follows:

Since

Since b is $= \frac{1}{n}$, and c is $= \frac{1}{n} \times \frac{n-1}{2n}$, and d is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and e is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$, and B is $= \frac{1}{n}$, and C is $= \frac{1}{n} \times \frac{n+1}{2n}$, and D is $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$, we shall have

$$\begin{aligned}
 E &= e + dB + cC + bD = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n} \\
 &+ \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n} \times \frac{n+1}{2n} + \frac{1}{n} \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \\
 &= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n} \times \frac{n+1}{2n} \\
 &+ \frac{1}{n} \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} = \frac{6n^3 - 11nn + 6n - 1}{24n^4} + \frac{2nn - 3n + 1}{6n^4} + \frac{nn - 1}{4n^4} + \frac{2nn + 3n + 1}{6n^4} \\
 &= \frac{6n^3 - 11nn + 6n - 1}{24n^4} + \frac{8nn - 12n + 4}{24n^4} + \frac{6nn - 6}{24n^4} + \frac{8nn + 12n + 4}{24n^4} \\
 &= \frac{6n^3 - 11nn + 22nn + 6n - 7 + 8}{24n^4} = \frac{6n^3 + 11nn + 6n + 1}{24n^4} = \frac{2nn + 3n + 1}{24n^4} \times \frac{3n + 1}{3n + 1} \\
 &= \frac{n + 1}{24n^4} \times \frac{2n + 1}{2n + 1} \times \frac{3n + 1}{3n + 1} = \frac{1}{n} \times \frac{n + 1}{2n} \times \frac{2n + 1}{3n} \times \frac{3n + 1}{4n} \\
 &= \frac{1}{n} \times \frac{n + 1}{2n} \times \frac{2n + 1}{3n} \times \frac{3n + 1}{4n}. \quad Q. E. I.
 \end{aligned}$$

Therefore the five first terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c \text{ ad infinitum}$, which is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bxx - \frac{2n+1}{3n}Cx^3 + \frac{3n+1}{4n}Dx^4$. Q. E. I.

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Art. 23. These five terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c \text{ ad infinitum}$, (which is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$), agree exactly with the five first terms of the same series obtained above,

above, in art. 4, by the operation of Algebraïck division there performed, and therefore afford a confirmation of the justness of the reasonings used in that first method of obtaining them. Indeed the second method is so nearly related to the first that it may be considered as the natural way of proving it, since it is founded on a multiplication of the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, (which is the divisor in the operation of division performed in the first method,) into the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (which is the quotient of the said division,) and supposing the product, or compound series S , thereby obtained, to be equal to 1 , (which is the dividend in the said operation of division performed in the first method,) and deducing the values of the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (which are the terms of the quotient,) from that supposition. For it thereby very much resembles the common method of proving the truth of an operation of division in Arithmetick by multiplying the divisor into the quotient to see if the product will be equal to the dividend; since in both cases a result that is obtained, first, by an operation of division, is afterwards confirmed by an operation of multiplication.

Of these two methods the first is the shorter and the easier, and enables us to find the foregoing five first terms of the series sought with much less trouble than the second. But the reasonings used in the second method may, perhaps, be thought clearer and more satisfactory than those in the first, inasmuch as the operation of multiplication is generally considered as plainer and more convincing than that of division.

The Investigation of the Sixth Term, Fx^5 , of the Series $1, Bx, Cx^2, Dx^3, Ex^4,$

*$Fx^5, Gx^6, Hx^7, \&c$ ad infinitum, which is equal to the Quantity $\frac{1}{1+x} \frac{-1}{n}$,
or to the Fraction $\frac{1}{1+x} \frac{1}{n}$.*

Art. 24. Since the sign $+$ is to be prefixed to Ex^4 , the fifth term of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$ is multiplied above in art. 14, it follows that the fifth horizontal row of terms in the compound series S , which is the product of that multiplication, will be $+ Ex^4 + bEx^5 - cEx^6 + dEx^7 - \&c$, in which the signs prefixed to the several terms are, respectively, the same as those which are prefixed to the corresponding terms of the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, which is multiplied into the said term $+ Ex^4$; agreeably to the rules

rules of Algebräick multiplication. Therefore the compound series S, obtained by the said multiplication, will, when the proper signs are prefixed to the terms of it's five first horizontal rows, be as follows, to wit,

$$\left\{ \begin{array}{l}
 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c \\
 - Bx - bBx^2 + cBx^3 - dBx^4 + eBx^5 - fBx^6 + gBx^7 - \&c \\
 + Cx^2 + bCx^3 - cCx^4 + dCx^5 - eCx^6 + fCx^7 - \&c \\
 - Dx^3 - bDx^4 + cDx^5 - dDx^6 + eDx^7 - \&c \\
 + Ex^4 + bEx^5 - cEx^6 + dEx^7 - \&c \\
 \quad Fx^5, \quad bFx^6, \quad cFx^7, \quad \&c \\
 \quad \quad Gx^6, \quad bGx^7, \quad \&c \\
 \quad \quad \quad Hx^7, \quad \&c \\
 \quad \quad \quad \quad \&c;
 \end{array} \right.$$

and consequently the last compound series obtained in art. 21, to wit, the compound series

$$\left\{ \begin{array}{l}
 - e + fx - gx^2 + bx^3 - \&c \\
 - dB + eBx - fBx^2 + gBx^3 - \&c \\
 - cC + dCx - eCx^2 + fCx^3 - \&c \\
 - bD + cDx - dDx^2 + eDx^3 - \&c \\
 \quad E, \quad bEx, \quad cEx^2, \quad dEx^3, \quad \&c \\
 \quad \quad Fx, \quad bFx^2, \quad cFx^3, \quad \&c \\
 \quad \quad \quad Gx^2, \quad bGx^3, \quad \&c \\
 \quad \quad \quad \quad Hx^3, \quad \&c \\
 \quad \quad \quad \quad \quad \&c
 \end{array} \right.$$

(which is derived from the compound series S by various operations which do not affect the signs that are to be prefixed to it's terms,) will be as follows, to wit,

$$\left\{ \begin{array}{l}
 - e + fx - gx^2 + bx^3 - \&c \\
 - dB + eBx - fBx^2 + gBx^3 - \&c \\
 - cC + dCx - eCx^2 + fCx^3 - \&c \\
 - bD + cDx - dDx^2 + eDx^3 - \&c \\
 + E + bEx - cEx^2 + dEx^3 - \&c \\
 \quad Fx, \quad bFx^2, \quad cFx^3 - \&c \\
 \quad \quad Gx^2, \quad bGx^3, \quad \&c \\
 \quad \quad \quad Hx^3, \quad \&c \\
 \quad \quad \quad \quad \&c.
 \end{array} \right.$$

Therefore

Therefore the last compound series will be equal to 0, or the sum of the several terms in it which are to be marked with the sign — will be equal to the sum of all the other terms in it, which are to be marked with the sign +.

But it has been shewn in art. 21 that the five terms — e — dB — cC — bD + E , which form the first vertical column of terms in this series, are equal to 0. Therefore, if those five terms are left out of this compound series, the remaining terms of it will also be equal to 0, that is, the compound series

$$\left\{ \begin{array}{l} + fx - gx^2 + bx^3 - \&c \\ + eBx - fBx^2 + gBx^3 - \&c \\ + dCx - eCx^2 + fCx^3 - \&c \\ + cDx - dDx^2 + eDx^3 - \&c \\ + bEx - cEx^2 + dEx^3 - \&c \\ \quad Fx, \quad bFx^2, \quad cFx^3, \quad \&c \\ \quad \quad Gx^2, \quad bGx^3, \quad \&c \\ \quad \quad \quad Hx^3, \quad \&c \\ \quad \quad \quad \quad \&c \end{array} \right.$$

will be equal to 0, or the sum of the several terms in it that are to be marked with the sign — will be equal to the sum of all the other terms of it, which are to be marked with the sign +. And this equality between these two sets of terms will still continue, when all the terms shall be divided by x . Therefore the compound series arising from such division, that is, the compound series

$$\left\{ \begin{array}{l} + f - gx + bx^2 - \&c \\ + eB - fBx + gBx^2 - \&c \\ + dC - eCx + fCx^2 - \&c \\ + cD - dDx + eDx^2 - \&c \\ + bE - cEx + dEx^2 - \&c \\ \quad F, \quad bFx, \quad cFx^2, \quad \&c \\ \quad \quad Gx, \quad bGx^2, \quad \&c \\ \quad \quad \quad Hx^2, \quad \&c \\ \quad \quad \quad \quad \&c \end{array} \right.$$

will be equal to 0, or the sum of the terms in it which are to be marked with the sign — will be equal to the sum of all the other terms of it, which are to be marked with the sign +. And this will always be true, of how small a magnitude soever we may suppose x to be taken: and therefore it will be true also when x is equal to 0. But, when x is equal to 0, all the terms in the said compound series which involve x will be equal to 0 likewise, and consequently the whole series will be reduced to the six terms contained in it's first vertical column,

column, to wit, the six terms $+f + eB + dC + cD + bE, F$. Therefore these six terms $+f + eB + dC + cD + bE, F$, taken together, will be equal to 0; and consequently, as the five first terms f, eB, dC, cD , and bE , are all marked with the sign $+$, the last term F must be marked with the sign $-$, or must be subtracted from the said five former terms, and must likewise be equal to the sum of the said five terms. Therefore the corresponding term, Fx^5 , in the compound series S (from which this last compound series is derived by various operations that do not affect the signs that are to be prefixed to it's terms,) must also have the sign $-$ prefixed to it: and so consequently (by art. 15,) must the same term Fx^5 in the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, by which the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - &c$ is multiplied in art. 14, and which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$. Therefore the

six first terms of the said series will be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5$, or $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}xx - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4 - Fx^5$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bxx - \frac{2n+1}{3n}Cx^3 + \frac{3n+1}{4n}Dx^4 - Fx^5$. Q. E. I.

Art. 25. To determine the magnitude of the co-efficient F , we must resolve the simple equation $F = f + eB + dC + cD + bE$; which may be done as follows:

Since b is $= \frac{1}{n}$, and c is $= \frac{1}{n} \times \frac{n-1}{2n}$, and d is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}$, and e is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n}$, and f is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}$, and B is $= \frac{1}{n}$, and C is $= \frac{1}{n} \times \frac{n+1}{2n}$, and D is $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$, and E is $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}$, it follows

In the 1st place, that f (which is $= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{4n-1}{5n}$), will be $(= \frac{n-1 \times 2n-1}{2n \times 3n^2} \times \frac{3n-1 \times 4n-1}{4n \times 5n} = \frac{2nn-3n+1}{6n^3} \times \frac{12nn-7n+1}{20nn}) = \frac{24n^4-50n^3+35nn-10n+1}{120n^5}$;

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And,

And, 2ndly, that eB will be $(= e \times \frac{1}{n} = \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times \frac{1}{n} = \frac{2nn-3n+1 \times 3n-1}{24n^5} = \frac{6n^3-11nn+6n-1}{24n^5} = \frac{5 \times 6n^3-11nn+6n-1}{5 \times 24n^5}) = \frac{30n^3-55nn+30n-5}{120n^5}$;

And, 3dly, that dC will be $(= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{1}{n} \times \frac{n+1}{2n} = \frac{2nn-3n+1 \times n+1}{12n^5} = \frac{2n^3-nn-2n+1}{12n^5}) = \frac{20n^3-10nn-20n+10}{120n^5}$;

And, 4thly, that cD will be $(= \frac{1}{n} \times \frac{n-1}{2n} \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} = \frac{nn-1 \times 2n+1}{12n^5} = \frac{2n^3+nn-2n-1}{12n^5}) = \frac{20n^3+10nn-20n-10}{120n^5}$;

And, 5thly, that bE will be $(= \frac{1}{n} \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} = \frac{2nn+3n+1 \times 3n+1}{24n^5} = \frac{6n^3+11nn+6n+1}{24n^5}) = \frac{30n^3+55nn+30n+5}{120n^5}$.

Therefore $f + eB + dC + bE$ will be $= \frac{24n^4-50n^3+35nn-10n+1}{120n^5} + \frac{30n^3-55nn+30n-5}{120n^5} + \frac{20n^3-10nn-20n+10}{120n^5} + \frac{20n^3+10nn-20n-10}{120n^5} + \frac{30n^3+55nn+30n+5}{120n^5} = \frac{24n^4+50n^3+35nn+10n+1}{120n^5} = \frac{4n+1 \times 6n^3+11nn+6n+1}{120n^5} = \frac{4n+1 \times 3n+1 \times 2n+1 \times n+1}{120n^5} = \frac{4n+1 \times 3n+1 \times 2n+1 \times n+1}{5n \times 4n \times 3n \times 2n \times n} = \frac{4n+1}{5n} \times \frac{3n+1}{4n} \times \frac{2n+1}{3n} \times \frac{n+1}{2n} \times \frac{1}{n} = \frac{4n+1}{5n} \times E.$

And consequently the co-efficient F (which is equal to $f + eB + dC + cD + bE$) will be $= \frac{4n+1}{5n} \times \frac{3n+1}{4n} \times \frac{2n+1}{3n} \times \frac{n+1}{2n} \times \frac{1}{n}$, or $\frac{4n+1}{5n} \times E.$ Q. E. I.

Therefore the fix first terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the quantity $1 + x^{\frac{-1}{n}}$, or to the fraction $\frac{1}{1+x^{\frac{1}{n}}}$, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3$

+

$$\frac{1}{1} \times \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} x^4 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n} x^5, \text{ or } 1 - \frac{1}{n} A x + \frac{n+1}{2n} B x^2 - \left[\frac{2n+1}{3n} C x^3 + \frac{3n+1}{4n} D x^4 - \frac{4n+1}{5n} E x^5. \right. \\ \left. Q. E. I. \right]$$

Art. 26. It is evident that, if we were to investigate the values of the following co-efficients G, H, I, K, L, &c of the terms of the series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - \&c$, (which

is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$) in the same manner as we

have done those of B, C, D, E, and F, the labour of making the necessary substitutions of the values of the former co-efficients B, C, D, E, F, G, H, I, K, L, &c that had been already discovered, and of the co-efficients $b, c, d, e, f, g, h, i, k, l$, &c of the former series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5$

$- gx^6 + hx^7 - ix^8 + kx^9 - lx^{10} + \&c$, (which is equal to $\frac{1}{1+x} \frac{1}{n}$),

in the simple equations by the resolution of which the values of the new co-efficients would be obtained, would soon become so great as to make this method of obtaining those co-efficients almost impracticable. I therefore shall content myself with having investigated in this manner the five co-efficients B, C, D, E, and F.

Art. 27. But, though it will not be convenient to investigate any more of the co-efficients of the terms of the series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6,$

Hx^7 , &c (which is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$), by this

second method founded on the multiplication of the series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - \&c$ into the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, on account of the labour of resolving the simple equations by means of which the values of the said co-efficients may be discovered; yet it will be worth our while to employ our attention upon the said method a little longer, in order to deduce from it a general demonstration that, to whatever number of terms the said series may be continued, the remaining terms will all be marked. (like the five terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5$, already investigated,) with the signs + and - in alternate succession, or that the said series will be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + \&c$ ad infinitum.

Art. 28. Now, in order to demonstrate this alternate succession of the signs + and — in the remaining terms of this series, it will be proper to set down the compound series S obtained above in art. 14, with the proper signs + and — prefixed to the several terms of it's first six horizontal rows of terms; which signs have been determined by the foregoing investigations. Now this may be done in the manner following:

The Multiplication of the Series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \text{\&c}$ ad infinitum into the Series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5, Gx^6, Hx^7, \text{\&c}$ ad infinitum, with the proper Signs + and — prefixed to the 2nd, 3d, 4th, 5th, and 6th Terms of the Second, or, Multiplying Series.

$$\begin{array}{r}
 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \text{\&c} \\
 1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5, \quad Gx^6, \quad Hx^7, \quad \text{\&c} \\
 \hline
 1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \text{\&c} \\
 - Bx - bBx^2 + cBx^3 - dBx^4 + eBx^5 - fBx^6 + gBx^7 - \text{\&c} \\
 + Cx^2 + bCx^3 - cCx^4 + dCx^5 - eCx^6 + fCx^7 - \text{\&c} \\
 - Dx^3 - bDx^4 + cDx^5 - dDx^6 + eDx^7 - \text{\&c} \\
 + Ex^4 + bEx^5 - cEx^6 + dEx^7 - \text{\&c} \\
 - Fx^5 - bFx^6 + cFx^7 - \text{\&c} \\
 \quad Gx^6, \quad bGx^7, \quad \text{\&c} \\
 \quad Hx^7, \quad \text{\&c} \\
 \quad \text{\&c.}
 \end{array}$$

Here we have the compound series S with the proper signs + and — prefixed to all the terms of the first six horizontal rows of terms. And from this exhibition of it we may deduce a proof that Gx^6 , or the first term of the seventh horizontal row of terms, must have the sign + prefixed to it, and that Hx^7 , or the first term of the eighth horizontal row of terms, must have the sign — prefixed to it, and that Ix^8 , Kx^9 , Lx^{10} , Mx^{11} , Nx^{12} , Ox^{13} , Px^{14} , Qx^{15} , &c *ad infinitum*, or the first terms of the 9th, 10th, 11th, 12th, 13th, 14th, 15th, 16th, and other following horizontal rows of terms in the said compound series, to whatever number of horizontal rows of terms the said series may be continued, will be marked with the sign + and the sign — alternately.

A Demonstration of the Alternate Succession of the Signs + and — to each other in the following Terms $Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$ ad infinitum of the said Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, or $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, \&c$, which is equal to $\frac{1}{1+x\frac{-1}{n}}$, or to the Fraction

$$\frac{1}{1+x\frac{-1}{n}}.$$

Art. 29. For, in the first place, it appears from the compound series S, as exhibited in the preceeding art. 28, that all the six terms that stand immediately above the term Gx^6 , or the first term of the seventh horizontal row of terms, to wit, the six terms $-gx^6 - fBx^6 - eCx^6 - dDx^6 - cEx^6 - bFx^6$, are marked with the sign $-$. But, by reasonings exactly similar to those by which we have investigated the signs to be prefixed to the former terms Bx, Cx^2, Dx^3, Ex^4 , and Fx^5 , and the values of the co-efficients B, C, D, E, and F, that are involved in them, it may be shewn that all the terms in the said vertical column of terms that involve x^6 , including the lowest term Gx^6 , must be equal to 0. Therefore the said lowest term Gx^6 must be marked with the contrary sign to that of all the terms that stand above it, and must be equal to the sum of all the said terms; that is, the said term Gx^6 must be marked with the sign $+$, and must be equal to the sum of all the terms that stand above it, or to $gx^6 + fBx^6 + eCx^6 + dDx^6 + cEx^6 + bFx^6$. Q. E. D.

And, 2ndly, since Gx^6 , the first term of the seventh horizontal row of terms, is marked with the sign $+$, it follows from what is shewn above in art. 15, that the same term Gx^6 in the multiplier, or simple series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5, Gx^6, Hx^7, \&c$, in art. 28, must likewise be marked with the same sign $+$. Therefore the seventh horizontal row of terms in the said compound series S, (which is the product of the multiplication of the multiplicand, or simple series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, into the term $+ Gx^6$), will, (according to the known rules of Algebraick multiplication,) have the same signs $+$ and $-$ prefixed to it's several terms as are prefixed to the corresponding terms of the multiplicand, or simple series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, respectively, and consequently will be as follows, to wit, $+ Gx^6 + bGx^7 - cGx^8 + dGx^9 - eGx^{10} + fGx^{11} - gGx^{12} + \&c$, in which the second term bGx^7 has the sign $+$ prefixed to it, as well as the first term Gx^6 . But all the terms in the same vertical column with the term bGx^7 , that stand above it, to wit, the six terms $+ bx^7 + gBx^7 + fCx^7 + eDx^7 + dEx^7 + cFx^7$, have the sign $+$ prefixed to them. Therefore all the seven terms that are

are placed immediately above Hx^7 , or the first term of the eighth horizontal row of terms, to wit, the seven terms $+bx^7 + gBx^7 + fCx^7 + eDx^7 + dEx^7 + cFx^7 + bGx^7$, will have the sign $+$ prefixed to them. But, by reasonings exactly similar to those by which we have investigated the signs to be prefixed to the former terms Bx , Cx^2 , Dx^3 , Ex^4 , and Fx^5 , it may be shewn that all the terms in the said vertical column of terms involving x^7 , including the lowest term Hx^7 , must be equal to 0. Therefore the said lowest term Hx^7 must be marked with the contrary sign to that of all the seven terms in the said vertical column that stand above it, and must be equal to the sum of all the said terms; that is, the said term Hx^7 must be marked with the sign $-$, and must be equal to $bx^7 + gBx^7 + fCx^7 + eDx^7 + dEx^7 + cFx^7 + bGx^7$.
Q. E. D.

And, 3dly, since the first term Hx^7 of the eighth horizontal row of terms in the said compound series S is to be marked with the sign $-$, it follows by art. 15 that the same term Hx^7 in the multiplicator, or simple series $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7$, &c, or $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7$, &c, in the multiplication exhibited in art. 28, must likewise be marked with the sign $-$, and consequently the first eight terms of the said multiplying series will be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7$. Therefore the eighth horizontal row of terms in the said compound series S , (which is the product of the multiplication of the multiplicand, or simple series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$ into the term $-Hx^7$;) will, (according to the known rules of Algebraick multiplication,) have the contrary signs $+$ and $-$ prefixed to it's several terms to those which are prefixed to the corresponding terms of the multiplicand, or simple series $1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$, respectively, and consequently will be as follows, to wit, $-Hx^7 - bHx^8 + cHx^9 - dHx^{10} + eHx^{11} - fHx^{12} + gHx^{13} - bHx^{14} + \&c$, in which the second term bHx^8 has the sign $-$ prefixed to it. But all the seven terms involving x^8 , that would be placed immediately above the term $-bHx^8$, if the multiplication exhibited in art. 28 had been carried to a greater number of terms, must evidently be marked with the same sign $-$; because all the seven terms of the next preceeding vertical column of terms, that stand in the same first seven horizontal rows of terms, immediately above the term $-Hx^7$, to wit, the terms $+bx^7 + gBx^7 + fCx^7 + eDx^7 + dEx^7 + cFx^7 + bGx^7$, are marked with the sign $+$, and the signs prefixed to the terms of every horizontal row of terms are, if we begin from the second term of every such row, alternately $+$ and $-$. Therefore in the vertical column of terms involving x^8 , of which the lowest term will be Ix^8 , the eight terms placed immediately above the said lowest term, (of which eight terms the lowest will be $-bHx^8$;) will all be marked with the same sign $-$. But, by reasonings exactly similar to those by which we have investigated the signs to be prefixed to the former terms Bx , Cx^2 , Dx^3 , Ex^4 , and Fx^5 , it may be shewn that all the terms in the said vertical column of terms involving x^8 , including the lowest term Ix^8 , must be equal to 0. Therefore the said lowest term

term must be marked with the contrary sign to that of all the eight terms in the same vertical column that stand above it, and must be equal to the sum of all the said terms; that is, the said lowest term $I x^8$ must be marked with the sign +, and must be equal to $i x^8 + b B x^8 + g C x^8 + f D x^8 + e E x^8 + d F x^8 + c G x^8 + b H x^8$. Q. E. D.

And, 4thly, in like manner it may be shewn in general that all the first terms $K x^9, L x^{10}, M x^{11}, N x^{12}, O x^{13}, P x^{14}, Q x^{15}$, &c of the following horizontal rows of terms in the compound series S , will be marked with the sign — and the sign + alternately, or will be $-K x^9 + L x^{10} - M x^{11} + N x^{12} - O x^{13} + P x^{14} - Q x^{15}$, &c, to whatever number of horizontal rows of terms the said compound series may be continued. Q. E. D.

For, whenever we have ascertained the sign that is to be prefixed to the first term of a new horizontal row of terms, which we will call the m th row, and that sign has been found to be contrary to the sign that was prefixed to the first term of the preceeding, or $m - 1$ th, horizontal row of terms, (as we have found to be the case in all the terms $B x, C x^2, D x^3, E x^4, F x^5, G x^6, H x^7$, and $I x^8$, which are marked with the sign — and the sign + alternately,) it will follow that the sign to be prefixed to the first term of the $m + 1$ th horizontal row of terms will be contrary to that which is prefixed to the first term of the m th horizontal row of terms, and consequently the same with that which is prefixed to the first term of the $m - 1$ th horizontal row of terms; as may be shewn in the following manner.

When the proper signs are prefixed to the terms of the m th horizontal row of terms, the second term of it will be marked with the same sign as the first term of it, and consequently with the contrary sign to that of the first term of the $m - 1$ th horizontal row. But in the $m - 1$ th horizontal row of terms the second term is marked with the same sign as the first term. Therefore the second term of the m th horizontal row of terms will be marked with the contrary sign to that of the second term of the $m - 1$ th horizontal row of terms. But the sign of the third term of every horizontal row of terms is contrary to the sign of its second term. Therefore the sign of the second term of the m th horizontal row of terms, (which has been shewn to be contrary to that of the second term of the $m - 1$ th horizontal row of terms, will be the same with that of the third term of the said $m - 1$ th horizontal row of terms. But the said second term of the m th horizontal row of terms stands in the same vertical column with, and immediately under, the said third term of the $m - 1$ th horizontal row of terms: and all the terms of the said vertical column with the said third term of the $m - 1$ th horizontal row of terms, and which stand above it, are marked with the same sign as the said third term. Therefore the said second term of the m th horizontal row of terms will be marked with the same sign as the

the said third term of the $\overline{m-1}$ th horizontal row of terms, and as all the terms in the same vertical column that stand above the said third term. But the said second term of the m th horizontal row of terms and the said third term of the $\overline{m-1}$ th horizontal row of terms are in the same vertical column with, and stand above the first term of the next, or $\overline{m+1}$ th, horizontal row of terms. Therefore all the terms that stand in the same vertical column with, and directly above, the first term of the next, or $\overline{m+1}$ th horizontal row of terms will be marked with the same sign, which will be the same as that of the first term of the m th horizontal row of terms. But all the terms that stand in the same vertical column with the first term of the $\overline{m+1}$ th horizontal row of terms, including the said first term, (which is the lowest term of the said vertical column,) must be equal to 0, for the reasons alledged in the foregoing investigations. Therefore the sign to be prefixed to the last, or lowest term of the said vertical column, or to the first term of the $\overline{m+1}$ th horizontal row of terms, must be contrary to the sign that is prefixed to all the other terms in the same vertical column, that stand directly above it, and consequently to the sign of the first term of the foregoing, or m th, horizontal row of terms.

Q. E. D.

And for the same reason the sign to be prefixed to the first term of the $\overline{m+2}$ th horizontal row of terms must be contrary to the sign that is to be prefixed to the $\overline{m+1}$ th horizontal row of terms, and the sign to be prefixed to the $\overline{m+3}$ th horizontal row of terms must be contrary to the sign that is to be prefixed to the $\overline{m+2}$ th horizontal row of terms; and so on *ad infinitum*. And consequently the terms $Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13}, Px^{14}, Qx^{15}$, &c *ad infinitum* will be $-Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + \&c$ *ad infinitum*, or will be marked with the sign $-$ and the sign $+$ alternately, to whatever number the horizontal rows of terms, in the compound series S, of which the said terms are the first terms, may be continued.

Q. E. D.

Art. 30. But, by art. 15, the signs $+$ and $-$ that are to be prefixed to the first terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13}, Px^{14}, Qx^{15}$, &c of the several horizontal rows of terms contained in the compound series S, are the same with those which are to be prefixed to the same terms in the multiplicator, or simple series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, in the operation of multiplication exhibited in art. 14 and art. 28. Therefore the signs $-$ and $+$ must be prefixed alternately to the second and other following terms, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, of the said series; and consequently the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, Ix^8, Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13},$

Px^{14}, Qx^{15} , &c (which is equal to $1 + x \frac{-1}{n}$, or to the fraction $\frac{1}{1+x \frac{1}{n}}$),

will,

will, when the proper signs + and — are prefixed to it's several terms, be as follows, to wit, $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} +$ &c *ad infinitum*, with the signs + and — succeeding each other alternately, to whatever number of terms the said series may be continued. Q. E. D.

Art. 31. We have now, I think, sufficiently considered the two first methods of investigating the terms of the infinite series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$

$Gx^6, Hx^7, &c$, which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{1+x} \frac{1}{n}$; both of which methods are derived from a knowledge of the series

$1 + bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - &c \text{ ad infinitum}$, or

$1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2 + \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \frac{6n-1}{7n} Gx^7 - &c \text{ ad infinitum}$, which is equal to $\frac{1}{1+x} \frac{-1}{n}$.

These two methods of investigating this series are, as I conceive, the easiest and most satisfactory methods that can be taken for the purpose. But it may, nevertheless, be agreeable to the reader to see a third method of investigating it, which does not require the previous knowledge of the series $1 + bx - cx^2$

$+ dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - &c$, or $1 + \frac{1}{n} Ax - \frac{n-1}{2n} Bx^2$

$+ \frac{2n-1}{3n} Cx^3 - \frac{3n-1}{4n} Dx^4 + \frac{4n-1}{5n} Ex^5 - \frac{5n-1}{6n} Fx^6 + \frac{6n-1}{7n} Gx^7 -$

&c, (which is equal to $\frac{1}{1+x} \frac{1}{n}$), but is derived from Sir Isaac Newton's

Theorem concerning the powers of a residual quantity in the more simple case of that theorem, when the index of the power is a whole number. This third method of obtaining the value of the said series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5,$

$Gx^6, Hx^7, &c$, which is equal to the quantity $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction

$\frac{1}{1+x} \frac{1}{n}$, may be explained in the following manner:

A Third Method of Investigating the Terms of the Infinite Series $1, Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the Quantity $\sqrt[n]{1+x}^{-1}$, or to the Fraction $\frac{1}{1+x)^{\frac{1}{n}}}$.

Art. 32. It is shewn above in art. 3 that the quantity $\sqrt[n]{1+x}^{-1}$, or the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$, will be equal to an infinite series of terms of which the first term will be 1, and the second and other following terms will involve in them the several powers of x in their natural order, without interruption, to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, combined with, or multiplied into, certain fixt numeral co-efficients, which may be conveniently denoted by the capital letters $B, C, D, E, F, G, H, \&c$; and that the whole of the said series must always be less than 1, or it's first term, and consequently that it's second term, Bx , must be subtracted from it's first term 1, and therefore have the sign — prefixed to it; and therefore that the said infinite series will be $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ *ad infinitum*. This therefore is the series in which we are now to endeavour to find the values of all the terms, except the first term 1, and to determine the signs + and — that are to be prefixed to all the terms after the second term Bx , to which we already know that the sign — is to be prefixed.

Art. 33. Now, since the whole of the said series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ is less than it's first term 1, it may be subtracted from the said first term. Let it be so subtracted. And the remainder will be the series $+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, in which the signs which ought to be prefixed to the terms $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, will be respectively contrary to those which ought to be prefixed to the same terms in the original series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$.

Let this remainder, or new series $+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ *ad infinitum*, be called y .

Then will $1 - y$ be equal to the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, and consequently to the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$. Therefore the n th power

of

of the residual quantity $1 - y$ will be equal to the n th power of the fraction

$$\frac{1}{1+x} \frac{1}{n}, \text{ that is, to the fraction } \frac{1}{1+x \frac{1}{n}}^n, \text{ or to the fraction } \frac{1}{1+x}.$$

But, by the residual theorem in the case of integral powers, the n th power of $1-y$ is equal to the series $1 - \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}yy - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c$; and the fraction $\frac{1}{1+x}$ is equal to the infinite series $1 - x + x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum*.

Therefore the series $1 - \frac{n}{1}y + \frac{n}{1} \times \frac{n-1}{2}y^2 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}y^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}y^4 - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}y^5 + \&c$ will be equal to the infinite series $1 - x + xx - x^3 + x^4 - x^5 + \&c$.

Let us now, for the sake of brevity, substitute b instead of $\frac{n}{1}$, and c instead of $\frac{n}{1} \times \frac{n-1}{2}$, and d instead of $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, and $e, f, g, h, \&c$ instead of $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, and $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, and the following co-efficients of y^6 and y^7 , &c, respectively, in the former of these two series. And the said series will thereby be converted into the series $1 - by + cy^2 - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + \&c$. And consequently we shall have the said series $1 - by + cy^2 - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + \&c$ equal to the infinite series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum*.

Art. 34. Further, since y is equal to the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, it follows that $yy, y^3, y^4, y^5, y^6, y^7, \&c$ will be equal to the square and cube, and fourth power, and fifth, sixth, seventh, and other following powers of the said series. Now these powers of the said series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ may be obtained by multiplying it repeatedly into itself in the manner following:

*Multiplications of the Series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$
into itself.*

$$+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c = y.$$

$$+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c = y.$$

$$\begin{array}{rcl}
 + B^2x^2, BCx^3, BDx^4, BEx^5, BFx^6, BGx^7, & \&c \\
 BCx^3, C^2x^4, CDx^5, CEx^6, CFx^7, & \&c \\
 BDx^4, CDx^5, D^2x^6, DEx^7, & \&c \\
 BEx^5, CEx^6, DEx^7, & \&c \\
 BFx^6, CFx^7, & \&c \\
 BGx^7, & \&c
 \end{array}$$

$$\left. \begin{array}{rcl}
 + B^2x^2, 2BCx^3, 2BDx^4, 2BEx^5, 2BFx^6, 2BGx^7, & \&c \\
 C^2x^4, 2CDx^5, 2CEx^6, 2CFx^7, & \&c \\
 D^2x^6, 2DEx^7, & \&c
 \end{array} \right\} = yy.$$

$$+ Bx, Cx^2, Dx^3, Ex^4, Fx^5, \&c = y.$$

$$\begin{array}{rcl}
 + B^3x^3, 2B^2Cx^4, 2B^2Dx^5, 2B^2Ex^6, 2B^2Fx^7, & \&c \\
 BC^2x^5, 2BCDx^6, 2BCEx^7, & \&c \\
 BD^2x^7, & \&c \\
 B^2Cx^4, 2BC^2x^5, 2BCDx^6, 2BCEx^7, & \&c \\
 C^3x^6, 2C^2Dx^7, & \&c \\
 B^2Dx^5, 2BCDx^6, 2BD^2x^7, & \&c \\
 C^2Dx^7, & \&c \\
 B^2Ex^6, 2BCEx^7, & \&c \\
 B^2Fx^7, & \&c
 \end{array}$$

$$\left\{ \begin{array}{rcl}
 + B^3x^3, 3B^2Cx^4, 3B^2Dx^5, 3B^2Ex^6, 3B^2Fx^7, & \&c \\
 3BC^2x^5, 6BCDx^6, 6BCEx^7, & \&c \\
 C^3x^6, 3BD^2x^7, & \&c \\
 3C^2Dx^7, & \&c
 \end{array} \right\} = y^3.$$

$$+ Bx,$$

$$+ Bx, Cx^2, Dx^3, Ex^4, \&c$$

$$\begin{array}{rcl}
 + B^4x^4, & 3B^3Cx^5, & 3B^3Dx^6, & 3B^3Ex^7, & \&c \\
 & & 3B^3C^2x^6, & 6B^3CDx^7, & \&c \\
 & & & BC^3x^7, & \&c \\
 & B^3Cx^5, & 3B^3C^2x^6, & 3B^3CDx^7, & \&c \\
 & & & 3BC^3x^7, & \&c \\
 & & B^3Dx^6, & 3B^3CDx^7, & \&c \\
 & & & B^3Ex^7, & \&c
 \end{array}$$

$$\begin{array}{rcl}
 + B^4x^4, & 4B^3Cx^5, & 4B^3Dx^6, & 4B^3Ex^7, & \&c \\
 & & 6B^3C^2x^6, & 12B^3CDx^7, & \&c \\
 & & & 4BC^3x^7, & \&c
 \end{array} \Bigg\} = y^4.$$

$$+ Bx, Cx^2, Dx^3, \&c$$

$$\begin{array}{rcl}
 + B^5x^5, & 4B^4Cx^6, & 4B^4Dx^7, & \&c \\
 & & 6B^4C^2x^7, & \&c \\
 & B^4Cx^6, & 4B^4C^2x^7, & \&c \\
 & & B^4Dx^7, & \&c
 \end{array}$$

$$\begin{array}{rcl}
 + B^5x^5, & 5B^4Cx^6, & 5B^4Dx^7, & \&c \\
 & & 10B^4C^2x^7, & \&c
 \end{array} \Bigg\} = y^5$$

$$+ Bx, Cx^2, \&c$$

$$\begin{array}{rcl}
 + B^6x^6, & 5B^5Cx^7, & \&c \\
 & B^5C^2x^7, & \&c
 \end{array}$$

$$+ B^6x^6, 6B^5Cx^7, \&c = y^6$$

$$+ Bx, \&c$$

$$+ B^7x^7, \&c = y^7.$$

Art. 35. Having thus found the values of y^2, y^3, y^4, y^5, y^6 , and y^7 in terms involving the powers of x combined with the numeral co-efficients B, C, D, E, F, G, H , &c, let the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c and it's several powers here computed, which are equal to y^2, y^3, y^4, y^5, y^6 , and y^7 , be substituted instead of y and it's said six following powers, in the equation $1 - by + cy^2 - dy^3 + ey^4 - fy^5 + gy^6 - hy^7, \&c = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum*, obtained above in art. 33. And we shall thereby obtain an equation between the compound series into which the simple series $1 - by + cy^2 - dy^3 + ey^4 - fy^5 + gy^6 - hy^7 + \&c$ will be converted by such substitution, and the said simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum*; which equation will be as follows; to wit, the compound series

$$\begin{aligned}
 & \left[\begin{aligned}
 & -bBx, \quad bCx^2, \quad bDx^3, \quad bEx^4, \quad bFx^5, \quad bGx^6, \quad bHx^7, \quad \&c \\
 & + cB^2x^2, \quad 2cBCx^3, \quad 2cBDx^4, \quad 2cBEx^5, \quad 2cBFx^6, \quad 2cBGx^7, \quad \&c \\
 & \quad \quad \quad cC^2x^4, \quad 2cCDx^5, \quad 2cCEx^6, \quad 2cCFx^7, \quad \&c \\
 & \quad \quad \quad \quad \quad cD^2x^6, \quad 2cDEx^7, \quad \&c \\
 & - dB^3x^3, \quad 3dB^2Cx^4, \quad 3dB^2Dx^5, \quad 3dB^2Ex^6, \quad 3dB^2Fx^7, \quad \&c \\
 & \quad \quad \quad 3dBC^2x^5, \quad 6dBCDx^6, \quad 6dBCEx^7, \quad \&c \\
 & \quad \quad \quad \quad \quad dC^3x^6, \quad 3dBD^2x^7, \quad \&c \\
 & \quad \quad \quad \quad \quad \quad \quad 3dC^2Dx^7, \quad \&c \\
 & + eB^4x^4, \quad 4eB^3Cx^5, \quad 4eB^3Dx^6, \quad 4eB^3Ex^7, \quad \&c \\
 & \quad \quad \quad 6eB^3C^2x^6, \quad 12eB^3CDx^7, \quad \&c \\
 & \quad \quad \quad \quad \quad 4eBC^3x^7, \quad \&c \\
 & - fB^5x^5, \quad 5fB^4Cx^6, \quad 5fB^4Dx^7, \quad \&c \\
 & \quad \quad \quad 10fB^3C^2x^7, \quad \&c \\
 & + gB^6x^6, \quad 6gB^5Cx^7, \quad \&c \\
 & - hB^7x^7, \quad \&c \\
 & \quad \quad \quad + \&c
 \end{aligned} \right.
 \end{aligned}$$

= the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum*. And from this equation we may, by comparing together the homologous terms, or terms involving the same powers of x , on the opposite sides of it, derive, one after another, the values of the several numeral co-efficients B, C, D, E, F, G, H , &c of the powers of x in the series $1 - Bx, Cx^2, Dx^3,$

Ex^4, Fx^5, Gx^6, Hx^7 , &c (which is equal to $1 + x \sqrt[n]{-1}$, or to the fraction $\frac{1}{1 + x \sqrt[n]{-1}}$), and determine the signs $+$ and $-$ with which the several terms $Cx^2,$

$Dx^3,$

$Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, of the said series (which follow the second term Bx , which we already know is to have the sign — prefixed to it,) are to be marked. This may be done in the manner following :

The Investigation of the Value of the Second Term, Bx , of the Series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, which is equal to the Quantity

$$\frac{1}{1 + a\sqrt[n]{x}}, \text{ or to the Fraction } \frac{1}{1 + x\sqrt[n]{x}}.$$

Art. 36. In these two serieses the first terms are the same, to wit, 1. And the latter series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum* is evidently less than it's first term 1. For, since x is supposed to be less than 1, the several powers of it, to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, x^8$, &c, will form a decreasing progression; and therefore the odd powers of x , to wit, x, x^3, x^5, x^7 , &c, which in this series are all marked with the sign —, or are subtracted from the first term 1, will be greater than the next following even powers of it, to wit, x^2, x^4, x^6, x^8 , &c, which in this series are marked with the sign +, or are added to the first term 1. Therefore every two terms of the series, after the first term 1, (as, for example, the two terms $-x + x^2$, and the two terms $-x^3 + x^4$, and the two terms $-x^5 + x^6$, and the two terms $-x^7 + x^8$,) will somewhat diminish the magnitude of the first term 1; and consequently the whole of the said series will be less than it's first term 1. Therefore the whole of the said series may be subtracted from it's first term 1. Let it be so subtracted. And the remainder will be the series $x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + \&c$ *ad infinitum*.

Further, since the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum* is less than it's first term 1, and the said simple series is equal to the compound series, set down in art. 35, it follows that the said compound series will also be less than 1, which is also it's own first term. Therefore the said compound series may be subtracted from 1, or from it's own first term. Let it be so subtracted. And then the remainder of such subtraction will be equal to the remainder of the foregoing subtraction, or to the series $+x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - x^8 + \&c$ *ad infinitum*; that is, the compound series

$$+ bBx,$$

$$\begin{aligned}
 & \left[\begin{aligned}
 & + bBx, bCx^2, bDx^3, bEx^4, bFx^5, bGx^6, bHx^7, \&c \\
 & - cB^2x^2, 2cBCx^3, 2cBDx^4, 2cBEx^5, 2cBFx^6, 2cBGx^7, \&c \\
 & \quad cC^2x^4, 2cCDx^5, 2cCEx^6, 2cCFx^7, \&c \\
 & \quad \quad cD^2x^6, 2cDEx^7, \&c \\
 & + dB^3x^3, 3dB^2Cx^4, 3dB^2Dx^5, 3dB^2Ex^6, 3dB^2Fx^7, \&c \\
 & \quad 3dBC^2x^5, 6dBCDx^6, 6dBCEx^7, \&c \\
 & \quad \quad dC^3x^6, 3dBD^2x^7, \&c \\
 & \quad \quad \quad 3dC^2Dx^7, \&c \\
 & - eB^4x^4, 4eB^3Cx^5, 4eB^3Dx^6, 4eB^3Ex^7, \&c \\
 & \quad 6eB^2C^2x^6, 12eB^2CDx^7, \&c \\
 & \quad \quad 4eBC^3x^7, \&c \\
 & + fB^5x^5, 5fB^4Cx^6, 5fB^4Dx^7, \&c \\
 & \quad 10fB^3C^2x^7, \&c \\
 & - gB^6x^6, 6gB^5Cx^7, \&c \\
 & + bB^7x^7, \&c \\
 & - \&c
 \end{aligned} \right.
 \end{aligned}$$

will be equal to the simple series $x - x^2 + x^3 - x^4 + x^5 - x^6 + x^7 - \&c$
ad infinitum.

Now let all the terms of this equation be divided by x .

And we shall then have the following equation, to wit, the compound series

$$\begin{aligned}
 & \left[\begin{aligned}
 & + bB, bCx, bDx^2, bEx^3, bFx^4, bGx^5, bHx^6, \&c \\
 & - cB^2x, 2cBCx^2, 2cBDx^3, 2cBEx^4, 2cBFx^5, 2cBGx^6, \&c \\
 & \quad cC^2x^3, 2cCDx^4, 2cCEx^5, 2cCFx^6, \&c \\
 & \quad \quad cD^2x^5, 2cDEx^6, \&c \\
 & + dB^3x^2, 3dB^2Cx^3, 3dB^2Dx^4, 3dB^2Ex^5, 3dB^2Fx^6, \&c \\
 & \quad 3dBC^2x^4, 6dBCDx^5, 6dBCEx^6, \&c \\
 & \quad \quad dC^3x^5, 3dBD^2x^6, \&c \\
 & \quad \quad \quad 3dC^2Dx^6, \&c \\
 & - eB^4x^3, 4eB^3Cx^4, 4eB^3Dx^5, 4eB^3Ex^6, \&c \\
 & \quad 6eB^2C^2x^5, 12eB^2CDx^6, \&c \\
 & \quad \quad 4eBC^3x^6, \&c \\
 & + fB^5x^4, 5fB^4Cx^5, 5fB^4Dx^6, \&c \\
 & \quad 10fB^3C^2x^6, \&c \\
 & - gB^6x^5, 6gB^5Cx^6, \&c \\
 & + bB^7x^6, \&c \\
 & - \&c
 \end{aligned} \right.
 \end{aligned}$$

= the

= the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \&c$ *ad infinitum*.

Now this equation will be always true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true, when x is equal to 0. But, when x is equal to 0, all the terms of the said compound series that involve x , that is, all it's terms except it's first term $+ bB$, will become equal to 0 likewise; and, in like manner, all the terms of the said simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \&c$ that involve x , that is, all it's terms except the first term 1, will also become equal

to 0. Therefore bB will be = 1; that is, $\frac{n}{1} \times B$ will be = 1. And

consequently B will be equal to $\frac{1}{n}$, as it was found to be by the two former

methods of investigation grounded on the division of 1 by the series $1 + \frac{1}{n}x - \frac{1}{n} \times \frac{n-1}{2n}x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n}x^3 - \&c$ (which is equal to

$\sqrt[n]{1+x}$), and on the multiplication of the same series into the series 1, Bx ,

Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , $\&c$, which is equal to $\sqrt[n]{1+x}^{-1}$, or to

the fraction $\frac{1}{\sqrt[n]{1+x}}$. Therefore the two first terms of the series $1 - Bx$,

Cx^2 , Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , $\&c$, which is equal to $\sqrt[n]{1+x}^{-1}$, or to the fraction $\frac{1}{\sqrt[n]{1+x}}$, will be $1 - \frac{1}{n}x$. Q. E. I.

The Investigation of the Value of the Third Term, Cx^2 , of the Series $1 - Bx$, Cx^2 ,

Dx^3 , Ex^4 , Fx^5 , Gx^6 , Hx^7 , $\&c$, which is equal to the Quantity $\sqrt[n]{1+x}^{-1}$,

or to the Fraction $\frac{1}{\sqrt[n]{1+x}}$, and of the Sign that is to be prefixed to it.

Art. 37. Since bB , or the first term of the compound series set down in the foregoing article, is equal to 1, or the first term of the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \&c$ *ad infinitum*; and the said simple series is

equal to the said compound series; and the whole of the said simple series is less than its first term 1; it follows that the whole of the said compound series will be less than its first term bB . Therefore it is possible to subtract each of these two series from its first term. Let these two series be so subtracted, that is, each of them from its own first term. And the remainders thence arising will be equal to each other; that is, the compound series

$$\begin{array}{r}
 \left\{ \begin{array}{l}
 bCx, \quad bDx^2, \quad bEx^3, \quad bFx^4, \quad bGx^5, \quad bHx^6, \text{ \&c} \\
 + \quad cB^2x, \quad 2cBCx^2, \quad 2cBDx^3, \quad 2cBEx^4, \quad 2cBFx^5, \quad 2cBGx^6, \text{ \&c} \\
 \quad \quad \quad cC^2x^3, \quad 2cCDx^4, \quad 2cCEx^5, \quad 2cCFx^6, \text{ \&c} \\
 \quad \quad \quad \quad cD^2x^5, \quad 2cDEx^6, \text{ \&c} \\
 - \quad dB^3x^2, \quad 3dB^2Cx^3, \quad 3dB^2Dx^4, \quad 3dB^2Ex^5, \quad 3dB^2Fx^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad 3dBC^2x^4, \quad 6dBCDx^5, \quad 6dBCEx^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad \quad dC^3x^5, \quad 3dBD^2x^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad \quad \quad 3dC^2Dx^6, \text{ \&c} \\
 + \quad eB^4x^3, \quad 4eB^3Cx^4, \quad 4eB^3Dx^5, \quad 4eB^3Ex^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad 6eB^2C^2x^5, \quad 12eB^2CDx^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad \quad 4eBC^3x^6, \text{ \&c} \\
 - \quad fB^5x^4, \quad 5fB^4Cx^5, \quad 5fB^4Dx^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad \quad 10fB^3C^2x^6, \text{ \&c} \\
 + \quad gB^6x^5, \quad 6gB^5Cx^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad - \quad hB^7x^6, \text{ \&c} \\
 \quad \quad \quad \quad \quad \quad + \text{ \&c}
 \end{array} \right.
 \end{array}$$

will be equal to the simple series $x - x^2 + x^3 - x^4 + x^5 - x^6 + \text{ \&c}$ *ad infinitum*.

Now let all the terms of this equation be divided by x .

And we shall then have the compound series

$$\begin{aligned}
 & bC, \quad bDx, \quad bEx^2, \quad bFx^3, \quad bGx^4, \quad bHx^5, \quad \&c \\
 + & cB^2, \quad 2cBCx, \quad 2cBDx^2, \quad 2cBEx^3, \quad 2cBFx^4, \quad 2cBGx^5, \quad \&c \\
 & \quad \quad cC^2x^2, \quad 2cCDx^3, \quad 2cCEx^4, \quad 2cCFx^5, \quad \&c \\
 & \quad \quad \quad cD^2x^4, \quad 2cDEx^5, \quad \&c \\
 - & dB^3x, \quad 3dB^2Cx^2, \quad 3dB^2Dx^3, \quad 3dB^2Ex^4, \quad 3dB^2Fx^5, \quad \&c \\
 & \quad \quad \quad 3dBC^2x^3, \quad 6dBCDx^4, \quad 6dBCEx^5, \quad \&c \\
 & \quad \quad \quad \quad dC^3x^4, \quad 3dBD^2x^5, \quad \&c \\
 & \quad \quad \quad \quad \quad 3dC^2Dx^5, \quad \&c \\
 + & eB^4x^2, \quad 4eB^3Cx^3, \quad 4eB^3Dx^4, \quad 4eB^3Ex^5, \quad \&c \\
 & \quad \quad \quad 6eB^2C^2x^4, \quad 12eB^2CDx^5, \quad \&c \\
 & \quad \quad \quad \quad 4eBC^3x^5, \quad \&c \\
 - & fB^5x^3, \quad 5fB^4Cx^4, \quad 5fB^4Dx^5, \quad \&c \\
 & \quad \quad \quad 10fB^3C^2x^5, \quad \&c \\
 + & gB^6x^4, \quad 6gB^5Cx^5, \quad \&c \\
 - & \quad \quad bB^7x^5, \quad \&c \\
 & \quad \quad \quad + \&c
 \end{aligned}$$

equal to the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum*. And this equation will be always true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true, when x is equal to 0. But, when x is equal to 0, all the terms of this last compound series that involve x , that is, all it's terms except the two terms bC and $+ cB^2$, will become equal to 0 likewise; and, in like manner, all the terms of the said simple series $1 - x + x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum* that involve x , that is, all it's terms except the first term 1, will be equal to 0 likewise. Therefore the two terms bC and $+ cB^2$ will, together,

be = 1. But, by art. 33, b is = $\frac{n}{1}$, and c is = $\frac{n}{1} \times \frac{n-1}{2}$. Therefore

bC will be = $\frac{n}{1} \times C$, and $+ cB^2$ will be = $+ \frac{n}{1} \times \frac{n-1}{2} \times B^2$, and

consequently $\frac{n}{1} \times C$ and $+ \frac{n}{1} \times \frac{n-1}{2} \times B^2$ will, together, be = 1.

But B has been already found to be = $\frac{1}{n}$. Therefore B^2 will be = $\frac{1}{n} \times$

$\frac{1}{n}$, and consequently $\frac{n}{1} \times \frac{n-1}{2} \times B^2$ will be ($= \frac{n}{1} \times \frac{n-1}{2} \times \frac{1}{n} \times$

$\frac{1}{n} = \frac{n-1}{2} \times \frac{1}{n}$) = $\frac{n-1}{2n}$. Therefore $\frac{n}{1} \times C$ and $+ \frac{n-1}{2n}$ will, to-

gether, be = 1; that is, $+ \frac{n-1}{2n}$ together with $\frac{n}{1} \times C$, either added to

it, or subtracted from it, as shall be found necessary, will be equal to 1. But $\frac{n-1}{2n}$ is evidently less than 1. Therefore $\frac{n}{1} \times C$ must be added to $\frac{n-1}{2n}$, and not subtracted from it, in order to make it equal to 1; and consequently the sign + must be prefixed to $\frac{n}{1} \times C$, or to the term bC in the last compound series. And the magnitude of C will be determined by resolving the simple equation $+ bC + cB^2 = 1$, or $\frac{n}{1} \times C + \frac{n-1}{2n} = 1$; which may be performed as follows:

$$\begin{aligned} \text{Since } \frac{n}{1} \times C + \frac{n-1}{2n} &= 1, \text{ we shall have } C + \frac{n-1}{2nn} = \frac{1}{n}, \text{ and } C \\ &= \frac{1}{n} - \frac{n-1}{2nn} = \frac{2n}{2nn} - \frac{n-1}{2nn} = \frac{2n - (n-1)}{2nn} = \frac{2n - n + 1}{2nn} = \frac{n+1}{2nn} = \frac{1}{n} \\ &\times \frac{n+1}{2n}. \quad \text{Q. E. I.} \end{aligned}$$

Art. 38. Since the sign + is to be prefixed to the term bC in the last compound series set down in the foregoing art. 37, it must likewise be prefixed to the term bCx in the next preceding compound series, from which the last compound series is derived by dividing all its terms by x . But the terms of this series must be marked with contrary signs to those with which they are to be marked in the foregoing compound series, which is the last compound series set down in art. 36, because it is derived from that last compound series in art. 36 by subtracting all the terms of it from its first term $+ bB$, which will occasion a change of the signs of all the terms that are subtracted. Therefore the sign that is to be prefixed to the term bCx in the last compound series set down in art. 36 will be the sign —. And consequently the sign that is to be prefixed to the term $bCxx$ in the preceding compound series set down in art. 36 (from which the last compound series in that article is derived by dividing all its terms by x), will also be the sign —. But this compound series is derived from the compound series set down in art. 35 (of which 1 is the first term,) by subtracting the whole of the said series from its first term 1, which occasions a change of the signs of all the terms subtracted. Therefore the sign that is to be prefixed to the term $bCxx$ in that compound series, set down in art. 35, will be the sign +, and consequently the two first terms of the second horizontal row of terms in the said compound series will be $-bBx + bCx^2$. But this second horizontal row of terms in the said compound series is obtained by subtracting the value of by , to wit, the series $bBx, bCx^2, bDx^3, bEx^4, bFx^5, bGx^6, bHx^7$, &c, or $b \times$ the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7$, &c, from 1; which occasions a change of the signs in all the terms of the series subtracted. Therefore the sign that is to be prefixed to the term bCx^2 in the subtracted series $bBx, bCx^2, bDx^3, bEx^4, bFx^5, bGx^6, bHx^7$, &c, which

which is equal to by , will be the sign $-$. And consequently the sign that is to be prefixed to the term Cx^2 in the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to y , will also be the sign $-$. Therefore the sign that is to be prefixed to the same term Cx^2 in the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, (from which the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, or $Bx - Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, is derived by the subtraction of the whole series from it's first term 1, which occasions a change in the signs of all the terms subtracted,) will be the contrary sign $+$, and consequently the three first terms of the series $1 - Bx, Cx^2, Dx^3, Ex^4,$

$Fx^5, Gx^6, Hx^7, \&c$, which is equal to $1 + x \sqrt[n]{\frac{-1}{n}}$, or to the fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$,

will be $1 - Bx + Cx^2$, or $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}x^2$, or $1 - \frac{1}{n}Ax$

$+ \frac{n+1}{2n}Bx^2$.

Q. E. I.

The Investigation of the Value of the Fourth Term, Dx^3 , of the Series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the Quantity

$\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$, or to the Fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$, and of the Sign that is to be prefixed to it.

Art. 39. Since the sign $+$ is to be prefixed to the third term Cx^2 of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ (which is equal to

$\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$, or to the fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$), or the three first terms of the said

series are $1 - Bx + Cx^2$, it follows that, when the whole of the said series is subtracted from it's first term 1, and the remainder of the said subtraction, to wit, the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, is called y , the second term Cx^2 of the said remainder must have the sign $-$ prefixed to it, that is, the two first terms of the series which is equal to y will be $Bx - Cx^2$. Therefore the two first terms of the series which is equal to y^2 , will be $+ Bx^2 - 2BCx^3$. Therefore the two first terms of the series which is equal to by will be $+ bBx - bCx^2$, and the two first terms of the series which is equal to Cy^2 will be $+ cB^2x^2 - 2cBCx^3$. Therefore, when the series which is equal to by is subtracted from 1, and the series which is equal to Cy^2 is added to it, the first terms of the compound series resulting from these operations will be

I * *
 $- b B x + b C x^2$
 $+ c B^2 x^2 - 2c B C x^3$; that is, the four terms $b B x$, $b C x^2$,
 $c B^2 x^2$ and $2c B C x^3$ in the first compound series set down in art. 35 will be

$- b B x + b C x^2$
 $+ c B^2 x^2 - 2c B C x^3$. Therefore in the compound series
 set down in art. 36, (which is derived from the former compound series by
 subtracting the whole series from the first term 1,) the same terms will have
 contrary signs to those they had before, and will be as follows, to wit,

$+ b B x - b C x^2$
 $- c B^2 x^2 + 2c B C x^3$. Therefore the corresponding
 terms to these in the next compound series set down in art. 36, (which is de-
 rived from the former, or second, compound series by only dividing all its
 terms by x ,) will be $+ b B - b C x$

$- c B^2 x + 2c B C x^2$. Therefore in the next,
 or fourth, compound series, (which is set down in art. 37, and is derived from
 the last, or third, compound series by subtracting the whole of the said series
 from it's first term $+ b B$,) the three terms $b C x$, $c B^2 x$, and $2c B C$ will have
 contrary signs to those they had before, and will be $+ b C x$

$+ c B^2 x - 2c B C x^2$.
 Therefore in the next, or fifth, compound series (which is derived from the
 last, or fourth, compound series by only dividing all it's terms by x ,) the
 corresponding terms to these three terms will be marked with the same signs
 that they are marked with, and will be $+ b C$
 $+ c B^2 - 2c B C x$.

But the term $d B^3 x$ in this fifth compound series is also marked with the
 sign $-$. Therefore the three terms that form the second vertical column of
 this fifth compound series, and that involve the simple power of x , will be
 $b D x - 2c B C x - d B^3 x$, of which the two latter are marked with the
 sign $-$, and the sign to be prefixed to the first term $b D x$ remains still to be
 determined.

Art. 40. It has been shewn in art. 37 that $+ b C + c B^2$ in the fifth com-
 pound series, or the last compound series set down in that article, is equal to 1,
 which is the first term of the simple series $1 - x + x^2 - x^3 + x^4 - x^5$
 $+ \&c$ *ad infinitum*, to which the said compound series is equal. But the
 whole of the said simple series is less than it's first term 1. Therefore the
 whole of the said compound series will be less than the binomial quantity
 $+ b C + c B^2$ which is equal to 1. And consequently the whole of the said
 compound series may be subtracted from the said binomial quantity $+ b C$
 $+ c B^2$.

Now let the whole of the said compound series be subtracted from the said
 binomial quantity $+ b C + c B^2$; and let the whole simple series $1 - x$

$+ x^2 - x^3 + x^4 - x^5 + \&c$ *ad infinitum*, (which is equal to the said compound series,) be subtracted from it's first term 1, which is equal to the said binomial quantity $+ bC + cB^2$. And the remainders of these subtractions will be equal to each other; that is, the compound series

$$\left[\begin{array}{llllll}
 bDx, & bEx^2, & bFx^3, & bGx^4, & bHx^5, & \&c \\
 + 2cBCx, & 2cBDx^2, & 2cBE x^3, & 2cBF x^4, & 2cBG x^5, & \&c \\
 & cC^2x^2, & 2cCDx^3, & 2cCE x^4, & 2cCF x^5, & \&c \\
 & & cD^2x^4, & 2cDE x^5, & \&c \\
 + dB^3x, & 3dB^2Cx^2, & 3dB^2Dx^3, & 3dB^2Ex^4, & 3dB^2Fx^5, & \&c \\
 & & 3dBC^2x^3, & 6dBCDx^4, & 6dBCE x^5, & \&c \\
 & & & dC^3x^4, & 3dBD^2x^5, & \&c \\
 & & & & 3dC^2Dx^5, & \&c \\
 - eB^4x^2, & 4eB^3Cx^3, & 4eB^3Dx^4, & 4eB^3Ex^5, & \&c \\
 & & 6eB^3C^2x^4, & 12eB^2CDx^5, & \&c \\
 & & & 4eBC^3x^5, & \&c \\
 + fB^5x^3, & 5fB^4Cx^4, & 5fB^4Dx^5, & \&c \\
 & & 10fB^3C^2x^5, & \&c \\
 - gB^6x^4, & 6gB^5Cx^5, & \&c \\
 + bB^7x^5, & \&c \\
 - \&c
 \end{array} \right]$$

will be equal to the simple series $x - x^2 + x^3 - x^4 + x^5 - \&c$ *ad infinitum*.

Now let all the terms of this equation be divided by x .

And we shall then have the compound series

$bD,$

$$\begin{aligned}
 & \left. \begin{aligned}
 & bD, \quad bEx, \quad bFx^2, \quad bGx^3, \quad bHx^4, \quad \&c \\
 + & 2cBC, \quad 2cBDx, \quad 2cBE x^2, \quad 2cBF x^3, \quad 2cBG x^4, \quad \&c \\
 & \quad \quad cC^2x, \quad 2cCDx^2, \quad 2cCE x^3, \quad 2cCF x^4, \quad \&c \\
 & \quad \quad \quad cD^2x^2, \quad 2cDE x^3, \quad \&c \\
 + & dB^3, \quad 3dB^2Cx, \quad 3dB^2Dx^2, \quad 3dB^2Ex^3, \quad 3dB^2Fx^4, \quad \&c \\
 & \quad \quad \quad 3dB^2C^2x^2, \quad 6dBCDx^3, \quad 6dBCE x^4, \quad \&c \\
 & \quad \quad \quad \quad dC^3x^3, \quad 3dB^2D^2x^4, \quad \&c \\
 & \quad \quad \quad \quad \quad 3dC^2Dx^4, \quad \&c \\
 - & eB^4x, \quad 4eB^3Cx^2, \quad 4eB^3Dx^3, \quad 4eB^3Ex^4, \quad \&c \\
 & \quad \quad \quad 6eB^2C^2x^3, \quad 12eB^2CDx^4, \quad \&c \\
 & \quad \quad \quad \quad 4eBC^3x^4, \quad \&c \\
 + & fB^5x^2, \quad 5fB^4Cx^3, \quad 5fB^4Dx^4, \quad \&c \\
 & \quad \quad \quad 10fB^3C^2x^4, \quad \&c \\
 - & gB^6x^3, \quad 6gB^5Cx^4, \quad \&c \\
 + & hB^7x^4, \quad \&c \\
 - & \&c
 \end{aligned} \right\}
 \end{aligned}$$

= to the simple series $1 - x + x^2 - x^3 + x^4 - \&c$ *ad infinitum*. And this equation will be always true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true when x is equal to 0. But, when x is equal to 0, all the terms of this last compound series that involve x , that is, all it's terms except the three terms $bD + 2cBC + dB^3$, will become equal to 0 likewise; and, in like manner, all the terms of the said simple series $1 - x + x^2 - x^3 + x^4 - \&c$ *ad infinitum* that involve x , that is, all it's terms except the first term 1, will be equal to 0 likewise. Therefore the three terms $bD + 2cBC + dB^3$, (of which the first term bD has no sign prefixed to it, because it has not yet been determined whether it is to be added to, or to be subtracted from, the sum of the two other terms $+ 2cBC + dB^3$;) will be equal to 1.

But, by art. 33, b is $= \frac{n}{1}$, and c is $= \frac{n}{1} \times \frac{n-1}{2}$, and d is $= \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$.

Therefore bD will be $= \frac{n}{1} \times D$, and $2cBC$ will be $(= \frac{n}{1} \times \overline{n-1} \times BC) = \overline{nn-n} \times BC$, and dB^3 will be $(= \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times B^3) = \frac{n^3-3nn+2n}{6} \times B^3$, and consequently the trinomial quantity $bD + 2cBC + dB^3$ will be $= nD + \overline{nn-n} \times BC + \frac{n^3-3nn+2n}{6} \times B^3$.

But

But it has been shewn that B is $= \frac{1}{n}$, and that C is $= \frac{1}{n} \times \frac{n+1}{2n}$.

Therefore $\overline{nn-n} \times BC$ will be $(= \overline{nn-n}) \times \frac{1}{n} \times \frac{1}{n} \times \frac{n+1}{2n} = \frac{n^3-n}{2n^3} = \frac{3n^3-3n}{6n^3}$, and $\frac{n^3-3nn+2n}{6} \times B^3$ will be $(= \frac{n^3-3nn+2n}{6} \times \frac{1}{n^3}) = \frac{n^3-3nn+2n}{6n^3}$, and consequently $\overline{nn-n} \times BC + \frac{n^3-3nn+2n}{6} \times B^3$ will be $(= \frac{3n^3-3n}{6n^3} + \frac{n^3-3nn+2n}{6n^3} = \frac{4n^3-3nn-n}{6n^3}) = \frac{4nn-3n-1}{6nn}$.

Therefore the trinomial quantity $bD + 2cBC + dB^3$ will be $=$ to $bD + \frac{4nn-3n-1}{6nn}$, or (because b is equal to n) will be $= nD + \frac{4nn-3n-1}{6nn}$.

But the trinomial quantity $bD + 2cBC + dB^3$ is $= 1$.

Therefore $nD + \frac{4nn-3n-1}{6nn}$ will also be $= 1$; that is, the fraction $\frac{4nn-3n-1}{6nn}$ together with the quantity nD , either added to it or subtracted from it, as shall be found necessary, will be equal to 1 .

But the fraction $\frac{4nn-3n-1}{6nn}$ is evidently less than the fraction $\frac{4nn}{6nn}$, and consequently, *a fortiori*, is less than the fraction $\frac{6nn}{6nn}$, which is greater than $\frac{4nn}{6nn}$, and therefore must be less than 1 , which is equal to $\frac{6nn}{6nn}$.

Therefore, in order to make the two terms nD and $\frac{4nn-3n-1}{6nn}$ be equal to 1 , the first term nD must be added to the second term $\frac{4nn-3n-1}{6nn}$, and consequently must have the sign $+$ prefixed to it. Therefore we shall have $+ nD + \frac{4nn-3n-1}{6nn} = 1$, and consequently (subtracting $\frac{4nn-3n-1}{6nn}$ from both sides,) $+ nD = 1 - \frac{4nn-3n-1}{6nn} (= \frac{6nn}{6nn} - \frac{4nn-3n-1}{6nn} = \frac{6nn-4nn+3n+1}{6nn}) = \frac{2nn+3n+1}{6nn}$; and $+ D = \frac{2nn+3n+1}{6n^3} = \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$. Therefore the fourth term Dx^3 of the infinite series $1 - Bx,$

$Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the quantity $\overline{1+x}^{\frac{-1}{n}}$, or to the fraction $\frac{1}{\overline{1+x}^{\frac{1}{n}}}$, will be $\frac{2nn+3n+1}{6n^3} \times x^3$, or $\frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times x^3$, or $\frac{2n+1}{3n} Cx^3$. Q. E. I.

Art. 41. It remains that we determine the sign $+$ or $-$ that is to be prefixed to the said fourth term Dx^3 , or $\frac{2n+1}{3n} Cx^3$, of the said infinite series $1 - Bx + Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, which is equal to the quantity $\overline{1+x}^{\frac{-1}{n}}$, or the fraction $\frac{1}{\overline{1+x}^{\frac{1}{n}}}$. This may be determined in the manner following:

We have seen that the sign $+$ is to be prefixed to the term nD , or bD , in the last compound series set down in the preceeding, or 40th, article. Therefore in the compound series next before the last, and from which the last is derived by only dividing all it's terms by x , the corresponding term bDx must have the same sign $+$ prefixed to it. Therefore in the next preceeding compound series, or the fifth compound series, or the last compound series that is set down in art. 37, (from which the sixth compound series is derived by the subtraction of all it's terms from the two terms $+bC + cB^2$;) the same term bDx must have the contrary sign $-$ prefixed to it; and so must the corresponding term bDx^2 in the next preceeding, or fourth, compound series set down in the beginning of art. 37, from which the fifth compound series was derived by only dividing all the terms by x . Therefore in the third compound series, (which is set down in art. 36, and from which the fourth compound series that is set down in the beginning of art. 37 is derived by subtracting all it's terms from it's first term bB ;) the corresponding term bDx^2 must be marked with the contrary sign $+$, and consequently the corresponding term bDx^3 in the next preceeding, or the second compound series, (from which the third compound series is derived by only dividing all it's terms by x) will also be marked with the sign $+$. Therefore the same term bDx^3 in the first compound series, (which is set down in art. 35, and from which the second compound series is derived by subtracting all it's terms from it's first term 1 ;) must have the contrary sign $-$ prefixed to it, as well as the term bBx , which is the first term of the horizontal row of which the term bDx^3 is the third. Therefore in the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ (which is equal to y ;) the term Dx^3 must have the contrary sign $+$ prefixed to it; because the said horizontal row of terms in which Bx and Dx^3 occur, represents the product of the multiplication of the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ into b subtracted from 1 ; or the first term of the said first compound series, which subtraction produces a change in the signs of all the quantities subtracted.

Lastly,

Lastly, since the sign $+$ is to be prefixed to the term Dx^3 in the series $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, or since the three first terms of the said series are $+ Bx - Cx^2 + Dx^3$, and the said series is equal to y , or to the remainder that arises by subtracting the whole first series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ from it's first term 1 , it follows that in the said first series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ the said term Dx^3 must have the contrary sign $-$ prefixed to it; that is, the fourth term Dx^3 of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ which is

equal to the quantity $1 + x \sqrt[n]{\frac{-1}{n}}$, or the fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$, will be $- Dx^3$,
 or $-\frac{2n+1}{3n} \times Cx^3$. Q. E. I.

Art. 42. We may therefore now conclude with certainty from this third method of investigating the terms of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ *ad infinitum*, which is equal to the quantity $\sqrt[n]{1 + x \sqrt[n]{\frac{-1}{n}}}$,

or to the fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$, that the four first terms of it are $1 - \frac{1}{n}x + \frac{1}{n}$

$\times \frac{n+1}{2n}x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \frac{2n+1}{3n}Cx^3$, as they were found to be above by the two former methods of investigating them.

Art. 43. By reasonings similar to those which have been used in art. 35, 36, 37, 38, 39, 40, and 41, for determining the values of the co-efficients B, C , and D , of the second, third, and fourth terms of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ (which is equal to the quantity $\sqrt[n]{1 + x \sqrt[n]{\frac{-1}{n}}}$,

or to the fraction $\frac{1}{1 + x \sqrt[n]{\frac{-1}{n}}}$), we might determine successively the several values

of the following co-efficients $F, G, H, \&c$, and might likewise determine the signs which are to be prefixed to the terms $Ex^4, Fx^5, Gx^6, Hx^7, \&c$ in the said series which involve the said co-efficients. And it is evident that every new vertical column of terms in the first compound series set down above in art. 35 would be equal to the corresponding term, or term involving the same power of x , in the simple series $1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + \&c$ *ad infinitum*, to which the said compound series is equal; which would afford us a new simple equation for the determination of the value of the new co-efficient that is involved in the first, or highest, term of the said vertical column.

column. Thus, for the determination of the value of the co-efficient E we should have the equation $b E x^4, 2c B D x^4, c C^2 x^4, 3 d B^2 C x^4, e B^4 x^4 = + x^4$, or (dividing all the terms by x^4 .) $b E, 2c B D, c C^2, 3 d B^2 C + e B^4 = 1$, by the resolution of which (after prefixing the proper signs + and - to the several terms $2c B D, c C^2$, and $3 d B^2 C$.) we may obtain the value of the co-efficient E. And the three following vertical columns of terms in the same compound series will afford us the like simple equations for determining the values of the three following co-efficients F, G, and H. But the resolution of these simple equations would be found to be a matter of great intricacy and difficulty, as may easily be imagined from the difficulty we found in determining the value of the last co-efficient D of the fourth term, $D x^3$, of the said series $1 - Bx + Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$, and the sign - that is to be prefixed to it. And to resolve any more of these simple equations after those involving the co-efficients F, G, and H, in order to obtain the values of the following co-efficients I, K, L, M, N, O, P, Q, &c of the following terms $Ix^8, Kx^9, Lx^{10}, Mx^{11}, Nx^{12}, Ox^{13}, Px^{14}, Qx^{15}, \&c$ would be so excessively tedious and laborious that it may well be deemed impracticable. And, further, if these equations could be resolved, it would still be impossible, by means of the values of the co-efficients E, F, G, H, I, K, L, M, N, O, P, Q, &c obtained by means of such resolutions, to shew that the following co-efficients R, S, T, V, &c, of the remaining terms of the series which had not been investigated, would be generated, or derived, from the former co-efficients (that had been investigated,) and from each other by the same law of derivation, or continuation, that had been observed to take place in the co-efficients that had been investigated. We will therefore now abandon this very laborious and imperfect method of investigating the values of the co-efficients B, C, D, E, F, G, H, &c of the powers of x in the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ *ad infinitum* (which is equal to the quantity

$\frac{-1}{1+x} \frac{1}{n}$, or the fraction $\frac{1}{1+x} \frac{1}{n}$), and endeavour to find some other method

of discovering the values of the said co-efficients, by which it may be made to appear that the law of the generation, or derivation, of these co-efficients one from another, which is observed to take place in the co-efficients that are actually investigated, must also take place with respect to the remaining co-efficients which have not been investigated. And then we may justly reckon the investigation, or demonstration, of this case of the binomial theorem to be complete.

Art. 44. We will, however, observe, before we enter upon this new, or fourth, method of investigation, that it follows from the last, or third, method of investigation, as well as from the first and second methods, that the signs to be prefixed to the second and other following terms, $Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$ of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, \&c$

&c (which is equal to $\frac{-1}{1+x)^{\frac{1}{n}}}$, or to the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$), will be alter-

nately — and +, to whatever number of terms the said series may be continued. This conclusion follows from the last, or third, method of investigation, because the investigation of the value of every new co-efficient in that method is performed by means of a subtraction of all the terms of the compound series involving the said new co-efficient from some of the first terms of it, which subtraction causes a change in the signs of all the terms subtracted. But it follows still more clearly from the first method of investigation, as is shewn above in the beginning of this discourse in art. 7, 8, 9, 10, and 11. And therefore in the following, or fourth, method of investigating the terms of the series $1 - Bx, Cx^2, Dx^3, Ex^4, Fx^5, Gx^6, Hx^7, &c$, (which is equal to the quantity $\frac{-1}{1+x)^{\frac{1}{n}}}$, or to the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$), I shall suppose this alternate succession

of the signs — and + in the terms of the said series to take place, and the said series to be $1 - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + &c$ *ad infinitum*, or to whatever number of terms the said series may be continued.

Art. 45. And, further, in this fourth method of investigation I shall likewise suppose it to be known that B, or the co-efficient of Bx , the second term of the said series, is $= \frac{1}{n}$, as it has been already shewn to be in each of the three foregoing methods of investigation. In this fourth method of investigation it is therefore supposed that the quantity $\frac{-1}{1+x)^{\frac{1}{n}}}$, or the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$, is equal to the following series, to wit, $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + &c$ *ad infinitum*, and the only object of the investigation is to discover the values of the co-efficients C, D, E, F, G, H, &c of the third, fourth, fifth, and other following terms of this series, and the law of their generation, or derivation, one from the other, and to shew that the said law will take place with respect to all the co-efficients of the remaining terms of the said series, as well as to those co-efficients that shall have been actually investigated.

A Fourth Method of investigating the Third Term Cx^2 and the other following Terms, after the third Term, of the infinite Series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ ad infinitum, which is equal to the Quantity $\sqrt[n]{1+x}$, or to the Fraction $\frac{1}{1+x)^{\frac{1}{n}}}$; in which the Law of the Continuation of the Terms of the said Series, or of their Generation, or Derivation, one from the other, is ascertained, and shewn to extend to all the Terms of the Series, as well as to those that have been actually investigated.

Art. 46. In the expression $\sqrt[n]{1+x}$, or the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$, the quantity x is supposed to be less than 1; and it is supposed also that, if it be of any magnitude less than 1, the said fraction $\frac{1}{1+x)^{\frac{1}{n}}}$ will be equal to the infinite series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, in which the capital letters C, D, E, F, G, H, &c denote certain numeral co-efficients hitherto unknown.

Therefore, if we suppose x to denote some particular quantity less than 1, and y to denote another quantity greater than x , but yet less than 1, it will follow that the fraction $\frac{1}{1+y)^{\frac{1}{n}}}$ will be equal to the infinite series $1 - \frac{1}{n}y + Cy^2 - Dy^3 + Ey^4 - Fy^5 + Gy^6 - Hy^7 + \&c$ ad infinitum:

Let the excess of y above x be denoted by the letter d , so that $x + d$ shall be equal to y . And let $x + d$ be substituted instead of y in the last equation. And we shall then have a third equation, which will be as follows; to wit,

$\frac{1}{1+x+d)^{\frac{1}{n}}} =$ the infinite series $1 - \frac{1}{n} \times \sqrt[n]{x+d} + C \times \sqrt[n]{x+d}^2 - D \times \sqrt[n]{x+d}^3 + E \times \sqrt[n]{x+d}^4 - F \times \sqrt[n]{x+d}^5 + G \times \sqrt[n]{x+d}^6 - H \times \sqrt[n]{x+d}^7 + \&c$ ad infinitum, or (by expanding the square and other following powers of $x + d$), $\frac{1}{1+x+d)^{\frac{1}{n}}} =$ the series

$$\begin{aligned}
1 &= \frac{1}{n} \times \overline{x+d} + C \times \overline{x^2+2xd+d^2} \\
&- D \times \overline{x^3+3x^2d+3xd^2+d^3} + E \times \overline{x^4+4x^3d+6x^2d^2+4xd^3+d^4} \\
&- F \times \overline{x^5+5x^4d+10x^3d^2+10x^2d^3+5xd^4+d^5} \\
&+ G \times \overline{x^6+6x^5d+15x^4d^2+20x^3d^3+15x^2d^4+6xd^5+d^6} \\
&- H \times \overline{x^7+7x^6d+21x^5d^2+35x^4d^3+35x^3d^4+21x^2d^5+7xd^6+d^7} \\
&+ \&c = \text{the compound series}
\end{aligned}$$

$$\left\{ \begin{aligned}
1 &- \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c \\
&- \frac{1}{n}d + 2Cxd - 3Dx^2d + 4Ex^3d - 5Fx^4d + 6Gx^5d - 7Hx^6d + \&c \\
&\quad + Cd^2 - 3Dxd^2 + 6Ex^2d^2 - 10Fx^3d^2 + 15Gx^4d^2 - 21Hx^5d^2 + \&c \\
&\quad \quad - Dd^3 + 4Exd^3 - 10Fx^2d^3 + 20Gx^3d^3 - 35Hx^4d^3 + \&c \\
&\quad \quad \quad + Ed^4 - 5Fxd^4 + 15Gx^2d^4 - 35Hx^3d^4 + \&c \\
&\quad \quad \quad \quad - Fd^5 + 6Gxd^5 - 21Hx^2d^5 + \&c \\
&\quad \quad \quad \quad \quad + Gd^6 - 7Hxd^6 + \&c \\
&\quad \quad \quad \quad \quad \quad - Hd^7 + \&c \\
&\quad \quad \quad \quad \quad \quad \quad + \&c.
\end{aligned} \right.$$

Art. 47. Now let f be $= 1 + x$.

Then will $f + d$ be $= 1 + x + d$, and $\overline{f+d} \frac{1}{n}$ will be $= \overline{1+x+d} \frac{1}{n}$,

and $\frac{\frac{1}{f}}{\overline{f+d} \frac{1}{n}}$ will be $= \frac{\frac{1}{1+x+d}}{\frac{1}{n}}$.

But $f + d$ is $= f \times \left(1 + \frac{d}{f}\right)$, and consequently $\overline{f+d} \frac{1}{n}$ will be $= f \frac{1}{n} \times \overline{1 + \frac{d}{f}} \frac{1}{n}$, and the fraction $\frac{\frac{1}{f}}{\overline{f+d} \frac{1}{n}}$ will be $=$ the fraction $\frac{\frac{1}{f \frac{1}{n} \times \overline{1 + \frac{d}{f}} \frac{1}{n}}}{\frac{1}{f \frac{1}{n} \times \overline{1 + \frac{d}{f}} \frac{1}{n}}} =$ the fraction $\frac{\frac{1}{f \frac{1}{n}}}{\frac{1}{f \frac{1}{n}}} \times$ the fraction $\frac{\frac{1}{\overline{1 + \frac{d}{f}} \frac{1}{n}}}{\frac{1}{\overline{1 + \frac{d}{f}} \frac{1}{n}}}$.

Now, because y , or $x + d$, is less than 1, it follows that d (which is less than $x + d$) must also be less than 1, and consequently, *à fortiori*, must be less

less than $1 + x$, or f . Therefore the fraction $\frac{d}{f}$ will be less than 1, and the binomial quantity $1 + \frac{d}{f}$ will be similar to the binomial quantity $1 + x$ in having its second term less than its first term 1. Therefore, since the fraction

$\frac{1}{1+x}^{\frac{1}{n}}$ is equal to the infinite series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c \text{ ad infinitum}$, it follows that the fraction

$\frac{1}{1+\frac{d}{f}}^{\frac{1}{n}}$ will be equal to the similar series $1 - \frac{1}{n} \times \frac{d}{f} + C \times \left(\frac{d}{f}\right)^2 - D \times \left(\frac{d}{f}\right)^3 + E \times \left(\frac{d}{f}\right)^4 - F \times \left(\frac{d}{f}\right)^5 + G \times \left(\frac{d}{f}\right)^6 - H \times \left(\frac{d}{f}\right)^7 + \&c$.

Therefore the fraction $\frac{1}{f^n} \times$ the fraction $\frac{1}{1+\frac{d}{f}}^{\frac{1}{n}}$ will be equal to the

fraction $\frac{1}{f^n} \times$ the series $1 - \frac{1}{n} \times \frac{d}{f} + C \times \left(\frac{d}{f}\right)^2 - D \times \left(\frac{d}{f}\right)^3 + E \times \left(\frac{d}{f}\right)^4 - F \times \left(\frac{d}{f}\right)^5 + G \times \left(\frac{d}{f}\right)^6 - H \times \left(\frac{d}{f}\right)^7 + \&c \text{ ad infinitum}$, and

consequently to the series $\frac{1}{f^{\frac{1}{n}}} - \frac{1}{n} \times \frac{1}{f^{\frac{1}{n}}} \times \frac{d}{f} + C \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^2 - D \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^3 + E \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^4 - F \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^5 + G \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^6 - H \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^7 + \&c \text{ ad infinitum}$.

Therefore the fraction $\frac{1}{f+d}^{\frac{1}{n}}$ (which is equal to the fraction $\frac{1}{f^{\frac{1}{n}}} \times$ the fraction $\frac{1}{1+\frac{d}{f}}^{\frac{1}{n}}$) will be equal to the series $\frac{1}{f^{\frac{1}{n}}} - \frac{1}{n} \times \frac{1}{f^{\frac{1}{n}}} \times \frac{d}{f}$

$+ C \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^2 - D \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^3 + E \times \frac{1}{f^{\frac{1}{n}}} \times \left(\frac{d}{f}\right)^4 -$

$F \times$

$$F \times \frac{1}{f^{\frac{1}{n}}} \times \left[\frac{d}{f} \right]^5 + G \times \frac{1}{f^{\frac{1}{n}}} \times \left[\frac{d}{f} \right]^6 - H \times \frac{1}{f^{\frac{1}{n}}} \times \left[\frac{d}{f} \right]^7 + \&c$$

ad infinitum.

Therefore, if we substitute $1+x$ in this last equation instead of f , (to which it is equal,) we shall have the fraction $\frac{1}{1+x+d)^{\frac{1}{n}}}$ = the series

$$\frac{1}{1+x)^{\frac{1}{n}}} - \frac{1}{n} \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d}{1+x} + C \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^2 - D \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^3 + E \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^4 - F \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^5 + G \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^6 - H \times \frac{1}{1+x)^{\frac{1}{n}}} \times \left[\frac{d}{1+x} \right]^7 + \&c \text{ ad infinitum} =$$

the series $\frac{1}{1+x)^{\frac{1}{n}}} - \frac{1}{n} \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d}{1+x} + C \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^2}{1+x)^2} - D \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^3}{1+x)^3} + E \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^4}{1+x)^4} - F \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^5}{1+x)^5} + G \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^6}{1+x)^6} - H \times \frac{1}{1+x)^{\frac{1}{n}}} \times \frac{d^7}{1+x)^7} + \&c \text{ ad infinitum.}$

Art. 48. But it has been shewn that the fraction $\frac{1}{1+x+d)^{\frac{1}{n}}}$ is equal to the compound series

$$\left\{ \begin{array}{l} 1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c \\ - \frac{1}{n}d + 2Cxd - 3Dx^2d + 4Ex^3d - 5Fx^4d + 6Gx^5d - 7Hx^6d + \&c \\ + Cd^2 - 3Dxd^2 + 6Ex^2d^2 - 10Fx^3d^2 + 15Gx^4d^2 - 21Hx^5d^2 + \&c \\ - Dd^3 + 4Exd^3 - 10Fx^2d^3 + 20Gx^3d^3 - 35Hx^4d^3 + \&c \\ + Ed^4 - 5Fxd^4 + 15Gx^2d^4 - 35Hx^3d^4 + \&c \\ - Fd^5 + 6Gxd^5 - 21Hx^2d^5 + \&c \\ + Gd^6 - 7Hxd^6 + \&c \\ - Hd^7 + \&c \\ + \&c. \end{array} \right.$$

Therefore the simple series $\frac{1}{1+x} \times \frac{1}{n} - \frac{1}{n} \times \frac{1}{1+x} \times \frac{d}{1+x} + C \times$

$$\frac{1}{1+x} \times \frac{d^2}{1+x^2} - D \times \frac{1}{1+x} \times \frac{d^3}{1+x^3} + E \times \frac{1}{1+x} \times \frac{d^4}{1+x^4}$$

$$- F \times \frac{1}{1+x} \times \frac{d^5}{1+x^5} + G \times \frac{1}{1+x} \times \frac{d^6}{1+x^6} - H \times \frac{1}{1+x} \times \frac{d^7}{1+x^7} + \&c \text{ ad infinitum (which is equal to the fraction } \frac{1}{1+x+d} \times \frac{1}{n} \text{)}$$

will be equal to the said compound series.

Therefore, if we subtract the said simple series from it's first term $\frac{1}{1+x} \times \frac{1}{n}$, (than which it is evidently less,) and at the same time subtract the said compound series (which is equal to the said simple series) from the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, which forms the upper horizontal row of terms in the said compound series, and which is equal to $\frac{1}{1+x} \times \frac{1}{n}$, the remainders of these two subtractions will be

equal to each other; that is, the simple series $\frac{1}{n} \times \frac{1}{1+x} \times \frac{d}{1+x} - C$

$$\times \frac{1}{1+x} \times \frac{d^2}{1+x^2} + D \times \frac{1}{1+x} \times \frac{d^3}{1+x^3} - E \times \frac{1}{1+x} \times \frac{d^4}{1+x^4}$$

$$\frac{d^4}{(1+x)^4} + F \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^5}{(1+x)^5} - G \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^6}{(1+x)^6} + H \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^7}{(1+x)^7} - \&c \text{ ad infinitum}$$

will be equal to the compound series

$$\left\{ \begin{aligned} & + \frac{1}{n}d - 2Cxd + 3Dx^2d - 4Ex^3d + 5Fx^4d - 6Gx^5d + 7Hx^6d - \&c \\ & - Cd^2 + 3Dxd^2 - 6Ex^2d^2 + 10Fx^3d^2 - 15Gx^4d^2 + 21Hx^5d^2 - \&c \\ & + Dd^3 - 4Exd^3 + 10Fx^2d^3 - 20Gx^3d^3 + 35Hx^4d^3 - \&c \\ & - Ed^4 + 5Fxd^4 - 15Gx^2d^4 + 35Hx^3d^4 - \&c \\ & + Fd^5 - 6Gxd^5 + 21Hx^2d^5 - \&c \\ & - Gd^6 + 7Hxd^6 - \&c \\ & + Hd^7 - \&c \\ & - \&c. \end{aligned} \right.$$

Therefore, if we divide all the terms of this last equation by d , we shall have the simple series $\frac{1}{n} \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{1}{1+x} - C \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d}{(1+x)^2}$

$$+ D \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^2}{(1+x)^3} - E \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^3}{(1+x)^4} + F \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^4}{(1+x)^5} - G \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^5}{(1+x)^6} + H \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{d^6}{(1+x)^7} - \&c \text{ ad}$$

infinitum equal to the compound series

$$\left\{ \begin{aligned} & + \frac{1}{n} - 2Cx + 3Dx^2 - 4Ex^3 + 5Fx^4 - 6Gx^5 + 7Hx^6 - \&c \\ & - Cd + 3Dxd - 6Ex^2d + 10Fx^3d - 15Gx^4d + 21Hx^5d - \&c \\ & + Dd^2 - 4Exd^2 + 10Fx^2d^2 - 20Gx^3d^2 + 35Hx^4d^2 - \&c \\ & - Ed^3 + 5Fxd^3 - 15Gx^2d^3 + 35Hx^3d^3 - \&c \\ & + Fd^4 - 6Gxd^4 + 21Hx^2d^4 - \&c \\ & - Gd^5 + 7Hxd^5 - \&c \\ & + Hd^6 - \&c \\ & - \&c. \end{aligned} \right.$$

Art. 49. This equation is always true, of how small a magnitude soever we may suppose the quantity d to be taken: and therefore it will also be true when d is equal to 0. But, when d is equal to 0, all the terms in the equation that involve d will be equal to 0 likewise; and consequently the equation will then be

as follows, to wit, $\frac{1}{n} \times \frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{1}{1+x} =$ the simple series $\frac{1}{n} - 2Cx +$

$3Dx^2 - 4Ex^3 + 5Fx^4 - 6Gx^5 + 7Hx^6 - \&c$ *ad infinitum*. And, if we multiply all the terms of this equation by n , we shall have the equation

$\frac{1}{\frac{1}{1+x} \frac{1}{n}} \times \frac{1}{1+x} =$ the series $1 - 2nCx + 3nDx^2 - 4nEx^3 +$

$5nFx^4 - 6nGx^5 + 7nHx^6 - \&c$ *ad infinitum*. And, if we multiply all the terms of this last equation into $1+x$, we shall have the equation

$\frac{1}{\frac{1}{1+x} \frac{1}{n}} =$ the compound series

$$\left\{ \begin{array}{l} 1 - 2nCx + 3nDx^2 - 4nEx^3 + 5nFx^4 - 6nGx^5 + 7nHx^6 - \&c \\ + x - 2nCx^2 + 3nDx^3 - 4nEx^4 + 5nFx^5 - 6nGx^6 + \&c. \end{array} \right.$$

But $\frac{1}{\frac{1}{1+x} \frac{1}{n}}$ is equal to the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum*.

Therefore the said series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum* will be equal to the compound series

$$\left\{ \begin{array}{l} 1 - 2nCx + 3nDx^2 - 4nEx^3 + 5nFx^4 - 6nGx^5 + 7nHx^6 - \&c \\ + x - 2nCx^2 + 3nDx^3 - 4nEx^4 + 5nFx^5 - 6nGx^6 + \&c. \end{array} \right.$$

Now by the help of this equation we may determine the values of the several co-efficients C, D, E, F, G, H, &c *ad infinitum*, by proceeding in the manner following:

The Investigation of the Co-efficient C.

Art. 50. Since the simple series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ is equal to the compound series

$$\left\{ \begin{array}{l} 1 - 2nCx + 3nDx^2 - 4nEx^3 + 5nFx^4 - 6nGx^5 + 7nHx^6 - \&c \\ + x - 2nCx^2 + 3nDx^3 - 4nEx^4 + 5nFx^5 - 6nGx^6 + \&c, \end{array} \right.$$

and the former of these serieses is less than it's first term 1, it follows that the latter series must likewise be less than 1. Therefore each of these serieses may be subtracted from 1.

Let them be so subtracted.

Then will the remainder of the first subtraction be equal to the remainder of the second subtraction; that is, the simple series $\frac{1}{n}x - Cx^2 + Dx^3 - Ex^4 + Fx^5 - Gx^6 + Hx^7 - \&c$ will be equal to the compound series

$$\left\{ \begin{array}{l} + 2nCx - 3nDx^2 + 4nEx^3 - 5nFx^4 + 6nGx^5 - 7nHx^6 + \&c \\ - x + 2nCx^2 - 3nDx^3 + 4nEx^4 - 5nFx^5 + 6nGx^6 - \&c, \end{array} \right.$$

and consequently (dividing all the terms of this equation on both sides by x), the simple series $\frac{1}{n} - Cx + Dx^2 - Ex^3 + Fx^4 - Gx^5 + Hx^6 - \&c$ will be equal to the compound series

$$\left\{ \begin{array}{l} 2nC - 3nDx + 4nEx^2 - 5nFx^3 + 6nFx^4 - 7nGx^5 + \&c \\ - 1 + 2nCx - 3nDx^2 + 4nEx^3 - 5nFx^4 + 6nGx^5 - \&c. \end{array} \right.$$

Now this equation will always be true, of how small a magnitude soever we may suppose x to be taken: and therefore it will also be true when x is equal to 0. But, when x is equal to 0, the several terms in the foregoing equation that involve x will be equal to 0 likewise; and consequently the whole of the said simple series will be equal to it's first term $\frac{1}{n}$, and the whole of the said compound

compound series will be equal to the two terms $2nC - 1$. Therefore $\frac{1}{n}$ will be $= 2nC - 1$, and therefore (adding 1 to both sides,) we shall have $\frac{1}{n} + 1 = 2nC$, and $C = \frac{\frac{1}{n} + 1}{2n} (= \frac{1}{2nn} + \frac{1}{2n} = \frac{1}{2nn} + \frac{n}{2nn} = \frac{1+n}{2nn}) = \frac{1}{n} \times \frac{n+1}{2n}$. Q. E. I.

Therefore the three first terms of the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$, which is equal to $\overline{1+x}^{\frac{-1}{n}}$, or to the fraction $\frac{1}{1+x}^{\frac{1}{n}}$, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n}x^2$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2$. Q. E. I.

The Investigation of the Co-efficient D.

Art. 51. Since the simple series $\frac{1}{n} - Cx + Dx^2 - Ex^3 + Fx^4 - Gx^5 + Hx^6 - \&c$ has been shewn in the last article to be equal to the compound series

$$\begin{cases} 2nC - 3nDx + 4nEx^2 - 5nFx^3 + 6nGx^4 - 7nHx^5 + \&c \\ -1 + 2nCx - 3nDx^2 + 4nEx^3 - 5nFx^4 + 6nGx^5 - \&c, \end{cases}$$

and the whole of the said simple series is evidently less than it's first term $\frac{1}{n}$, because the second term Cx is marked with the sign $-$, it follows that the whole of the said compound series must also be less than $\frac{1}{n}$, and consequently than the two terms $2nC - 1$, which have been shewn to be equal to $\frac{1}{n}$. Therefore the whole of the said simple series may be subtracted from it's first term $\frac{1}{n}$, and the whole of the said compound series may be subtracted from the two terms $2nC - 1$.

Let them be so subtracted.

Then

Then will the remainder of the first subtraction be equal to the remainder of the second subtraction; that is, the simple series $+ Cx - Dx^2 + Ex^3 - Fx^4 + Gx^5 - Hx^6 + \&c$ will be equal to the compound series

$$\begin{cases} + 3nDx - 4nEx^2 + 5nFx^3 - 6nGx^4 + 7nHx^5 - \&c \\ - 2nC + 3nDx - 4nEx^2 + 5nFx^3 - 6nGx^4 + \&c. \end{cases}$$

Therefore, if we divide all the terms of this equation by x , we shall have the simple series $C - Dx + Ex^2 - Fx^3 + Gx^4 - Hx^5 + \&c =$ the compound series

$$\begin{cases} 3nD - 4nEx + 5nFx^2 - 6nGx^3 + 7nHx^4 - \&c \\ - 2nC + 3nDx - 4nEx^2 + 5nFx^3 - 6nGx^4 + \&c. \end{cases}$$

And this equation will always be true, of how small a magnitude soever we may suppose x to be taken: and therefore it will also be true when x is equal to 0. But, when x is equal to 0, all the terms that involve x will be equal to 0 likewise, and consequently the whole of the said simple series will be equal to it's first term C , and the whole of the said compound series will be equal to the two terms $3nD - 2nC$. Therefore we shall have $C = 3nD - 2nC$, and consequently $3nD = C + 2nC = (2n+1) \times C$, and $D = \frac{(2n+1) \times C}{3n} = \frac{2n+1}{3n} \times \frac{n+1}{2n} \times \frac{1}{n}$. Q. E. I.

Therefore the four first terms of the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum*, which is equal to $\frac{1}{1+x} \frac{-1}{n}$, or to the fraction $\frac{1}{(1+x) \frac{1}{n}}$, will be $1 - \frac{1}{n}x + \frac{1}{n} \times \frac{n+1}{2n} \times x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} x^3$, or $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \frac{2n+1}{3n}Cx^3$. Q. E. I.

The Investigation of the Co-efficient E.

Art. 52. Since the simple series $C - Dx + Ex^2 - Fx^3 + Gx^4 - Hx^5 + \&c$ has been shewn in the last article to be equal to the compound series

$$\begin{cases} 3nD - 4nEx + 5nFx^2 - 6nGx^3 + 7nHx^4 - \&c \\ - 2nC + 3nDx - 4nEx^2 + 5nFx^3 - 6nGx^4 + \&c, \end{cases}$$

and

and the whole of the said simple series is evidently less than it's first term C, because the second term Dx is marked with the sign $-$, it follows that the whole of the said compound series must also be less than C, and consequently than the two terms $3nD - 2nC$ in the said compound series, which are equal to C. Therefore the whole of the said simple series may be subtracted from it's first term C, and the whole of the said compound series may be subtracted from the two terms $3nD - 2nC$. Let them be so subtracted. Then will the remainder of the former subtraction be equal to the remainder of the latter subtraction; that is, the simple series $Dx - Ex^2 + Fx^3 - Gx^4 + Hx^5 - \&c$ *ad infinitum* will be equal to the compound series

$$\left\{ \begin{array}{l} 4nEx - 5nFx^2 + 6nGx^3 - 7nHx^4 + \&c \\ - 3nDx + 4nEx^2 - 5nFx^3 + 6nGx^4 - \&c. \end{array} \right.$$

Therefore, if we divide all the terms of these two serieses by x , we shall have the simple series $D - Ex + Fx^2 - Gx^3 + Hx^4 - \&c$ *ad infinitum* equal to the compound series

$$\left\{ \begin{array}{l} 4nE - 5nFx + 6nGx^2 - 7nHx^3 + \&c \\ - 3nD + 4nEx - 5nFx^2 + 6nGx^3 - \&c. \end{array} \right.$$

And this equation will always be true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true when x is equal to 0. But, when x is equal to 0, all the terms in this equation that involve x will be equal to 0 likewise; and consequently the whole of the said simple series will be equal to it's first term D, and the whole of the said compound series will be equal to the two terms $4nE - 3nD$. Therefore D will be $= 4nE - 3nD$, and $4nE$ will be $= D + 3nD = \overline{3n+1} \times D$, and E will be $= \frac{3n+1}{4n} D$, or (because D is $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}$) E will be $= \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}$. Q. E. I.

Therefore the five first terms of the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c$ *ad infinitum*, which is equal to the quantity $\overline{1+x}^{-\frac{1}{n}}$, or to the fraction $\frac{1}{\overline{1+x}^{\frac{1}{n}}}$, will be $1 - \frac{1}{n}x + \frac{1}{n}$

$$\times \frac{n+1}{2n}x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4,$$

$$\text{or } 1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \overline{\frac{2n+1}{3n}}Cx^3 + \frac{3n+1}{4n}Dx^4. \quad \text{Q. E. I.}$$

The

The Investigation of the Co-efficient F.

Art. 53. Since the simple series $D - Ex + Fx^2 - Gx^3 + Hx^4 - \&c$ *ad infinitum* has been shewn in the last article to be equal to the compound series

$$\left\{ \begin{array}{l} 4^n E - 5^n Fx + 6^n Gx^2 - 7^n Hx^3 + \&c \\ - 3^n D + 4^n Ex - 5^n Fx^2 + 6^n Gx^3 - \&c, \end{array} \right.$$

and the whole of the said simple series is less than it's first term D , it follows that the whole of the said compound series must also be less than D , and consequently than the two terms $4^n E - 3^n D$ in the said compound series, which are equal to D . Therefore the whole of the said simple series may be subtracted from it's first term D , and the whole of the said compound series may be subtracted from the two terms $4^n E - 3^n D$, which are equal to D . Let them be so subtracted. Then will the remainder of the former subtraction be equal to the remainder of the latter subtraction; that is, the simple series $Ex - Fx^2 + Gx^3 - Hx^4 + \&c$ *ad infinitum* will be equal to the compound series

$$\left\{ \begin{array}{l} 5^n Fx - 6^n Gx^2 + 7^n Hx^3 - \&c \\ - 4^n Ex + 5^n Fx^2 - 6^n Gx^3 + \&c. \end{array} \right\}$$

Therefore, if we divide all the terms of these two serieses by x , we shall have the simple series $E - Fx + Gx^2 - Hx^3 + \&c$ *ad infinitum* = the compound series

$$\left\{ \begin{array}{l} 5^n F - 6^n Gx + 7^n Hx^2 - \&c \\ - 4^n E + 5^n Fx - 6^n Gx^2 + \&c. \end{array} \right\}$$

And this equation will always be true, of how small a magnitude soever we may suppose the quantity x to be taken: and therefore it will also be true when x is equal to 0. But, when x is equal to 0, all the terms in this equation that involve x will be equal to 0 likewise, and consequently the whole of the said simple series will be equal to it's first term E , and the whole of the said compound series will be equal to the two terms $5^n F - 4^n E$. Therefore E will be $= 5^n F - 4^n E$, and $5^n F$ will be $= E + 4^n E = 4^n + 1 \times E$, and consequently F will be $= \frac{4^n + 1}{5^n} \times E$. Q. E. I.

Therefore the six first terms of the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c \text{ ad infinitum}$, which is equal to the quantity $\overline{1+x}^{-\frac{1}{n}}$, or to the fraction $\frac{1}{1+x\frac{1}{n}}$, will be $1 - \frac{1}{n}x + \frac{1}{n}$

$$\begin{aligned} & \times \frac{n+1}{2n}x^2 - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n}x^3 + \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n}x^4 \\ & - \frac{1}{n} \times \frac{n+1}{2n} \times \frac{2n+1}{3n} \times \frac{3n+1}{4n} \times \frac{4n+1}{5n}x^5, \text{ or } 1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 \\ & - \frac{2n+1}{3n}Cx^3 + \frac{3n+1}{4n}Dx^4 - \frac{4n+1}{5n}Ex^5. \quad \text{Q. E. I.} \end{aligned}$$

Art. 54. After these full investigations of the four co-efficients C, D, E, and F, I believe it will be evident to the reader that, by reasonings similar to those by which we have discovered the values of these four co-efficients, we might obtain the two following simple equations, to wit, $F = 6nG - 5nF$, and $G = 7nH - 6nG$, for the determination of the values of the two following co-efficients G and H, and consequently might infer that $6nG$ would be $= 5nF + F = \overline{5n+1} \times F$, and therefore that G would be $= \frac{5n+1}{6n} \times F$, and that $7nH$ would be $6nG + G = \overline{6n+1} \times G$, and consequently that H would be $= \frac{6n+1}{7n} \times G$. And therefore we may now

conclude that the first eight terms of the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c \text{ ad infinitum}$, which is equal to the quantity $\overline{1+x}^{-\frac{1}{n}}$, or to the fraction $\frac{1}{1+x\frac{1}{n}}$, will be $1 - \frac{1}{n}Ax + \frac{n+1}{2n}Bx^2 - \frac{2n+1}{3n}Cx^3 + \frac{3n+1}{4n}Dx^4 - \frac{4n+1}{5n}Ex^5 + \frac{5n+1}{6n}Fx^6 - \frac{6n+1}{7n}Gx^7$. Q. E. I.

Art. 55. And, further, after these full investigations of the four co-efficients C, D, E, and F, of which we have found the values by resolving the four simple equations $\frac{1}{n} = 2nC - 1$, $C = 3nD - 2nC$, $D = 4nE - 3nD$, and $E = 5nF - 4nE$, I apprehend it will be easy to perceive not only that the two following co-efficients G and H may be determined in the same manner by the resolution of the two following simple equations $F = 6nG - 5nF$ and $G = 7nH - 6nG$, but also that the following co-efficients I, K, L, M, N, O, P,

P, Q, &c (to whatever number of terms we may think proper to continue the series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + \&c$ *ad infinitum*, which is equal to the quantity $\frac{-1}{1+x)^{\frac{1}{n}}}$, or to the fraction $\frac{1}{1+x)^{\frac{1}{n}}}$) may be determined in the same manner by the resolution

of the like simple equations deduced from the equality of the said series to the compound series obtained in the latter part of art. 49, to wit, the compound series

$$\left\{ \begin{array}{l} 1 - 2nCx + 3nDx^2 - 4nEx^3 + 5nFx^4 - 6nGx^5 + 7nHx^6 - 8nIx^7 + \&c \\ + \quad x - 2nCx^2 + 3nDx^3 - 4nEx^4 + 5nFx^5 - 6nGx^6 + 7nHx^7 - \&c, \end{array} \right.$$

which simple equations will be as follows, to wit, 1st, $H = 8nI - 7nH$, 2ndly, $I = 9nK - 8nI$, 3dly, $K = 10nL - 9nK$, 4thly, $L = 11nM - 10nL$, 5thly, $M = 12nN - 11nM$, 6thly, $N = 13nO - 12nN$, 7thly, $O = 14nP - 13nO$, 8thly, $P = 15nQ - 14nP$, and the like equations for the following co-efficients, whatever may be their number, the first, or left-hand, side of each equation being the co-efficient of some power of x in the aforefaid simple series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + \&c$, and the second, or right-hand, side of the same equation being the excess of the co-efficient of the same power of x contained in the upper row of terms of the said compound series, which is equal to the said simple series, above the co-efficient of the same power of x in the second, or lower, row of terms in the same compound series. Therefore, by adding the last term in each of these equations (which is marked with the sign $-$, or subtracted from the second term,) to both sides of it, we shall obtain a second set of equations, which will be as follows; to wit,

$$1st, \quad 6nG = F + 5nF = \overline{5n+1} \times F;$$

$$2ndly, \quad 7nH = G + 6nG = \overline{6n+1} \times G;$$

$$3dly, \quad 8nI = H + 7nH = \overline{7n+1} \times H;$$

$$4thly, \quad 9nK = I + 8nI = \overline{8n+1} \times I;$$

$$5thly, \quad 10nL = K + 9nK = \overline{9n+1} \times K;$$

M 2

6thly,

$$6\text{thly, } 11nM = L + 10nL = \overline{10n + 1} \times L;$$

$$7\text{thly, } 12nN = M + 11nM = \overline{11n + 1} \times M;$$

$$8\text{thly, } 13nO = N + 12nN = \overline{12n + 1} \times N;$$

$$9\text{thly, } 14nP = O + 13nO = \overline{13n + 1} \times O;$$

$$\text{and } 10\text{thly, } 15nQ = P + 14nP = \overline{14n + 1} \times P;$$

and the like equations for the following co-efficients, whatever may be their number; and lastly, by dividing both sides of each of these last equations by the several quantities $6n, 7n, 8n, 9n, 10n, 11n, 12n, 13n, 14n, 15n$, &c, which are the multipliers of the several co-efficients, G, H, I, K, L, M, N, O, P, Q, &c, we shall obtain a third set of equations which will express the several values of the said co-efficients by their relations to the co-efficients immediately preceeding them, to wit, the equations

$$G = \frac{5n+1}{6n} \times F, H = \frac{6n+1}{7n} \times G, I = \frac{7n+1}{8n} \times H, K = \frac{8n+1}{9n}$$

$$\times I, L = \frac{9n+1}{10n} \times K, M = \frac{10n+1}{11n} \times L, N = \frac{11n+1}{12n} \times M, O =$$

$$\frac{12n+1}{13n} \times N, P = \frac{13n+1}{14n} \times O, \text{ and } Q = \frac{14n+1}{15n} \times P, \text{ \&c, in which}$$

the law of continuation of the several fractions $\frac{5n+1}{6n}, \frac{6n+1}{7n}, \frac{7n+1}{8n},$

$\frac{8n+1}{9n}, \frac{9n+1}{10n}, \frac{10n+1}{11n}, \frac{11n+1}{12n}, \frac{12n+1}{13n}, \frac{13n+1}{14n},$ and $\frac{14n+1}{15n}$, by which

the several co-efficients G, H, I, K, L, M, N, O, P, and Q are generated from the co-efficient F and from each other, is evidently this, "*that every new generating fraction is derived from that which immediately preceeds it by adding n to both it's numerator and it's denominator.*" And this law of continuation must necessarily take place in all the following generating fractions of the series as well as in those that have been here set down, because the said generating fractions are derived, in the manner that has been above described, from the

simple series $1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 -$

$Hx^7 + Ix^8 - Kx^9 + Lx^{10} - Mx^{11} + Nx^{12} - Ox^{13} + Px^{14} - Qx^{15} + \text{\&c ad infinitum}$, and from the compound series obtained in art. 49, which is equal to it, in both which series the terms succeed each other, and in the latter series are generated one from another, in a known and regular manner, of which the said law of continuation in the said generating fractions is a necessary consequence.

Art. 56. We may therefore now conclude with certainty that the quantity $\sqrt[n]{1+x}^{-1}$, or the fraction $\frac{1}{1+x}^{\frac{1}{n}}$, will be equal to the infinite series

$$1 - \frac{1}{n} A x + \frac{n+1}{2n} B x^2 - \frac{2n+1}{3n} C x^3 + \frac{3n+1}{4n} D x^4 - \frac{4n+1}{5n} E x^5 +$$

$$\frac{5n+1}{6n} F x^6 - \frac{6n+1}{7n} G x^7 + \frac{7n+1}{8n} H x^8 - \frac{8n+1}{9n} I x^9 + \frac{9n+1}{10n} K x^{10} -$$

$$\frac{10n+1}{11n} L x^{11} + \frac{11n+1}{12n} M x^{12} - \frac{12n+1}{13n} N x^{13} + \frac{13n+1}{14n} O x^{14} -$$

$$\frac{14n+1}{15n} P x^{15} + \&c \text{ ad infinitum, or, in other words, that Sir Isaac Newton's}$$

binomial theorem is true in the case of the quantity $\sqrt[n]{1+x}^{-1}$, or the fraction $\frac{1}{1+x}^{\frac{1}{n}}$, as well as in the first and simplest case of $\sqrt[n]{1+x}^m$, and in the case of $\sqrt[n]{1+x}^{\frac{1}{n}}$.

Q. E. D.

Art. 57. In this last, or fourth, method of obtaining the values of the co-efficients C, D, E, F, G, H, &c, it is evident that the latter co-efficients F, G, H, &c are obtained with nearly the same ease as the first co-efficient C, to wit, by multiplying E, F, and G, &c into the generating fractions $\frac{4n+1}{5n}$, $\frac{5n+1}{6n}$, $\frac{6n+1}{7n}$, &c respectively, as C is obtained by multiplying $\frac{1}{n}$, or B, into the generating fraction $\frac{n+1}{2n}$; whereas in the three former methods of obtaining the values of these co-efficients, and more especially in the third method, the labour of investigating every new co-efficient was much greater than that of investigating the co-efficient immediately preceeding it, and the difficulty of investigating the values of the co-efficients G and H would be so great as to be almost insuperable. This fourth method, therefore, of obtaining the values of these co-efficients is, in a practical view, very greatly superiour to either of the three former methods of obtaining them; though the processes employed in it in order to obtain the compound series consisting of two rows of terms, which is set down at the end of art. 49, are very numerous, and the reasonings that accompany them are seemingly somewhat remote and subtle. But the great facility with which the co-efficients C, D, E, F, G, H, &c (to whatever number we may think proper to continue them,) may be obtained in this method must be considered as an ample compensation for the great number of Algebraick operations, and for the subtlety and remoteness of the reasonings made use of in investigating the said series.

Art. 58.

Art. 58. This last, or fourth, method of investigating the terms of the series

$$1 - \frac{1}{n}x + Cx^2 - Dx^3 + Ex^4 - Fx^5 + Gx^6 - Hx^7 + \&c,$$

which is equal to the quantity $\sqrt[n]{1+x}^{-1}$, or to the fraction $\frac{1}{\sqrt[n]{1+x}}$,

has been drawn up in exact imitation of the investigation that is given of the

infinite series that is equal to $\sqrt[n]{1+x}^{\frac{1}{n}}$ in the second volume of the *Scriptores Logarithmici*, pages 225, 226, 227, &c --- 241, and which is either the same with, or derived from, the investigation that had been given of that series in a more concise manner by the late learned Mr. John Landen of Peterborough: so that the reader of these tracts is in some degree indebted to Mr. Landen for the

present investigation as well as for that relating to $\sqrt[n]{1+x}^{\frac{1}{n}}$ in the said former volume. But neither Mr. Landen, nor any other writer of Algebra whose works have fallen under my observation, has ever, as far as I can recollect, given an express investigation of the binomial theorem in this case of it, or when the index of the power of $1+x$ is $-\frac{1}{n}$, or the quantity to be expressed in an

infinite series of simple terms is $\sqrt[n]{1+x}^{-1}$, or the fraction $\frac{1}{\sqrt[n]{1+x}}$. Yet

after this famous theorem has been published more than an hundred years, it seems to be high time that it should be demonstrated in all it's cases.

ANALYSIS FLUXIONUM.

AUCTORE GUIL. HALES, D. D.

RECTORE DE KILLESANDRA,

ET NUPER TRIN. COLL. DUBLIN SOCIO, AC LINGUARUM ORIENTALIUM
PROFESSORE.

Sanctos ausus recludere fontes. VIRG.

ANALYSIS OF FLUXION

BY J. H. COOPER, M.A.

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FRANCISCO MASERES,

CURIAE SCACCARII REGIS MAGNAE BRITANNIAE IN ANGLIA BARONI QUINTO;

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AMICO FIDO;

VINDICIAS HASCE NEWTONIANAS—

METAPHYSICAS ET MATHEMATICAS,

PHYSICAS ET THEOLOGICAS;—

QUA PAR EST OBSERVANTIA,

GRATUS LUBENSQUE

D. D. D.

GUILLIELMUS HALES.

FRANCISCO MARRER

FRANCISCO MARRER REGIS BRITANNICAE IN ANGLIA BARONI COUNTESS

ANALYSTE REGIS

MATHEMATICO PATRONO

VIRI HONORI

AMICO FIDELI

VINDICIAS BASCE NEWTONIANAS

METAPHYSICAS ET MATHEMATICAS

PHYSICAS ET THEOLOGICAS

QUIA PARS EST OMNIVM

GRATIAS AGIT

1711

GUILIELMUS HALL

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P R Æ F A T I O.

INTER THEORIAM ÆQUATIONUM ALGEBRAÏCARUM atque FLUXIONALIUM tanta intercedit Analogia, eadēque tam arctā affinitate, imò cognatione, inter sese consociantur et conjunguntur; ut absque hujus ope, illa jure imperfecta, et quasi manca, existimari debeat: Nam per calculum FLUXIONALEM, seu, ut ab exteris designatur, DIFFERENTIALEM, (vix nisi nomine et notatione à se invicem discrepantes) optimè promovenda est ALGEBRA. Sicut unico patere poterit exemplo, THEOREMATIS BINOMII, quod, licet indoles ejus sit omninò fluxionalis, omnes tamen MATHESIOS regiones permeat et adauget. Hanc igitur ANALYSIN FLUXIONUM, ANALYSI ÆQUATIONUM pridem editæ (*), planè gemellam, edere jam libet; ut pro virili fungar munere interpretis fidi ANALYSEOS ANTIQUÆ illius, veræ et genuinæ:—"Quæ cum Algebrâ Speciosâ facilitate contendit, evidentia verò et demonstrationum elegantia eam longè superare videtur;"—Teste, peritissimo Geometrâ simul ac Analystâ, EDMUNDO HALLEY; qualem eam optimè exponit PAPPUS ALEXANDRINUS (in APPENDICE, I. § 136.) ex scriptis Veterum Analystarum:—ARCHIMEDIS, in Quadraturâ Parabolæ; APOLLONII, in tractatu de Sectione Rationis; EUCLIDIS, in Libris Datorum et Porismatum; ERATOSTHENIS, de Mediis Proportionalibus, &c.—Hanc verò ANALYSIN ANTIQUAM pænè amissam, (CARTESIO, ejusque sectatoribus, haud minùs MATHESIN quàm PHILOSOPHIAM inquinantibus, adhibendo demonstrationes, antiquis quidem breviores,

(*) Tractatum priorem (ANALYSIN ÆQUATIONUM) impensis propriis edidi anno 1784; hunc verò, pro singulari suo in MATHESIN, PHILOSOPHIAMQUE BRITANNICAM studio, impensis suis editurus est vir cl. FR. MASERES, vel ipso flagitante. Neque, ut spero, in amici dignissimi modestiam peccasse censebor, si in lucem, (etiam illo invito), excerpta quadam deprompserim ex epistolâ ejus, hanc rem spectantia; Sept. 17. anno 1799 datâ:

—"I am glad to find that you have been so actively and usefully employed, since your return to IRELAND, as in revising and enlarging your *Analysis Fluxionum*; which I hope you will soon publish. If you will send it to me, I will print it for you, either in the *Fifth* Volume of the *Scriptores Logarithmici*, or by itself;—or half the number of copies in the *Scriptores Logarithmici*, and the other half by themselves, as you shall direct. And I believe that last manner of printing it would be the best:—that is, to print 250 copies in the *Scriptores Logarithmici*, and another 250 separately; as, by this way, it will come sooner into the hands of the Learned."

—"I wish you every opportunity of propagating your taste for sound literature and science; and thereby of becoming eminently useful to your COUNTRY: than which, nothing can afford you greater satisfaction."

See § 4, Note (a); § 17, Note (d); and § 131.

at longè laxiores atque infirmiores) curiosa felicitate restituit, et instauravit et perfecit NEWTONUS, Veterum evidentiam et certitudinem, Recentiorum brevitatem et facilitatem, rarissimâ conjunctione affectus; si modò rite expendatur et probè intelligatur imperatoria brevis quâ METHODUM FLUXIONUM delibando transcurrit Inventor ille sagacissimus.

Hujus Methodi fons et origo est Doctrina RATIONUM PRIMARUM ET ULTIMARUM; cujus ope demonstrantur LEMMATA illa præclara seu PRINCIPIA MATHEMATICA, perbrevia et subtilissima, in primâ sectione doctissimi et celeberrimi illius libri cui titulus, Philosophiæ Naturalis Principia Mathematica. Hæc verò Doctrina, qualem ipse auctor eam orbi litterato in diversis suis scriptis proposuit, pulcherrimam exhibet synopsis pariter et extensionem METHODI EXHAUSTIONUM à veteribus geometris usurpatæ et excultæ, quam vir mirandâ sagacitate præditus uno eodémque tempore in compendium redegit et ad novas et maximè arduas speculationes felicissimè promovit et applicuit.

Ceterum fatendum est hanc Doctrinam RATIONUM PRIMARUM ET ULTIMARUM, ex intimis METAPHYSICÆ fontibus haustam, et "quâ potuit brevitate" à NEWTONO traditam, obscuritate non facilè discutiendâ involvi: quocircâ expositio ejusdem, satis lucida, evidens, et accurata (non obstantibus tot tantisque Interpretibus Analyseos Newtonianæ) hætenus desiderari videtur; istque conditio sine quâ non clarè cernere atque animo complecti licebit fundamentum solidum et nullis cavillationibus dimovendum, cui adstruitur tota THEORIA FLUXIONUM, neque diluere objectiones doctorum quorundam virorum, qui etiâ his seris temporibus ejusdem evidentiam et certitudinem impugnant, sive ex insitiâ vel imperitiâ, vel denique ex invidiâ et partium quodam studio, oriundas.

Itaque, ne falsis criminibus ulterius premi videatur Methodus Fluxionum, in PRIMA PARTE, ausus sum fontes ipsos METAPHYSICOS recludere ex quibus derivantur PRINCIPIA MATHEMATICA FLUXIONUM, in SECUNDA PARTE; Analyseos Antiquæ et veræ quasi custos rigidusque Satelles; et magni mirandique ANALYSTÆ BRITANNICI æquus et candidus vindex; excussis quàm fidelissimè et quàm accuratissimè potuerim, objectionibus Analystarum magni nominis, BERKELEY, LANDEN, D'ALEMBERT, TORELLI, DE LA GRANGE, &c. et nuperrimè, CENSORIS (ut ferunt, Professorii et Academici,) apud THE MONTHLY REVIEW, Appendix, April 1798, Vol. 28. N. S.

In hac Disquisitione, subtilissimâ sane et reconditissimâ, et inter laborem "tardè eruendi quæ tam altè jacent," Demonstrationem Theorematis Binomii, à doctissimo viro, Johanne Landen, inventam, (quam ego dudum admiratus fueram et in ANALYSI ÆQUATIONUM adhibueram et excolueram,)

eram,) iterum ad examen revocare visum est; et, cum ista demonstratio vix, et ne vix quidem, in principiis in quibus fundatur, à computo fluxionali discrepare videtur, aliam demonstrationem ejusdem Theorematis omnino, novam, et in principiis Methodi Fluxionum fundatam, huic Landenianæ substituendam esse judicavi; quæ quidem est ipsa longè facilior, et simplicior, et ad naturam Theorematis demonstrandi magis accommodata, utpote quæ derivatur ex analogiâ quâdam elegantissimâ, quæ intercedere observatur inter SERIEM BINOMIAM et SERIEM FLUXIONUM ORDINUM SUPERIORUM, quarum serierum termini correlativi sunt ad se invicem semper in datâ quâdam ratione.

HIS VINDICIIS NEWTONIANIS—METAPHYSICIS et MATHEMATICIS—in APPENDICE II. alias quoque—PHYSICAS et THEOLOGICAS—subjungere visum est; ut laus debita illi PHILOSOPHO CHRISTIANO, integra servetur et illibata; quam inquinare non verentur, falsò atque ignavè, MINUTI quidam PHILOSOPHI, hoc “SÆCULO perperam sanè ab iis, eorumque sectatoribus, RATIONIS dicto. Pudètque inter hujusmodi obtrectatores, inveniri ipsius NEWTONIANÆ PHILOSOPHIÆ (ad THEOLOGIAM sobriè et humiliter hactenus ancillantem,) non solum Discipulum sed etiàm Professore, ROBISON, apud EDINBURGENSES; temerè inculpantem theoriam illam sublimem at longè subtilissimam “De pulsibus mediæ ætheriæ, vibrando propagatis à corporibus tremulis ad sensorium usque; ibique, à substantiâ sensitivâ, seu vi sentiendi, intelligendi, et agendi perceptis:”—cujus objectiones penitiùs et fusiùs, ex NEWTONI scriptis, et testimoniis PHILOSOPHIÆ PRIMARIÆ et SACRÆ, antiquitùs in honore habitæ, in hac secundâ Appendice excutiuntur.

Quàm cautè verò, et quàm timidè procedendum in luce sublustri Analyseos sublimioris,—NATURAM DIVINAM HUMANAMQUE explorantis, ideâsque claras et determinatas INFINITATIS arripere gestientis,—monet MACLAURIN optimus:

“In reasoning concerning finite quantities, we apprehend that GEOMETRICIANS cannot be too scrupulous in admitting or treating of Infinites; of which our ideas are so very imperfect.—PHILOSOPHY probably will always have its mysteries; but these are to be avoided in GEOMETRY (*), and

(*) GEOMETRY has its mysteries, no less than PHILOSOPHY and RELIGION: Witness the elementary doctrine of *Parallels*, resting on the disputed foundation of the *eleventh* axiom of *Euclid*, the intuitive evidence of which is denied by *Tacquet* and others; and the *quadrature of the circle*, which can only be solved by approximation, still remains the reproach of Geometricians. In *ALGEBRA*, the *irreducible case* of cubick equations exhibits two of the roots under an impossible form, when all are real: which, notwithstanding, can be accurately extracted in some particular cases; and in the rest approximately, by means of the *Binomial Theorem*. And in *FLUXIONS*, too, the famous *Problem of the Three Bodies*, so necessary to determine the accelerations or retardations of the motions of the planets and the derangements

and we ought to guard against abating from its strictness and evidence. the rather, that *an absurd PHILOSOPHY is the natural product of a vitiated GEOMETRY.*"

KILLESANDRA,
Oct. 23, 1799.

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ments of their orbits, by their mutual attractions, (of which approximate solutions have been given by *Euler*, *Clairaut* and *D'Alembert*), is thus represented by the first of these three great men, with all the candour of a profound Mathematician:—" *Hujus problematis enodatio completa omnes ANALYSEOS vires transcendere videtur.*"—The difficulty consists in integrating three differential equations of the second order; or, in the language of Newton's doctrine of Fluxions, in finding the fluents of three fluxionary equations of the second order.—Indeed the whole theory of the *Resolution of the Higher Equations in Algebra*, and the *Inverse Method of Fluxions*, which corresponds thereto, is extremely imperfect still, in this OUR INFANCY OF SCIENCE.—See *Maclaurin's* dying words, *Appendix*, II. § 162.

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ANALYSIS FLUXIONUM.

PARS PRIMA.

DE FLUXIONUM INVENTIONE.

§ 1. *THEORIÆ Fluxionum* præstantia insignis, omnis *Matheseos* fines peragrans, promoventis et illustrantis, prorsus postulare videtur ut præmittantur aliqua, more historico, de ejus inventione; qualia extant in *Commercio Epistolico Johannis Collins* et aliorum, anno Domini 1712, edito, atque iterum auctius anno Domini 1722; in *Raphson's History of Fluxions*; in diversis editionibus *Principiorum Newtoni*; *Hutton's Mathematical Dictionary*; et *Monthly Review*, Append. Vol. 28, 1799.

2. Utrum *Newtono* an *Leibnitzio* tribuenda sit laus hujus inventionis, et uter horum eam prius meditando extuderat; diù atque acriter contestatum est inter fautores *calculi fluxionalis* et *calculi differentialis*, vix ab invicem nisi merâ notatione discrepantium; et adhuc sub judice lis est.

3. Veruntamen hanc controversiam penitiùs inspicienti, et sine partium studio perpendenti, satis constabit, utrumque horum illustrium Geometrarum pro ejusdem inventore jure censi posse; utrumque, præcellenti ingenio et mirâ sagacitate præditum, utrumque eodem "*labore improbo*," et "*cogitandi vi indefessâ*" functum, ex iisdem fontibus scientiæ utrique patentibus, tandem eadem *elementa metaphysica*, eadem *principia mathematica*, methodi sive *fluxionalis* sive *differentialis*, hausisse; atque sectatoribus suis delibasse, principio, carptum, ac sine demonstrationibus edita. Hæc quidem *Newtonus* postea, obiter tantum, et quasi invitus, longo post tempore supplevit, parçè et "*quâ potuit brevitate*," in *Lemmate Secundo Lib. II. Princip.*; in *Quadraturâ Curvarum*; et in *Analyfi per Æquationes numero terminorum infinitas*:—Hanc curam *Leibnitzius* (vir omnimodæ scientiæ avidus, et à Geometriæ studio nimis sæpe ad alias speculationes ingenii vires transfereus) prorsus neglexit, discipulisque suis celeberrimis, *Jacobo* et *Johanni Bernoulli*, eorumque successoribus *Euler*, *D'Alembert*, *La Grange*, et *La Place*, legavit; qui munus sibi demandatum, demonstrandi et in immensum promovendi fines fluxionum, longè ultrâ limites *Newtonianos* et *Leibnitzianos*, ingenti cum emolumento ipsius *MATHESEOS*, exsecuti sunt.

4. Glo-

4. Gloriabatur certè *Leibnitz*,—"se *Newtono* nil debuissè;" nec inficiabatur *Newtonus*: pro suâ modestiâ inexpugnabili nihil sibi arrogans, et tantum postulans jus æquum partitionis, patiens et commodus; "cui omnium mortalium minime necessarium fuit *laudem aliunde mutuari*:—*Hanc sibi ipsi peperit*." (a). Audiamus ipsum, in primâ editione *PRINCIPIORUM*, anno 1687, p. 253. de *Fluxionum inventione* differentem:

"In literis quæ MIHI cum Geometrà peritissimo GULIELMO, GOTHOFREDO, LEIBNITIO annis abhinc decem [1677] intercedebant, cum significarem me compotem esse methodi determinandi Maximas et Minimas, ducendi Tangentes, et similia peragendi; quæ in terminis surdis æquè ac in rationalibus procederet; et literis transpositis hanc sententiam continentibus—"Datâ æquatione, quocunque fluentes quantitates involvente, fluxiones invenire"—eandem celarem: rescripsit vir clarissimus 'se quoque in ejusmodi methodum incidissè;' et methodum suam communicavit, à meâ vix abludentem, præterquam in verborum et notarum formulis," ["et ideâ generationis quantitatum"—inseruntur hæc quatuor verba in editione secundâ *PRINCIPIORUM* anno 1713]—"Utriusque fundamentum continetur in hoc lemmate." [II. Lib. II. PRINCIP.]

5. Et certè fatendum est, femina non obscura methodi suæ differentialis, possuisse *Leibnitzium*, in *Theoriâ Notionum Abstractarum*, *Academiæ Regiæ Parisiensis* dicatâ, anno 1671; antequam cum *Newtono* commercium aliquod epistolicum instituisset. Hanc methodum postea publici juris fecit, in epistolâ quâdam in *Actis Lips.* Jan. 21, 1677, editâ, verum absque demonstratione prolatam; quam postea suppleverunt, celeberrimi fratres, *Daniel et Johannes Bernoulli*, famæ *Leibnitzianæ* vindices acerrimi, iisdemque fontibus analyticos sublimioris haud levitè imbuti.

(a) Dr. Halley, in his letter to Mr. Newton, dated May 22, 1686, informing him that the Royal Society had resolved to print his "*Incomparable Treatise* intitled *PHILOSOPHIÆ NATURALIS PRINCIPIA MATHEMATICA*," at their own charge, in a large quarto of a fair letter—"judging that so excellent a work ought not to have its publication delayed"—concludes with stating Mr. Hooke's claim—"as having some pretensions upon the invention of the rule of gravity being reciprocally as the squares of the distances from the centre. He says you had the notion from him, though he owns the demonstration of the curves generated thereby to be wholly your own.—How much of this is so, you know best; as likewise, what you have to do in this matter. Only Mr. Hooke seems to expect you should make some mention of him in the Preface, which it is possible you may see reason to prefix."

"I must beg your pardon that it is I that send you this ungrateful account; but I thought it my duty to let you know it, that so you might act accordingly: being in myself fully satisfied, that the greatest candour imaginable is to be expected from a person who has of all men the least need to borrow reputation."

See the whole letter, and the proceedings of the Royal Society on this business, in *Birch's History of the Royal Society*, Vol. IV. p. 484.

In consequence of this unfounded claim of Dr. Hooke, and his own aversion to controversy, *Newton* withdrew the third book of the *Principia*, which treated of the *System of the World* in a popular way; wishing not to let it go abroad into the world, without a strict and rigorous demonstration; as it appeared afterwards:—

"De hoc argumento composueram librum tertium [DE MUNDI SYSTEMATE] methodo populari, ut à pluribus legeretur. Sed quibus principia posita satis intellecta non fuerint, ii vim consequentiarum minime percipient, neque præjudicia deponent, quibus à multis retrò annis insueverunt: Et propterea, ne res in disputationes trahatur, summam libri illius transtuli in Propositiones, more mathematico, ut ab iis solis legantur qui principia prius evolverint."

Prefat. in Lib. III. Princip.

6. Nec extant testimonia, *Newtonum* methodum suam fluxionum cuius communicasse ante annum 1672; regerit enim *Newtonus*, in brevi illâ historiâ de fluxionum inventionem, *Schol. Lem. II. Lib. II. Princip. 3^{tiæ} edit. 1726, p. 245*, in his verbis.

“In epistolâ quâdam ad *D. Collinsum* nostratem 10 Decemb. 1672 datâ, cum descripsissem *methodum Tangentium* quam suspicabar eandem esse cum *methodo Slusii* (tum nondum communicatâ), subiunxi:”

—“*Hoc est unum particulare, vel COROLLARIUM potius, METHODI GENERALIS quæ extendit se citrà molestum ullum calculum, non modò ad ducendas TANGENTES ad quasvis curvas, sive geometricas sive mechanicas; vel quomodocunque rectas lineas, aliasve curvas, respicientes; verùm etiâ ad resolvendum alia abstrusiora problematum genera DE CURVITATIBUS. AREIS, LONGITUDINIBUS, CENTRIS GRAVITATIS CURVARUM, &c. Neque (quemadmodum HUDDENII methodus de MAXIMIS ET MINIMIS) ad solas restringitur æquationes quæ quantitibus sordis sunt immunes. HANC METHODUM intextui alteri isti [DE ANALYSI PER ÆQUATIONES NUMERO TERMINORUM INFINITAS, quam anno 1671 de his rebus scripseram] quâ æquationem exegesi instituo reducendo eas ad series infinitas.*”—

7. Alteram quoque epistolam, 13 Jun. 1676, scripsit *Newtonus* ad *Oldenburgum*, cum *Leibnitzio* communicandam, in quâ celeberrimum *Theorema Binomium*, primò, absque demonstratione proposuit, (nec unquam postea demonstratione ipse munivit) cujus ope nimirum, “*æquationum illam exegesi*” potissimum, instituerat. Hoc autem, ex intimis penetralibus fluxionum haustum, *Newtono*, saltem ante *Pestem Londinensem*, anno 1665, innotuisse, antea memoravimus, *ANALYS. ÆQUAT. p. 33. not.* Vide quoque § 96 hujus.—Cæterum hæ duæ epistolæ, *Leibnitzianam* illam Jan. 21, 1677, tempore antegressæ, *primi inventoris* honorem *Newtono* servare videntur; nec inficiabitur æqua posteritas.

8. De famâ *Leibnitzii* plus æquo forsân detrectans, cætus *Analystarum Anglicorum* adjudicavit, “*That in the letter of 10th December 1672, from Newton to Collins, the method of fluxions was sufficiently described to any intelligent person.*”—Verùm, si “*hæc methodus satis in istâ epistolâ descripta fuit ut ab aliquo harum rerum modicè perito posset intelligi, quæri potest,*” cur *Newtonus*, anno 1677, ad æmulum suum “*intelligentissimum* certè atque *peritissimum*” *Leibnitzium* scribens, anagrammate quodam *methodum suam fluxionum directam* diutius celaret?—Disputationes sanè quæ adhuc, etiâ nostro ævo, vigent de evidentiâ et certitudine hujus methodi, inter *Analystas* perspicacissimos, scholæ tam *Newtonianæ* quam *Leibnitzianæ*, (non obstantibus, vindiciis, expositionibus, et commentariis discipulorum *Newtoni* celeberrimorum) satis ostendunt cur *tam tardè eruantur, quæ tam alè jacent.*—*Seneca.*

Adde quod, si verum fuisset, quod ex scriptis *Newtonianis* primos theoriæ suæ igniculos elicuerat *Leibnitzius*, haud mediocris laus idcirco illi concedenda videretur, qui arcanum molestissimum et longè difficillimum in lucem ex tenebris proferre poterat:

Quem vituperare ne inimici quidem possunt, nisi ut simul laudent.

9. Veruntamen tutò colligere licet, neque *Newtonum* neque *Leibnitzium* pro inventore absoluto hujusce methodi habendum esse; nam si manifestò constar, Vol. V.

O

utrumque

utrumque methodum suam à methodo *ducendi Tangentes Barrovianâ* mutuâsse; nonne utrique laudem *pleni perfectique ab integro inventoris*, detrahât æquus controversiæ iudex?

Audiamus *Leibnitzium* in epistolâ quâdam April 9, 1716, datâ :

“ M. *Newton* hazarde icy un accusation qui va tomber sur lui-même : Il prétend que ce que j’ay écrit pour lui à M. *Oldenbourg* en 1677 est un *déguisement de la méthode de M. Barrow*. Mais, comme M. *Newton* avoue dans la page 253 et 254 de la première édition de ses principes :—“ Me, [*Leibnitzium*] *ipsi* [*Newtono*] *tunc methodum communicâsse à methodo ipsius vix abludentem præterquam in verborum et notarum formulis*”—“ il s’ensuivra que sa méthode aussi n’est qu’un *déguisement de celle de M. Barrow*.”—Et non infeliciter sanè retorquetur à *Leibnitzi* ; *Newtonum* enim potius mutuâsse ab amico, socio, studiorumque patrono, ipsiusque prædecessore in munere professorio *Lucasiano Cantabrigiæ*, optimo eruditissimoque *Barrow* ; quàm *Leibnitzium* suam methodum ab eodem *Barrow* derivâsse, scriptore, scilicet, quoad *Leibnitzium* peregrino et fortasse parùm noto, longè verisimilius est.

10. Quantum verò debuit *Newtonus* antecessoribus suis, ipse haud rarè ingenuè fatetur—modestiâ quàm ingenio non minùs spectabilis. Debuit *S’uso* quidem in *methodo Tangentium*, imprimisque *Barrow*, in *methodo ducendi Tangentes per triangulum arithmeticum* ; *Nicolao Mercatori*, in *Logarithmotechniâ*, quam à *divisione fractionum ad extractionem radicum per theorema binomium*, extendit *Newtonus* ; *Hugenio*, “ in eximio suo tractatu de *horologio oscillatorio*,” cujus methodum determinandi vires centripetas et centrifugas corporum revolvantium in *circulis*, ad vires corporum revolvantium in *curvis quibuscunque*, solertissimè extendit *Newtonus* per *circulos ejusdem curvaturæ* ; debuit etiâ *Wallisio*, in *quadraturâ curvarum*, reducendo *ordinatas irrationales ad rationales*, per *series infinitas* ; *Fermatio*, in *solutione problematum de maximis et minimis* ; *Cavallerio*, in *methodo indivisibilium* ; atque ipsi *Cartesio*, in *applicatione Algebræ ad Geometriam*, quam ipse tantoperè auxit, atque ad *mechanicam rationalem*, (seu “ *scientiam motuum qui ex viribus quibuscunque resultant, et virium quæ ad motus quoscunque requiruntur, accuratè propositam ac demonstratam*”)—solertissimè extendit et accommodavit.

11. *Magnus* sanè habendus est ille *inventor*, qui obscura et involuta scientiæ arcana, arte suâ indagatrice, enucleavit et detexit, lucem ex tenebris promens ; *major* tamen esse videbitur, si ipse *expositor*, quæ detexit, ea *noverit et voluerit* certâ ac idoneâ ratione demonstrare ;—“ *analyfi brevissimâ et simul perspicuâ, synthesi concinnâ et minimè operosâ*,” qualem exercebat olim magnus ille *Geometra Apollonius*, in libro suo eximio de *Sectione Rationis*, testante *Halley* ; atque minimè deficit ab ejus vestigiis *Newtonus*, in *Lemmate II. Lib. II. Princip.* imperatoriâ quâdam brevitate, et lucido ordine, fundamentum methodi suæ generalis fluxionum positurus.

12. Veruntamen fatendum est, quod *Newtonus*, nimis forsàn brevitatis studiosus et moram et molestiam interpretandi refugiens, non nisi “ *peritos alloquens*,” semina tantùm expositionis veræ atque genuinæ sevit, à posteris serò explicanda in frugem

frugem uberiorem justæ atque legitimæ demonstrationis, ubi “*Opinionum commenta deleverit dies*,” et tandem, post lapsum annorum centum, consensu communi sectatorum scholæ *Newtonianæ* et *Leibnitzianæ*, methodi ipsius generalis sive *fluxionis* sive *differentialis*, rudimenta meliùs intelligantur, ejùsque elementa prima “*quæ tam altè jacent*,” imis in penetralibus scientiæ *metaphysicæ*, cautiùs atque curatiùs “*eruerè*,” et argumentis idoneis et ad naturam scientiæ illustrandæ accommodatis tandem confirmare, fas sit.

13. Neutiquam cuivis homini contingit inventa sua, à primordiis ad ultimum perfectionis statum, spatio vitæ vel longissimæ perducere. Artis scientiæve cuiusvis elementa simul invenire, et inventa demonstrare, vel maximis ingeniis rarissimè concessit Deus. Gradatim et seriatim fit VERITATIS exploratio.—“*Plantabat Paulus, irrigabat Apollos, sed Deus augebat*”—haud minùs verum est in *Philosophiâ* quam in *Religione*.

“*Felix*” quidem censendus est, “*qui potuit*” vel paucos annulos arripere tantum, eosdèmq̃ue *infimos*, immensæ illius catenæ VERITATUM ÆTERNARUM, quam à cœlis ad tellurem usque demisit, ægris mortalibus et hallucinantibus, benignus ILLE DEUS, qui variis gaudet nominibus—Ο ΛΟΓΟΣ ΤΟΥ ΘΕΟΥ, Ἡ ΣΟΦΙΑ, seu “*RECTA RATIO*,” “*SAPIENTIA*,” “*VERITAS*,”—tàm apud philosophos profanos quàm apud sacros; et qui hanc catenam non viris ingenii et scientiæ suæ opinione tumidis, sed potiùs iis qui humiliùs de se sentiunt et vitæ et morum sanctitate insignes fiunt, porrigit,—paucis quidem “*quos æquus amavit JUPITER*”—“*Divûm hominûmq̃ue æterna potestas*; tales versùs sese, lenitèr et indefinentèr attrahens atque attollens:—alligatur enim summus hujùsce catenæ annulus ad basin ipsius *throni divini*. Vid. *John* xii. 32.

Και ἐδικαιώθη Ἡ ΣΟΦΙΑ ἀπὸ τῶν τεχνῶν αὐτῆς ἀπάντων. *Luk.* vii. 35.

Licèt enim pusilli *Sophistæ*, seu *Pseudo-Philosophi*, *veritates æternas* haud raro obfuscent, ipsàmque “*ALMAM SAPIENTIAM*” vilipendant et obtrecent ingrati; “*indies nihilominus*,” animosè vindicatur et “*justificatur, liberis suis omnibus*” verè philosophantibus, à primordiis rerum; quales identidem prodierunt, “*lucida sidera*,” præclara lumina mundi: (b) ENOCH, NOAH, JOB, ABRAHAM, MOSES, DANIEL *Archimagus*, et PAULUS *Tarsensis*; THALES, PYTHAGORAS, SOCRATES,

(b) 1. Enoch autediluvianus, et “*ab Adamo septimus*,” transcendentem ob pietatem vivus translatus est in cœlum, ætatis suæ anno 365mo; veros dies anni *solaris* mirè exponente; DEO forsan, illi, quasi *astronomo*, id revelante; sicut illi, quasi *prophetæ*, revelavit *statum futurum*, et *judicium generale*, quæ prædixit Enoch, denunciavitque impiis, *Jude* xiv.—Hoc enim (licèt inficiantur *semi-philosophi* et *semi-theologi* hodierni) dogma fuit SAPIENTIÆ PRIMÆVÆ antiquissimum:

Ὅτι ὁ ΘΕΟΣ ΕΣΤΙ, καὶ τοῖς ἐκζητοῦν
αὐτοῦ ΜΙΣΘΑΠΟΔΟΤΗΣ ΓΙΝΕΤΑΙ.

“*Quod DEUS EST, et exquirentibus Ipsum
REMUNERATOR fit.*” *Heb.* xi. 6.

Et “*ultimis quoque diebus*,” τῇ παλιγγένειᾳ,—ubi “*magnus ab integro sæclorum nascitur ordo*”—“*dicit vir*” omnis verè philosophans:

SOCRATES, PLATO, ARCHIMEDES, et HIPPARCHUS ὁ φιλαληθὲς ἄνθρωπος; COPERNICUS, GALILEO, ROGERUS BACON et FRANCISCUS BACON, PASCAL, BOYLE, BARROW, NEWTON, MACLAURIN, COTES, DANIEL BERNOULLI, &c. &c. &c. et novissimè quidem doctissimus vir, Gulielmus Jones, eques auratus, Curiae Supremæ Britannicæ apud Calcuttam in Indiâ Justiciarius, (Gulielmi Jones, celeberrimi Mathematici sub initio hujus sæculi, filius,) *Indorum Apostolus*: qui, suis in sæculis non magis claruerunt omnes, famâ scientiæ reconditoris, quam virtutis pietatisque sublimioris; et etiamnum, unusquisque eorum ἀποθανὼν ἐτι λαλεῖται, “*licet mortuus, adhuc laudantium in ore est;*” honosque eorum in æternum vigeat.

De Objectionibus allatis contrâ Methodum Fluxionum à præcipuis Auctoribus qui hætenùs impugnant ejusdem Evidentiam et Certitudinem.

14. CONFESTIM à primâ expositione perbrevis *Methodi Fluxionum*, novæ, subtilioris et reconditoris, objectiones variæ suboriebantur, Analystarum tam peritorum

“*Certe fructus est justus:*

Certe DEUS, judicans in terrâ.”

Psal. lviii. 11.

“*An JUDEX TOTIUS TERRÆ non faciet judicium?*”

Gen. xviii. 25.

2. NOAH, magnus ille propheta et instaurator generis humani, “*qui gratiam invenit apud DEUM,*” à diluvio servatus; quod prædixit, frustrâque denunciavit, 2 *Pet. ii. 5.* et *fortunas Noachidarum* cecinit moribundus, *Gen. ix. 25.*—Et “*Institutionis Menu*” apud *Indos*, (qui *Manes* est dictus apud *Ægyptios*,) etiamnum traditione celebrantur.—*Cyclos magnus* 600 annorum, Antidiluvianus, à *Josepho* notatus, ætatem *Noæ* tempore diluvii mirè servat.

3. *Jeb*, à Diluvio septimus (Vide *Abulfaragi*, p. 13.)—ὁ μέγας οἶκος ἐκεῖνος καὶ γενναῖος τῆς ἀληθείας ἀγωνιστής—“*magnus ille et nobilis VERITATIS vindex*,”—*Suidas*—peritissimisque astronomis, qui constellationes cardinales sphaeræ primitivæ recenseret, *Jeb ix. 9.* et xxxviii. 32. *ASH*, *CHIMAH*, *CHESIL* et *MAZAROTH*, sive “*Ursa Major, Taurus, Scorpio et Canis*” respectivè; ut videre licet in opere quodam an. 1799 edito, cui titulus, *THE INSPECTOR—London, White.*

4. ABRAHAM,—“*Amicus ille DEI,*” et “*Pater fidelium;*” idemque peritissimus astronomus; sicut per totam Orientem adhuc celebratur:—Ἀσκηθεὶς, κατὰ τὸν παλῆον νόμον, τὰς τῶν ἐπεφανῶν ἀστέρων κινήσεις καὶ σχοῶσμεν, ὥς ἂν ἐν τέλει ἴσται το μέγαλουργόν τῆς φαινόμενης ταύτης κίσεως, ἀλλ’ ἐχει τίνα τὸν ΔΗΜΙΟΥΡΤΟΝ καὶ κινεῖν καὶ διευθυνοῖν τὴν ἐναρμόνιον τῶν ἀστέρων πορείαν, καὶ τὸ κόσμῳ πάντος τὴν κάταστασιν.

“*Exercitatus, patrio more Chaldaeorum, in astrorum motibus observandis: atque conjecturâ affecutus est, quod non in iis ipsis consistit vis magna efficiæ apparentis hujusce creationis; sed quod potius habet aliquem OPIFICEM PUBLICUM, qui et movet et dirigit harmonicum astrorum cursum et mundi universi constitutionem.*”——*Suidas.*

5. MOSES,—maximus ille legislator, “*in omni sapientia ÆGYPTIORUM instructus;*” qui institutione *Paschæ*, (Deo forsàn revelante,) cyclum illum decennovalem lunæ, celeberrimum tradidit, in computandis noviluniis, et *paschalibus* pleniluniis, apprimè necessarium; cujus ope, *Meton*, longo post tempore, *Calendarium Atheniense* emendavit.

6. DANIEL

peritorum quàm imperitorum, validitatem ejus denegantium; ac, (non obstantibus magnis illis Newtoni discipulis,—COLSON, qui anno 1736, *Newtoni Methodum Fluxionum Analyti Aequationum per series infinitas intertextam*, versione Anglicâ et commentario copioso donavit et illustravit;—MACLAURIN, qui anno 1742, *Tractatum suum Fluxionum*, Anglicè edidit, fundamentum totius methodi generalis à primordiis ad mentem Newtoni in *Lemmate Secundo*, Lib. H. Princip. explicare professus;—et STEWART, qui anno 1745, duo ipsius Newtoni celeberrima opera, *Analytin Aequationum per series numero terminorum infinitas*, et *Quadraturam Curvarum*, versione Anglicâ et commentario uberrimo instruxit;—aliisque scholæ Newtonianæ discipulis celeberrimis, *Simpson*, *Saunderson*, *Emerson*, *Waring*, &c.) adhuc vigent objectiones tam nostratium *Berkeley*, *Landen*, &c. ac nuperimè scriptoris doctissimi in libro censorio, singulis mensibus edito, cui titulus, *The Monthly Review*; quàm exterorum *D'Alembert*, *Torrelli*, *La Grange*, &c. optimorum, scilicet, expositorum *Methodi Differentialis*, et acerrimorum obrectatorum *Methodi Fluxionum*.

Et certè fatendum est, non obstantibus tot commentariis tantorum Analystrarum, qui fines Fluxionum magnoperè extenderunt, hætenùs desiderari ipsius *Textus Newtoniani*, (brevissimi ac reconditissimi,) interpretem fidum atque perspicuum, methodi et lucidi ordinis ab ipso auctore instituti strenuè sequacem, et expositione nec nimis curtâ, nec nimis prolixâ, difficilia illustrantem; (vitia, quæ utcunque sibi mutuò contraria videantur, tamen haud rarò simul et semèl inveniuntur;) ac de sensu nativo ipsius authoris eruendo, quàm de suâ eruditione aut facundiâ ostendandâ, magis sollicitum.

6. DANIEL ille *Archimagus*, in omni sapientiâ et disciplinâ *Magorum* et *Chaldeorum* instructus, atque ita celebratus ut "*sapientior Daniele*" in proverbium est evehctum, per totam Orientem, *Ezek.* xxviii. 3.—Et in Sacris Scripturis honoratissimè distinguuntur—"Noah, Daniel et Job"—quasi præpotentes *intercessores* cum DEO—*Ezek.* xiv. 14.—Noah, pro se suâque familiâ, liberatis ab excidio diluvii—Daniel, pro reditu *Judeorum* Captivorum;—et Job, pro amicis suis temerè philosophantibus, et PROVIDENTIAM DIVINAM ignavè vindicantibus.

7. PAULUS *Tarsensis*—"apud pedes *Gamalielis*, celeberrimi præceptoris *Judeorum*, educatus;" *Act.* xxii. 3. et in omni disciplinâ scholarum philosophicarum *Alexandriæ*, *Athenis* et *Romæ*, non levitèr instructus.

8. THALES, eclipsin illam solarem prædixit, quæ diremit pugnam inter *Lydios Medosque* commissam, *Julii* 30, anno 607, ante *eram Christi vulgarem*, (ut posthac forsân probare mihi dabitur occasio in *dissertatione* quâdam *De Chronologia Antiquâ*, mox edendâ—si DEUS det vitam, det opes)—epochâ, nimirum, ab astronomis chronologice hætenùs non satis notatâ. Hanc verò eclipsin, tantoperè litigatam, facile computare potuisset *Thales*, ope *Cycli* illius *Decennovialis*, ex *Judeis Chaldeisque* mutuati—à *Græcis* et *Metone*—qui accuratius constabat ex *lunationibus* 223, spatio annorum 18 cum diebus 11, peractis—quem figuratè et aptè innuit *Theocritus*, *Idyll.* XV.

Οὐλὼναιδεναιὲς ἢ εννεκαιδεν' ὁ γαυρὸς—

De nuptiis Adonidis loquens, seu conjunctionibus Solis et Luna—"oëdecimo vel nonodecimo anno."

9. PYTHAGORAS, ille *Samius*, cujus allegoriam eximiam de semitis vitæ divergentibus ad veritatem vel ad errorem, per furculas literæ T, designatis, notat *Perfius*, *Sat.* III. 56.

Et tibi quæ SAMIOS deduxit litera ramos,
Surgentem DEXTRO monstravit limite callem.

Quam lucubentissimè exponere non dedignatus est ipse JESUS CHRISTUS, *Matt.* vii. 13.—Verum quoque mundi systema, seu *Copernicanum*, aut detexisse, aut saltèm ex *Chaldeis* didicisse, videtur.—Vide *MacLaurin's Account*, *Chap.* 2.

15. Audi.

15. Audiamus THE MONTHLY REVIEW, in eruditâ admodum recensione suâ libri cui titulus *Theorie des Fonctions Analytiques*, par La Grange, Paris 1798. Art. 1. Append. Vol. 28.

“*The Method of Fluxions* had never so acute, so learned, and so judicious a defender as *Maclaurin*.—His work is most valuable for the variety of information which it contains, and for many species of clear and forcible argumentation: yet whoever consults it, and sees the various *artifices* which he is obliged to employ for the purpose of demonstrating the several parts of the method; and the grounds of it not established in less than *fifteen long theorems*; and, in an enquiry *purely mathematical*, finds the author speaking of “causes and effects,” “of the springs and principles of things,” and proposing to deduce “the relations of quantities by comparing the powers which are conceived to generate them;”—will be convinced that this could only happen from so able a mathematician having failed to seize *the right principle* and the *true mode of explanation*.”

16. Atque hujus acutissimi censoris sententiæ multis nominibus accedo: nam, primò, à verâ interpretandi viâ deflectens, *Maclaurinus* illustrationibus geometricis, à naturâ *motûs* petitis, nimis immoratur in libro primo; longo sermone luxurians, et argumentationis ambagibus ad fastidium usque indulgens, à genio *Newtoniano* prorsus alienis; qui *methodum suam Fluxionum* exco-gitavit, “*ut effugeret tædium deducendi longas demonstrationes, more veterum geometrarum, ad absurdum*;”—et “*contractiores*” his, et simul accuratiores illis quæ per *methodum indivisibilium* à Cavallerio ejusque discipulis eliciebantur, magno scientiarum emolumento, substituit. Contrâ, dum brevis esse laborat, obscurus fit *Maclaurinus* in libro secundo; et, quasi interpretandi pertæsus, primo hujus capite, stricte tantum, principiis veris ac genuinis methodi universalis, (ex naturâ *relationum* quantitatum variabilium abstractè consideratarum, manantibus,) vix, aut leviter tantum notatis, *elementa prima* seu *metaphysica*, à *Newtono* posita, festinanter transcurrit; et *principia mathematica* quoque nimis et præproperâ celeritate absolvit; et denique, ipsius inventoris lucidum ordinem statim ab initio deferens, multa mutat; et quæ mutat, corrumpit: nam,

2. Prop. 1. Totius theoriæ suæ fundamentalem, § 707, “*fluxionem quadrati A^2 , æquari $2aA$* ,” non nisi imperfectè demonstrat; unde enim hanc fluxionem *assumpserat* (c), minimè ostendens, transit, per saltum, ad demonstrandam veram fluxionem hâc, nec *majorem* esse, nec *minorem*, argumento ad absurdum prolixiore usus.

(c) The defect of *Maclaurin*’s demonstration may thus be supplied, from his own principles rightly conceived, though rather confusedly expressed, § 701—706.

§ 707. Prop. 1. The fluxion of the root A being supposed equal to a , the fluxion of the square AA will be equal to $2aA$.

Let $A - a$, A , and $A + a$, be successive values of the variable root; and the corresponding values of the variable square will be $A^2 - 2aA + a^2$, A^2 , and $A^2 + 2aA + a^2$; of which the first

usus. Et inde, § 708, "*fluxionem rectanguli* $A + B$ *æquari* $aB + bA$," pro corollario, deducit. At artifex ipse, longè elegantius atque peritiùs, auspicando à *casu primo*, et generaliore,—*fluxione nempe rectanguli*, inde particularem,—*fluxionem nempe quadrati*,—facillimè deducit, *Caf. 3.*

3. Per varios anfractus procedens, *lemmata* compluria, ad demonstrandas propositiones insequentes, tortuosè introducit *Maclaurinus*, à quibus immunis erat *Methodus Newtoniana*.

4. Ad methodum inventoris jam serò reversus, quam adeò infelicitè initio deferuerat, brevius et melius demonstrat "*fluxionem* A^n *æquari* naA^{n-1} ," more *Newtoniano*, § 716, quam imperfectè et prolixè demonstraverat antea, § 712, secutus *Prop. 1.*

5. Negligentiâ minimè condonandâ, *Theorematis illius generalissimi, Lem. II. Lib. II. Princip. ultimum casum*, quem à præcedentibus, directo atque immediato nexu, summo acumine, summâque peritiâ Algebraicâ, deduxerat *Newtonus*, nempe, *fluxionem* $A^m B^n$ *æquari* $maB^n A^{m-1} + nbA^m B^{n-1}$, prorsus omittit; itémque *tria corollaria* insequentia; elegantissima atque uberrima quidem, quæ calculo fluxionali concinnando, ubi de *proportionalibus* agitur, et de *logarithmis* præsertim, magnoperè inserviunt, eâdem negligentiâ omisit.

is the least, and the last the greatest. From this last value of the square subtract the first, and, the extreme terms A^2 and a^2 being destroyed, there will remain the *excess* $+4aA$; which will denote the *whole* increase of the square from the first $(A-a)^2$ to the last $(A+a)^2$, with the increment a , *twice*, or $2a$; (which is the difference between the extreme values of the root, $A-a$ and $A+a$); and consequently *half the excess*, namely $2aA$, will denote the increase with the increment a *once*; or will *measure the rate of variation*, or be the *fluxion* of the variable square, from its *least* to its *mean* state AA .

If you deny it, let the true fluxion be either greater or less than $2aA$, by any assignable difference D , or equal to $2aA \pm D$. But if greater, the last increment must necessarily be greater than $+2aA + a^2$; if less, the first increment must be less than $+2aA - a^2$ (or the difference between the first and mean values of the variable square), contrary to the hypothesis. Q. E. D.

In numbers the case is plain:

Let the root $A = 10$; and its momentary increment $a = 1$.

Val.	Diff.
$(A - a) \times (A - a) = 9 \times 9 = 81.$	} 19
$A \times A = 10 \times 10 = 100$	
$(A + a) \times (A + a) = 11 \times 11 = 121$	} $\frac{21}{40}.$
But $4aA = 4 \times 10 = (40) = 19 + 21.$	
Therefore $2aA = 2 \times 10 = (20) = \frac{19 + 21}{2}.$	

Consequently $2aA$ expresses the *fluxion* or *rate* of increase of the variable square in its advance to the mean state $A \times A$.—The demonstration might be abridged upon *Newton's* principles, by taking only the half increment:—For the least value $(A - \frac{1}{2}a)^2$ becomes $A^2 - aA + \frac{1}{4}a^2$, and the greatest, $(A + \frac{1}{2}a)^2 = A^2 + aA + \frac{1}{4}a^2$. But the excess, $+2aA$, gives the *fluxion* at once, during the *whole* momentary increase. See § 84 and 85.

17. Diluendis igitur objectionibus eorum qui methodi fluxionum generalis evidentiam atque certitudinem impugnant, omninò imparem esse commentarium *Maclaurini*, quamvis valdè eruditi, et in hisce scientiis versatissimi, et (quod auctoritatem majorem ipsi parere debere super hoc argumento videatur,) in ipsius *Newtoni* amicitiam (d) admissi, quis mathematicus non videt?—At non levi detrimento theoriæ fluxionum, iste commentarius fusior, qualis qualis fuerit, mox supplantavit originalem textum, paulò subtiliorem atque difficiliorem intellectu, ob *imperatoriam* auctoris brevitàem, non nisi auribus rite purgatis capiendam—

ΦΑΝΩΤΟΣ ΟΥΝΕΤΟΙΟΙ.

Ex hoc fonte igitur profluxerunt vel perspicaciorum peritiorumque analytarum objectiones, scholæ *Leibnitzianæ* præsertim, adversus ipsam fluxionum methodum; quasi methodo *differentiali*, evidentia et certitudine, cedentem, et in principiis à geometriâ alienis fundatam.

18. Audiamus *D'Alembert*, analytam adèò celebratum, in *Mathesi* perspicacissimum, in religione hebetissimum, argumento paulò subtiliore validitatem exponendi fluxiones per velocitates, impugnantem:

“Introduire ici le mouvement, c'est y introduire une idée étrangère, et qui n'est point nécessaire à la démonstration: d'ailleurs, on n'a pas d'idée bien nette de ce que c'est la vitesse d'un corps à chaque instant, lorsque cette vitesse est variable. La vitesse n'est rien de réel, c'est le rapport de l'espace au tems, lorsque la vitesse est uniforme; mais lorsque le mouvement est variable, ce n'est plus le rapport de l'espace au tems, c'est le rapport de la différentielle de l'espace à cette

(d) The following letters of *Newton* reflect high honour on his liberality of mind and delicacy of friendship towards the excellent *Maclaurin*, when a candidate for the joint *Professorship of Mathematics* in the University of *Edinburgh*.—In one of these letters, addressed to Mr. *Maclaurin*, (with permission to shew it to the Patrons of the University,) Sir *Isaac* expresses himself thus:

“I am very glad to hear that you have a prospect of being joined to Mr. *James Gregory* in the Professorship of the Mathematics at *Edinburgh*, not only because you are my friend, but principally because of your abilities; you being acquainted as well with the new improvements of Mathematics, as with the former state of those sciences: I heartily wish you good success, and shall be very glad of hearing of your being elected. I am, with all sincerity, your faithful friend and most humble servant.”

In a second letter, which was addressed to the then Lord Provost of *Edinburgh*, (which Mr. *Maclaurin* knew nothing of till some years after Sir *Isaac's* death,) he thus writes:

“I am glad to understand, that Mr. *Maclaurin* is in good repute amongst you for his skill in Mathematics; for I think he deserves it very well: And, to satisfy you that I do not flatter him, and also to encourage him to accept the place of assisting Mr. *Gregory*, in order to succeed him, I am ready (if you please to give me leave,) to contribute twenty pounds per annum towards a provision for him, till Mr. *Gregory's* place becomes vacant, if I live so long, and I will pay it to his order in *London*.”

See an *Account of the Life and Writings of Maclaurin*, prefixed to his *Account of Sir Isaac Newton's Philosophical Discoveries*, in which are many curious and valuable particulars of the life of that CHRISTIAN PHILOSOPHER, especially in his last moments; so different from those of that miserable infidel *D'Alembert*.—APPENDIX II. § 161 & 164.

du tems; rapport dont on ne peut donner d'idée nette que par celle des *limites*. Ainsi il faut nécessairement en revenir à cette dernière idée, pour donner une idée nette des *fluxions*." —

19. Quinquaginta retrò annis circitèr, talia objecit vir clarissimus D'Alembert, cujus auctoritas nunc tanti æstimatur. Ac nuperrimè quidem, ("vires enim acquirunt eundo,") maximus ille promotor Algebrae puræ, nec non analyseos sublimioris, De La Grange, in opere quodam elementari, cui titulus, THEORIE DES FONCTIONS ANALYTIQUES, 1798, Paris, autumat se ferò tandem vera et genuina principia calculi differentialis extudisse, ab omni consideratione sive quantitatum infinitè parvarum, seu evanescentium, sive limitum vel fluxionum, libera atque immunia; usurpando tantùm Quantitatum finitarum Analysin Algebraicam.

20. Hiscè magni nominis viris consentiens Landen nostras, ita doctrinam fluxionum perstringit in *Analysi suâ residuâ*, anno 1758—Analysta et ipse ex peritissimis.

"It seems more proper, in the investigation of propositions by *Algebra*, to proceed upon the *anciently-received principles* of that art, than to introduce therein, *without any necessity*, the *new fluxionary principles* derived from a consideration of *motion*: and the rather, as the introduction of these new principles is not attended with any *peculiar advantage*."

21. Tantis opponentibus accedit quartus,—nimirum censor ille, in hiscè scientiis non levitèr versatus, (quisquis fuerit,) apud THE MONTHLY REVIEW, *Append.* Vol. 28, 1799,—ex cathedrâ moderatoris, ita pronuncians judicium, in *Newtonum* ejûsq; commentatores et discipulos:

"Of the *Method of Fluxions*, properly so called, *Newton* is the sole and undisputed inventor; to him, beyond all doubt and controversy, is to be attributed the *establishment* of that doctrine on the consideration of *Motion*."

"Had not *Newton* been the inventor of *this* method of considering Fluxions, we are of opinion, that very few persons would have contended either for its *metaphysical* or *mathematical* excellence: for, on the ground of perspicuity and evidence, the understanding is not much assisted by being directed to consider *all* quantities as generated by *Motion*. *Lines* we can conceive generated by the motion of *points*; *surfaces*, by the motion of *lines*; &c. But when such quantities as *Weight*, *Density*, *Force*, *Resistance*, &c. become the objects of enquiry, and are said to have their *flux* and *velocity*, then the true end of the *figurative* mode of speech—*Illustration*—is lost: On the ground of *mathematical* convenience—that the definition of Fluxions, as given, is commodious to deduce thence their properties and affections—nothing is gained; since the enquiry is reduced "*ultimately*" (*en dernière analyse*) to the determination of *the ultimate ratio* of vanishing quantities, or of *the limits* of such quantities: which latter method is only the former, *algebraically* expressed.

"That which happened to *Aristotle* has happened to *Newton*: His followers have bowed so implicitly to his *authority*, that they have not exercised their *reason*.—If the *English* Mathematicians first adopted *Newton's* Method from veneration to him, or *for want of better information*, they have since persevered in it, (WE may almost say) *against conviction*.—

"It has been said, that *the true principles of any invention are most satisfactorily and truly stated by the inventors themselves*;—yet it is at the distance of more than a hundred years from the discovery of the *Fluxionary Calculus*, that its principles are first clearly and rigorously established. It has been reserved for the Mathematicians of *these days*, to effect what was denied to a *Newton* and a *Leibnitz*; and to supply to their theory that evidence and exactness, the want of which makes us attribute its invention rather to the felicity of the times in which they lived, than to the excellence of their genius:—to esteem it, not *pro partu ingenii*, but *pro partu temporis*."

22. Eruditorum fanè graves sunt hæ objectiones, et gravitèr pronunciatæ. Veruntamen, non obstantibus tantis iudiciis, et posthabita omni *authoritate*, (quæ in MATHESI minimè valere debet,) fidentèr affirmare ausim, propriis ex investigationibus, *Nullius additus jurare in verba magistri*—à nemine, ne vel ab ipso *De La Grange*,—evidentiùs, accuratiùs, solidiùs, invictiùs et generaliùs, strui fundamentum totius *Theoriæ Fluxionum*, quàm à celeberrimo inventore, *sagacissimi ingenii* viro; si modò, missis aliorum commentis, adeamus ipsius *textum originale*; nam, si attentè et circumspèctè hic textus inspiciatur, et, contextu examinato, cæterisque auctoris documentis criticè collatis, probè intelligatur, terminis technicis aptè definitis, atque ellipsis ritè suppletis, (quæ causæ sunt ex quibus præcipuè difficultates in *Newtoni* scriptis oriuntur) censebit, nì fallor, omnis æquus peritùsque controversiæ hujus judex, objectiones hactenùs prolatas, vel gravissimas, ex incitiâ potiùs, ex imperitiâ, præjudicio, vel livore, quàm ex justis ratiociniis et veritatis perceptione ortas esse; imò in ipsos objectores jure retorqueri posse.

23. Analystas *exteros* quidem, eosdèmq; *Gallos*, *nomini Britannico* olim infensos et nunc acerbissimè,—*D'Alembert* aut *De La Grange* (e)—ipsius *Newtoni* æmulos minimè impares, de laude ejus detrectare voluisse, non est mirum; qui
nec

(e) DE LA GRANGE.—Quindecim retrò annis, hæc notabam fidus *NEWTONI* discipulus. ANALYS. ÆQUAT. § 259.

"Ex dictis facillè judicabit lector, an valent multi nominis viri *La Grange* objectiones. *Dissert. Acad. Berlin*, an. 1768: nempe quod methodo *Newtoniana* [extrahendi radices æquationum quàm proximè, in *Analysi Æquationum per series infinitas*, quàm demonstratione generali munieram § 249—259.] ignoratur quantum ad veritatem accedit tum valor (*k*) primò assumptus, tum valor cuiusque complementi insequentis (*p*), (*q*), &c.—Methodus certè quam ut accuratiorem commendat, *Newtonianæ* facilitate longè cedit; quippe quæ ex inventione quantitatis $\sqrt{I}$, § 234, non nisi per tentamina,

nec rarò nec parcè, quamvis furtim, inventiones ejus reconditissimas enucleare compotes, ad se suòque transfulerunt: Cæterùm hîsce temerè consentire Analystas Britannicos, auctoritati alienæ ignavè succumbentes, pudet dolètque. Horum sanè adscribi ordinibus *Landen* patiamur forsàn, qui suis studiis, suisque opinionibus nimis et pertinacitèr erat addictus, ita quòd *Analysin* suam *residualem*, (quâ tantoperè delectabatur,) ipsi *Methodo Fluxionum*, (cujus larva tantùm nabenda sit, ut mox ostenditur,) arrogantèr et falsò anteposuerit; et qui, cum Analystæ, hujus ævi, celeberrimi, *D'Alembert* et *Euler*, adstipulantibus *Wildbore* et *Frisk*, communi consensu, ex diversis principiis, determinârunt eandem veram quantitatem *motûs corporis circâ axem liberè rotantis, ubi actione vis cujusdam extraneæ, aut percutientis aut accelerantis, perturbatur*—suam conclusionem, ab eorum conclusione non modicè dissentientem, eorum auctoritati quadruplici opponere ad mortem usque non desit; pertinaciæ exemplum præbens, quòd maximis ingeniis haud rarò contingit, plus æquo “*sibi fidentibus aliâque spernentibus*.” At verò *censorem publicum*, apud M. R.; ipsûmque, ut ferunt, *Professorem Matheseos Academicum*, ita peccâsse, minimè condonandum est. Valdè equidem suspicor, illum, *λογω ἀγγυρ*, “*ignavâ ratione*” seu negligentia, ex nimia et præproperâ mentis festinatione, hæc opinionum commenta

tentamina, valdè laboriosa in æquationibus altiorum demensionum, eruendæ, auspicatur; per inventionem *Limitum Æquationum* ex binomiis compluribus successivè substituendis, procedit; et per combinationem plurium *fractionum continuarum*, ex his operationibus elicitarum, concludit.”

Quantâ autem comitate, quanto animi candore sustinuit *Vindicias* hæcæ *Newtonianas*, haud temerè excitas, animadversionesque certè non leves; ex insequenti patebit epistolâ, quam mihi, homini obscuro, post acceptam *Analysin Æquationum*, scribere non dedignatus est vir clarissimus. Hanc verò, in seriniis meis hætenûs latentem, edere jam libet, minimè jactantiæ gratiâ,—studiis meis diù conversis, *MATHESE* relicta, in *MUSAS HÆBRÆAS*—“*quarum sacra sero, ingenti percussus amore*”—“*smut with the love of sacred song*”—sed quâ par est observantiâ; ne cuius *Vindicias secundas* legenti nascatur suspicio, me, animo utcumque infenso, similitate quâvis, aut odio hostili, denegare velle grata “*præmia laudi*” tam insigni: quem ex peritissimis Newtoni successoribus, (excepto forsan *La Place*) maximèque *Algebra puræ* et *Analyses sublimioris* hoc ævo cultoribus, facile principem, concedere cogor *Britannus*, et non inficiabitur æqua posteritas:

MONSIEUR,

Berlin, Nov. 1. 1785.

Agréez mes très sincères remerciemens de l'honneur que vous m'avez fait en m'envoyant votre ANALYSE DES ÆQUATIONS; que j'ai lue avec toute la satisfaction possible. La clarté et la précision qui regnent dans cet ouvrage, et la réunion qu'il présente des METHODES ELEMENTAIRES et des THEORIES SUELMES doivent lui donner un des premiers rangs parmi ceux de son genre; et par l'intérêt que je prends aux progrès de la GEOMETRIE, je partage la reconnoissance que vos COMPATRIOTES vous doivent pour ce travail. Je desirerois fort pouvoir mériter l'opinion avantageuse que vous voulez bien avoir de moi, mais je ne puis y répondre, que par la véritable estime, et la considération distinguée avec laquelle, j'ai l'honneur d'être,

MONSIEUR,

Votre très humble et très obéissant serviteur,

DE LA GRANGE.

Neque existit ratio quævis, cur (in Mathematicis saltèm) animo infenso judiciis censorum eruditum M. R. meum opponerem, qui satis supèrque laudabant *ANALYSIN ÆQUATIONUM* in recensione tractatûs illius, eodem circitèr anno; sed cum ambitiosè in lucem proferuntur *Exteriorum* opera Mathematica—*Cursus Mathematici Sauri et Bezart*; *Institutiones Analyticæ Euleri et Clairaut*, *De La Place* et *De La Grange*; at *Nostratum*, fastidiosè in umbram relegantur, iniquâ forsan comparatione, à censoribus *Britannis*; cur non mihi æquo saltèm jure liceret vicissim, asserre tanti censoris *Galli* testimonium, de unâ saltèm institutione elementariâ—etiâ *Hibernâ*?

ex aliis potius temerè assumpsisse: idque vel ex meâ ipsius fortunâ in hâc re verè simile mihi videtur, cui contigit adolescenti *Rudimenta Fluxionum* ex *Maclaurin's Treatise of Fluxions* didicisse; et qui non nisi anno superiore (1798), tandem, cum fortè curiosius ipsum textum *Newtonianum* inspexissem, atque ipsius auctoris simplicitatem elegantiamque demonstrandi, cum commentatoris prolixitate et ambagibus, contulissem, sententiam super hâc re mutare sum co-actus, et à commentatoris partibus in ipsius auctoris partes transire, et cum admiratione exclamare,

*Homo homini quid præstet!—Inventor
Interpreti quanto antecellit!*

24. Et mecum quidem et sentit et loquitur *Colson*, (*Methodi Fluxionum non sordidus auctor*) hîcce verbis:

“The principles of the method here taught have been *scrupulously examined* and sifted, have been *vigorously opposed*, and, we may say, *ignorantly rejected* as insufficient, by *some* mathematical gentlemen, [*Berkeley, &c.*] who seem not to have derived their knowledge of them from *their only true source*; that is, our *author's own treatise*, written expressly to explain them.”

25. Placet igitur, resuscitatis hodiè objectionibus obsoletis, novisque ingruentibus, et per *Angliam*, orbemque literarum, à *Galliâ* grassantibus, totam methodum fluxionum ab ipsis fontibus, tam *metaphysicis* quam *mathematicis*, ab integro ad examen rigidum revocare; ut, his perspectis et per scrutinium investigatis, ne diutius falsis laboret criminibus evidentia ejus et certitudo; neque *injurioso pede proruant stantem columnam laudis NEWTONIANÆ*, vel imperiti vel invidi, sublimem et subtilissimam methodum ex sacris fontibus ANALYSEOS ANTIQUÆ haustam promotamque, quasi “*partum temporis*” abortivum, ex elementis incongruis conflatum, atque caducum, insimulantes, et non potius “*partum ingenii*,” divinitus conceptum, atque ad statum parturæ maturum atque perfectum, plenitudine temporis in lucem editum, *hæredem immortalitatis*, venerantes.

26. Nec minùs urget impellitque sollicitudo, curaque non levis, NE QUID DETRIMENTI CAPIAT RESPUBLICA—influxu hoc infauſto *opinionum Gallicarum*, tanto studio, tantisque illecebris, tantoque auctoritatis pondere, PHILOSOPHIAM BRITANNICAM invadentium; et nisi maturè et strenuè et peritè, “*zelo secundum scientiam*” repellantur animosè, et rejiciantur indignantèr, IPSAM unà cum RELIGIONE ILLA, (cui modèstè et sobriè hæctenus ancillari solita est) sub erroris et impietatis fluctibus obruere minitantium.

ESTO PERPETUA!
STATE SUPER VIAS ANTIQUAS!
PRO FIDE SANCTIS COMMISSA, CONTENDITE!

27. Nec

27. Nec operam JUVENTUTI STUDIOSÆ, (nostrati potissimum) ingratam aut inutilem me præstiturum spero, si studia *Mathematica*, jamdudum intermissa, recolens, SUBSIDIA QUÆDAM ELEMENTARIA ad *Physicam Newtonianam* (et præsertim *Astronomiam*) PHILOSOPHIÆ PRIMÆ, SACRISQUE LITERIS, optime contentientem, pro virili, breviter, fideliter, et tamen simpliciter perspicuèque, proponere aggrediar :

“ To illustrate *Newton* is to cultivate *real science*; and to make his discoveries easy and familiar, will be no small improvement in MATHEMATICS and PHILOSOPHY.”——*Colson*.

De Methodo Rationum Primarum et Ultimarum, seu Limitum.

28. TOTIUS Methodi Fluxionum fons atque origo est doctrina *Rationum Primarum et Ultimarum*, si Geometricè loquamur, seu *Limitum*, si Algebricè; ut rectè perhibent *D'Alembert* et *Censor* ille professorius, *Newtonum* ipsum vel secuti, vel cum eo (in scii forsàn,) consentientes; qui ita loquitur :

——“ Præmissi verò hæc *Lemmata*, ut effugerem *tedium* deducendi longas demonstrationes, more veterum Geometrarum, ad absurdum. Contractiores enim redduntur demonstrationes per *methodum indivisibilem*: sed quoniam durior est *indivisibilem* hypothesis, et propterea methodus illa minus geometrica censetur, malui demonstrationes rerum sequentium ad ultimas quantitatum evanescentium summas et rationes, primasque nascentium; id est, ad limites summarum et rationum deducere; et propterea limitum illorum demonstrationes, quâ potui brevitate, præmittere: his enim idem præstat quod per *methodum indivisibilem*; et principiis demonstratis jam tutius utemur.”——“ Vimque talium demonstrationum ad *methodum præcedentium lemmatum* semper revocavi.”

29. Pendet sanè omnis hæc *Methodus Limitum*, ex *Lemmate primo*, Lib. I. Princip.

“ QUANTITATES, ut et QUANTITATUM RATIONES quæ ad æqualitatem tempore quovis finito constantè tendunt; et ante finem temporis illius propius ad invicem accedunt quàm pro datâ quâvis differentiâ; sunt ULTIMO ÆQUALES.”

“ Si negas, fiant ultimò inæquales; et sit earum ultima differentia *D*: Ergò, nequeunt propius ad æqualitatem accedere, quàm pro datâ differentiâ *D*: Contrà hypothesin.”

Q. E. D.

Ultima

Ultimæ rationes igitur, ipso definiente *Newtono*, et quàm accuratissimè, et distinctissimè, sunt "*limites, ad quos quantitatum sine limite decrecentium rationes, 1. semper appropinquant; et 2, quas propius assequi possunt quàm pro datâ quâvis differentiâ; 3. nunquàm verò transgredi; 4. nequè prius attingere quàm quantitates diminuuntur vel augentur in infinitum.*"

30. Sedulò autem et sollicitè distinguit *Newtonus* inter harum quantitatum magnitudines ultimas, et rationes ultimas.—"In sequentibus igitur si quando facili rerum conceptui consulens, dixerò quantitates quàm minimas," vel "*evanescentes*" vel "*ultimas*," cave intelligas quantitates magnitudine determinatas sed cogita semper diminuendas *sine limite*."—Atque iterum:—si quando quantitates tanquam *ex particulis constantes* consideravero, vel si pro *reâis* usurpavero *lineolas curvas*; nolim *indivisibilia* sed *evanescentia divisibilia*, non summas et rationes partium determinatarum, sed summarum et rationum *limites* intelligi.

31. Nec minùs sollicitè diluit *Newtonus* objectiones impugnantium evidentiam et certitudinem doctrinæ limitum.

Obiectio est "*Analystæ*" (*Berkley*), quòd "*quantitatum evanescentium nulla sit ultima proportio*; quippe quæ antequam evanuerunt non est ultima; ubi evanuerunt, nulla est."—Sed et eodem argumento, regerit *Newtonus*, æquè contendì posset "*nullam esse corporis ad certum locum ubi motus finiatur pervenientis, velocitatem ultimam*: hanc enim, antequam corpus attingit locum, non esse ultimam; ubi attingit, nullum esse." At responsio facilis est: Per "*velocitatem ultimam*" intelligi, eam, quâ corpus movetur—neque antequam attingit locum ultimum et motus cessat, neque postea—sed tunc, cum attingit; id est, illam ipsam velocitatem quâcum corpus attingit locum ultimum, et quâcum motus cessat. Et similiter, per "*ultimam rationem quantitatum evanescentium*," intelligendam esse rationem quantitatum, non antequam evanescent, non postea, sed quâcum evanescent. Paritèr et *ratio prima nascentium*, est ratio quâcum nascuntur. Et *summa prima et ultima*, est quâcum esse (vel augeri aut minui) incipiunt et cessant. Extat *limes* quem velocitas in fine motûs attingere potest, non autem transgredi; hæc est velocitas ultima: Et par est ratio limitis quantitatum et proportionum omnium incipientium et cessantium. Cùm hic *limes* sit certus et definitus, problema est verè geometricum eundem determinare. Geometrica verò omnia in aliis Geometricis determinandis ac demonstrandis, legitimè usurpantur.

32. "Contendì etiàm potest, quòd "*si dentur ultimæ quantitatum evanescentium rationes, dabuntur et ipsæ magnitudines ultimæ.*" Hæc verò consequentia non est necessaria. Nam fieri potest ut claram ideam habere possimus *relationis* ultimò intercedentis inter duas quantitates quàm minimas, licet *absoluta* magnitudo ipsarum quantitatum nec inveniri nec concipi potest. Res clariùs intelligetur in infinitè magnis: si quantitates duæ quarum data est differentia, augeantur in infinitum, dabitur harum ratio ultima, nempe ratio æqualitatis; nec tamen idcirco dabuntur et ipsæ quantitates ultimæ, seu maximæ, quarum ista est

ratio: hæ enim, cùm in immensum excreverint, omnem calculum, imò omnem *absolutæ* magnitudinis conceptum, longè transiliunt et prorsus refugiunt.

33. Illustrari possunt hæc, paulò subtiliùs forsàn et abstractiùs disputata quam pro tyronum captu, ex considerationibus *Algebraïcis* et *Geometricis*, et *Aritbmeticis*.

1. In reductione æquationum *quadraticarum*, *Analys. Æquat.* § 30. prodiit duplex valor radicis, nempe $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$, eritque valor uterque realis, ubi $\frac{1}{4}p^2$ fit *major* quam q ; uterque verò impossibilis, ubi fit *minor*; et ipso in *limite*, ubi $\frac{1}{4}p^2$ fit *æqualis* q , erit unicus valor radicis; evanescente jam quantitate radicali $\sqrt{\frac{1}{4}p^2 - q}$.

2. In resolutione *cubicarum* quoque, § 307, ipso in *limite*, ubi $\frac{3}{4}p^2$ fit *æqualis* q , duplex valor factoris quadratici, $x = -\frac{1}{2}p \pm \sqrt{-\frac{3}{4}p^2 + q}$ in unicum pariter coalescit.

34. Et ipse *Landen*, in demonstrationem suam *Theorematis Binomii*, purè *Algebraïcam* (ut refert) clanculùm introducit methodum *limitum*, seu fluxionum, ubi fingit quòd “*salvâ æquatione variari potest y ad libitum, donec fiat y = x.*”—*ANALYS. ÆQUAT.* p. 38.

Nec minùs *De La Grange*, naturam radicum æquationis optimè exponens, et theoriàm æquationum *Algebraïcarum* peritissimè promovens, ut docuimus *ANALYS. ÆQUAT.* § 145—148, methodum *limitum* feliciter fanè introducit.

Hinc pendet omnis inventio *limitum* vel *radicum æquationis*, vel extremarum, vel intermediarum; quorum inventio investigandis ipsis radicibus apprimè utilis est, per varias approximandi methodos.

35. In *sectionibus conicis* quoque luculentissimè obtinet *methodus limitum*; inde enim pendet omnis theoria *circularum* et *parabolarum* curvas alias *osculantium*, ingenti cum commodo theoriæ motuum planetarum atque cometarum adhibita, à summis *Analysis*, *Newtono*, *Halley*, *Clairaut*, &c.

Nam, si concipiantur singulæ *sectiones conicæ* gigni ex intersectionibus plani per axem conì perpetuò rotantis, cum superficie conicâ vel superficiebus conicis:

1. Sectio plani generantis, aut paralleli ad basin conì, aut subcontrariè positi, erit *circulus*.

2. *Circulus*, rotante plano, statim migrabit in *ellipsin*; cujus eccentricitas, rotando, continuè augebitur, donec planum fiat parallelum lateri opposito conì ipsius; in hòc verò statu mutationis, ellipsis vicissim, in *parabolam*, centro in infinitum abeunte, migrabit: deinde, rotante adhuc plano, parabola statim migrabit in *hyperbolam* atque *hyperbolas oppositas*, centro sectionis jam extrinsecus translato.

translato inter vertices; et hyperbolarum eccentricitas augebitur, donec, plano cum ipso axi tandem coincidente, hyperbolæ oppositæ in *triangula verticalia* migrabunt. Atqui eadem contingent, sed ordine contrario, donec, totâ rotatione peractâ, restituitur *circulus*.

3. Hinc constat *circulum* esse limitem nascentium ellipsium; *parabolem*, evanescentium ellipsium et nascentium hyperbolarum; et *triangulum*, evanescentium hyperbolarum. Sed calculus limitum longè facilius prodit quàm calculus sectionum intermediarum; quoniam *circuli* omnes sunt figuræ similes; necnon *parabolæ omnes*; sed singulæ ellipsium vel hyperbolarum species, rotando genitæ, sunt inter se dissimiles, et diversos calculos, nempe, unaquæque species calculum sibi proprium, requirunt.

36. Ex *rationibus arithmetiis*, hætenus posita quoque illustrare fas est.

1. Si *data quantitas* ($2a$) dividatur *inæqualitèr*, *exponente* (x) *partem intermediam variabilem*; erit *ratio facti sub partibus inæqualibus* $(a + x) \times (a - x)$ *ad quadratum semissis*, *ratio minoris inæqualitatis*; propter defectum quadrati x^2 . Per *Prop. 5. Lib. 2. Elem.*

Primò quidem, *ratio nascentis facti seu rectanguli*, (quantitate quàm maximè *inæqualitèr* divisâ) ad hoc quadratum, ferè infinita evadit, propter x ferè æqualem a ; dein, *decescente* x , *ratio rectanguli ad quadratum* perpetuò accedet ad æqualitatem; et ultimò, *evanescente* x , ipso in limite, fiet *ratio æqualitatis*.

Ex. gr. Sit $a = 101$, et primò, *variabilis* $x = 100$; erit *rectangulum ad quadratum* in ratione 201×1 ad 101×101 ; seu 201 ad 10201 , seu 1 ad 50.7 , hoc est, in ratione longè minore. Dein, sit $x = 10$, et *ratio rectanguli ad quadratum* jam fiet 111×91 ad 101×101 ; seu 10101 ad 10201 , seu 1 ad 1.009 ; propè ad æqualitatem accedente. Postremò sit $x = 1$; evadèturque *ratio*, 102×100 ad 101×101 , seu 10200 ad 10201 , seu 1 ad $1.000,09$, proximè ad æqualitatem accedente.

37. "Res clariùs intelligetur in infinitè magnis."

Si *eadem quantitas* ($2a$) *perpetuò augeatur*, *designante jam* x , *augmentum variabile*, erit *ratio rectanguli* $(2a + x) \times x$ *ad quadratum* $(a + x)^2$, *ratio quoque minoris inæqualitatis*. Propter defectum quadrati constantis, a^2 . Per *Prop. 6. Lib. 2. Elem.*

In hoc casu quoque, *crescente* x , *ratio rectanguli ad quadratum*, perpetuò ad æqualitatem accedit; et ultimò, *in infinitum aucto* x , fit *ratio æqualitatis*.

Ex. gr. Sit $a = 1$; et primò sit *variabilis* $x = 1$; erit *rectangulum ad quadratum* in ratione 3×1 ad 2×2 ; seu 3 ad 4 , seu 1 ad 1.3333 , &c. Dein, sit $x = 10$; evadèturque *ratio* 12×10 ad 11×11 ; seu 120 ad 121 , seu 1 ad 1.008 ; propiùs ad æqualitatem accedens: denuò sit $x = 100$; et fiet *ratio* 102×100 ad 101×101 ; seu 10200 ad 10201 , seu 1 ad $1.000,08$; longè

longè propiùs quàm in casu ultimo. Jam verò sit $x = 1000$; et ratio ejusdem rectanguli ad dictum quadratum quam proximè ad rationem æqualitatis accedet; erit enim æqualis rationi 1002×1000 ad 1001×1001 , seu 1002000 ad 1002001 , seu 1 ad $1,0000008$. Denique sit valor x *infinitè magnus*; et fiet ultimò ratio rectanguli ad quadratum ratio æqualitatis; differentiâ constanti, $a^2 = 1$, tandem pro nihilo evadente respectu magnitudinum ipsarum adedò immensarum: hæc igitur differentia *quasi* evanescit, licèt nunquam *actu* evanescit. Claram igitur habere possumus ideam *relationis* inter duas quascunque magnitudines utcunque auctas vel diminutas in infinitum, licèt ipsarum magnitudinum *absolutarum* ideam determinatam seu positivam nullatenus effingere fas sit, mortalibus, *finitas* tantùm *facultates* sortientibus.

38. Quò meliùs intelligatur natura *rationum primarum et ultimarum*, quam hìc pressius exposui; principia *Newtoniana*, brevissima et subtilissima quidem, quâ potui fidelitate evolvendo; placet insupèr sententias interpretum fusiùs adducere; non solùm illustrandi gratiâ, verùm etiâ ex abundanti confirmandi hætenus exposita, atque obrectatorum inscitiam plenissimè redarguendi, ex peritissimis *Newtoni* discipulis.

I.—Colson's Account of the Method of Fluxions.

39. "THE chief principle upon which the Method of Fluxions is built is this very simple one, taken from the *Rational Mechanics*; which is, that *mathematical quantity*, particularly *extension*, may be conceived as generated by continued *local motion*; and that *all quantities* whatever, (at least by *analogy* and *accommodation*,) may be conceived as generated after a like manner: consequently, there must be *comparative velocities* [or rates] of increase and decrease during such generations; whose *relations* are fixed and determinable, and may therefore (problematically) be proposed to be found.

"This problem our author solves by the help of *another principle* not less evident; which supposes that quantity is *infinitely divisible*, or that it may (*mentally* at least) *so far* continually diminish, as at last, before it is *totally* extinguished, to arrive at quantities which may be called *vanishing quantities*, or which are *infinitely little*, and less than any *assignable quantity*: or it supposes that we may form a notion, not indeed of *absolute*, but of *relative* and *comparative infinity*.

"It is a very just exception to the Method of *Indivisibles* [of *Cavalieri*], as also to the foreign *Infinitesimal Method* [of *Leibnitz*], that they have recourse at

once to infinitely little quantities, and infinite orders and gradations of these, not relatively but absolutely such: they assume these quantities *simul et semel*, without any ceremony, as quantities that *actually* and *obviously* exist, and make computations with them accordingly; the result of which must needs be as *precarious* as the absolute existence of the quantities they assume.

“*Absolute* infinity, as such, can hardly be the object either of our conceptions or calculations; but *relative* infinity may, under a proper regulation. Our author observes this distinction very strictly, and introduces none but *infinitely little* quantities that are *relatively* so: which he arrives at, by beginning with finite quantities, and proceeding by a gradual and necessary progress of diminution: his computations always commence by *finite* and *intelligible* quantities; and then, at last, he inquires what will be the result, in *certain circumstances*, when such or such quantities are diminished *in infinitum*. This is a constant practice even in *Common Algebra* and *Geometry*; and is no more than descending from a *general proposition* to a *particular case* which is certainly included in it.”

40. “And from these *easy* principles, managed with a vast deal of skill and sagacity, he deduces his *Method of Fluxions*: which if we consider only *so far* as he himself has carried it, together with the *application* he has made of it, either here or elsewhere, directly or indirectly, expressly or tacitly, to the most curious discoveries in *art* and *nature*, and to the sublimest *theories*, we may deservedly esteem it as the *greatest work of genius*, and as the noblest effort that ever was made by the human mind.”

41. “Indeed it must be owned, that many useful *improvements* and *new applications* have been *since* made by others, and probably will be *still* made, every day: for it is no mean excellence of this method, that it is doubtless still *capable* of a greater degree of perfection, and will always afford an inexhaustible fund of curious matter, to reward the pains of the ingenious and industrious *analyst*.”

42. “It should be well considered by these *gentlemen objectors*, that the great number of *examples* they will find here, to which the Method of Fluxions is successfully applied, are so many *vouchers* for the truth of the principles on which that method is founded: for the deductions are *always conformable* to what has been derived from other *uncontroverted* principles; and therefore must be acknowledged as *true*.

“This argument should have its due weight even with such as *cannot*, as well as with such as *will not* enter into the proof of the principles themselves. And the *hypothesis* that has been advanced to evade this conclusion—of *one error in reasoning being still corrected by another equal and contrary thereto*—and that, *so regularly, constantly, and frequently*, as it must be supposed to do here;—this hypothesis, I say, ought not to be seriously refuted, because I can hardly think it is seriously proposed.”—*Preface to Colson's Method of Fluxions*, p. 11.

II.—Mac-

II.—Maclaurin's Account of the Method of Fluxions.

43. "WE would not be understood to affirm that the Methods of *Indivisibles* and *Infiniteimals*, by which *so many uncontested truths* have been discovered, are without a foundation: we acknowledge further, that there is something *marvellous* in the doctrine of *Infinities*, that is apt to please and transport us; and that the Method of *Infiniteimals* has been prosecuted *of late*, with an acuteness and subtlety not to be paralleled in any other science. But GEOMETRY is best established on *clear and plain principles*; and these speculations are ever obnoxious to some difficulties: If the greatest accuracy has been always required in this science, in reasoning concerning *finite* quantities, we apprehend that Geometricians cannot be *too scrupulous* in admitting or treating of *infinities*, of which our ideas are *so imperfect*.

44. "They who have made use of *infinities* and *infiniteimals* with the greatest liberty, have not agreed as to the *truth* and *reality* they would ascribe to them.—*Cavalierius*, the ingenious author of the Method of *Indivisibles*, discovered a method which he found to be of very extensive use, and of an easy application for measuring and comparing planes and solids, and would not deprive the world of so valuable an invention. In proposing it, he strove to avoid the supposing magnitude to consist of *indivisible* parts, and to abstract from the consideration of *infinity*:—"Quoad continui compositionem, manifestum est ex præostensis, ad ipsum ex *indivisibilibus* componendum, nos minimè cogi: solum enim *continuum sequi indivisibilium proportionem*, et è *converso*; intentum fuit: quòd quidem cum utràque positione stare potest. Tandem verò dicta *indivisibilium aggregata* non ita pertractavimus, ut *infininitatis rationem* propter infinitas líneas seu plana subire videantur, &c."—*Cavalerii Geom. Indivis. Lib. 7. Præf.*—But he acknowledged that there remained *some difficulties* in this matter, which he was not able to solve:—and he speaks as if he foresaw that it should be afterwards delivered in an unexceptionable form, which might satisfy the most scrupulous Geometrician; and leaves this *gordian knot* (as he expresses himself) to some *Alexander*."

45. "The celebrated Mr. *Leibnitz*, justly esteemed for his various writings, owns *infinities* to be no more than *fictions*:—"On s'embarasse dans les séries des nombres qui vont à l'*infini*: On conçoit un *dernier terme*, un *nombre infini*, où *infiniment petit*: mais tout cela ne sont que *des fictions*. Tout nombre est *fini* et *assignable*; toute ligne l'est le même."—*Essai de Théodicée*, disc. prélim. § 70.—They who treated of *infinities* before him proceeded, as he observes, with a *timorousness* which the contemplation of such an object naturally inspires:—

Q 2

"Quand

“Quand on y étoit arrivé, on s’arrestoit avec un espèce d’effroy et de sainte terreur—on regardoit l’infini comme un mystère, qu’il falloit respecter, et qu’il n’étoit pas permis d’approfondir.”—“They stopt when they came to infinity with a sort of holy dread, and respected it as an unfathomable mystery.”—He adventures farther, in order to discover the source, and penetrate into the first principles of GEOMETRICAL TRUTH. INFINITY, according to him, is the great trunk from which its various branches are derived, and to which they all lead. In this great pursuit, he displays infinite and finite, with a freedom that puts us in mind of the ancient poet [Homer] and his gods—whom he represents with the passions of men, and mingles in their battles.”

“We doubt not that if a full and perfect account of all that is most profound in the HIGH GEOMETRY could have been deduced from the Doctrine of Infinites, it might have been expected from this author: but our ideas of infinites are too obscure and inadequate to answer this end: and there are many things advanced by all those who have applied them with great freedom in GEOMETRY, that give ground to a remark like to M. de St. Evremond’s, when he observes, that “it is surprising to find the ancient poets so scrupulous to preserve probability in actions purely human; and so ready to violate it, in representing the actions of the Gods.”—Some have not only admitted infinites and infinitesimals of infinite orders; but have distinguished even nothings into various kinds!—and if such liberties continue, it is not easy to foresee what absurdities may be advanced as discoveries in what is called the SUBLIME GEOMETRY.”—“PHILOSOPHY probably will always have it’s mysteries; but these are to be avoided in GEOMETRY, [as far as is possible, and as much as in us lieth]; and we ought to guard against abating from it’s strictness and evidence the rather, that an ABSURD PHILOSOPHY is the natural product of a VITIATED GEOMETRY.”

46. “Sir ISAAC NEWTON accomplished what CAVALERIUS had wished for, [and avoided the rock on which the daring Leibnitz had split] by inventing the Method of Fluxions; and proposing it in a way that admits of strict demonstration: which requires the supposition of no quantities but such as are finite, and easily conceived. The computations in this method are the same as in the Method of Infinitesimals; but it is founded on accurate principles agreeable to the ANCIENT GEOMETRY: In it, the premisses and conclusions are equally accurate; no quantities are rejected as infinitely small*, and no part of a curve is supposed to coincide with a right line. The excellency of this method has not been so fully described, or so generally attended to, as it seems to deserve; and it has been sometimes represented as on a level, in all these respects, with the Method of Infinitesimals. The chief design of this treatise, is to shew it’s advantages in a clearer and fuller light, and to promote the design of the great inventor, by

* Compare Newton’s Demonstration of the Fluxion of a Rectangle, § 84, 85, with Euler’s, derived from the Infinitesimal Method, § 87, to which it is infinitely superior in accuracy and elegance.

establishing

establishing the HIGHER GEOMETRY on plain principles, perfectly consistent with each other, and with those of the ancient Geometricians."—Maclaurin's *Introduction to Fluxions*.

*De Applicatione Methodi Rationum Primarum et Ultimorum seu Limitum, ad
Quadraturam Curvarum, atque ad Mechanicam Rationalem.*

47. Hujus applicationis elegantia quædam proferuntur exempla à Newtono in Lemmatibus insequentibus: LEM. II. et III. Si polygonorum rectilineorum figuræ cuiusvis curvilineæ similiter inscriptorum et circumscriptorum augeatur numerus laterum correspondentium, et diminuatur magnitudo in infinitum, ita ut differentia polygoni inscripti et circumscripti tandem fiat minor quâvis datâ: ultimæ rationes quas habent polygonum inscriptum et circumscriptum, ad se invicem, (et multò magis ad figuram curvilineam intermediam) erunt rationes æqualitatis.

Immediatè et nullo negotio sequitur hoc, ex Lemmate primo fundamentali; idque sive polygonorum latera sint æqualia, ut in Lemmate secundo; sive inæqualia, ut in Lemmate tertio.

Cor. 1. Hinc polygonâ inscriptâ et circumscriptâ, tandem evanescente differentia, ultimò coincident omni ex parte, cum figurâ curvilineâ.

Cor. 2. Et propterea, polygonâ ultimâ, quoad perimetros suos, non sunt rectilinea, sed rectilineorum limites curvilinei.

48. Atqui hæc est ipsa Methodus Exhaustionum quam exercebant Geometri veteres, in compendium redacta, et longè promota, ab hoc artifice summo; sicut luculentius ostenditur in Introductione Maclauriniana:

"The Method of Exhaustions was that by which the Ancients demonstrated all their theorems for measuring and comparing *curvilinear* figures. It is often said that *curve lines* have been considered by them as *polygons of an infinite number of sides*; but this principle no-where appears in their writings: we never find them resolving any figure or solid into *infinitely small elements*: on the contrary, they seemed to avoid such suppositions; as if they judged them *unfit* to be received into GEOMETRY, when it was obvious that their demonstrations might have been sometimes *abridged* by admitting them."

"They considered *curvilinear areas* as the *limits* of circumscribed or inscribed figures of a *more simple kind*, which approach to these limits (by a bisection of lines

lines or angles that is continued at pleasure); so that the difference betwixt them may become less than any given quantity: The inscribed or circumscribed figures were always conceived to be of a number and magnitude that is assignable; and from what had been shewn of these figures, they demonstrated the mensuration or the proportions of the *curvilinear limits themselves*, by arguments *ab absurdo*. They had made frequent use of demonstrations of this kind from the beginning of the ELEMENTS [as where the affections of *similar polygons*, found in the first six books, are transferred to *circles* in the twelfth book.] And these are in a particular manner adapted for making a transition from *right-lined* figures to such as are bounded by *curve lines*. By admitting them only, they established the more difficult and sublime parts of their *Geometry*, on the same foundation as the first elements of the science. Nor could they have proposed to themselves a more perfect model."

49. "We have already observed how solicitous *Archimedes* appears to be, that his demonstrations should be found to depend on those principles only that had been *universally received* before his time. In his Treatise of the *Quadrature of the Parabola*, he treats of a progression whose terms decrease constantly in the proportion of 4 to 1 (which we expressed by the trapezia $CDba$, $DGcb$, $GRdc$); but he does not suppose this progression to be continued to infinity, or mention the sum of an infinite number of terms; though it is manifest, that all which can be understood by those who assign that sum, was fully known to him. He contented himself with demonstrating this plain property of such a progression: That the sums of the terms continued at pleasure, added to the third part of the last term, amount always to four-thirds of the first term: (as, for instance, in the *Quadrature of the Parabola*, *Introduct.* p. 27—29. Pl. 3. Fig. 13. the sum of the trapezia $CDba$, $DGcb$, $GRdc$, added to one-third part of the last trapezium $GRdc$, amounts always to four-thirds of the first trapezium $CRda$, or to its limit, the triangle CAa ;) but while this first trapezium approaches continually to its limit, the inscribed polygon at the same time is approaching to the *parabolic* segment its limit; and these are equal to each other, as there demonstrated, or in *Hamilton's Conic Sections*, (*De Quadraturâ Parabola*, according to *Archimedes*.)—After all the methods that have been proposed for demonstrating the *Quadrature of the Parabola*, this of the inventor himself seems to have a particular accuracy and elegance: he does not suppose the chords of the curve to be bisected to infinity; so that, after an infinite bisection, the inscribed polygon might be said to coincide with the parabola. These suppositions had been new to the geometricians in his time; and such he appears to have avoided carefully.—On this occasion, he shews how to find the sum of any number of terms that decrease constantly in the proportion of four to one: and by this example of a geometrical progression, (as it is commonly called,) opened up a subject, which has been treated at great length by the modern geometricians."

50. "But

50. "But even *Archimedes* himself has not escaped the censures of some writers, *Hobbes*, *Joseph Scaliger*, &c. who, being *unskilful* in Geometry, and *unable to reconcile their own conceits with his demonstrations*, have represented him as in an error, and "misleading mathematicians by his authority."—*Plutarch*, however, celebrates the *simplicity* and *plainness* with which he treats the most difficult and abstruse questions :

Ου γὰρ ἐστὶν ἐν Γεωμετρίας χαλεπωτέρας καὶ βωρυτέρας ὑποθέσεις ἐν ἀπλῶς τοῖς λαβοῦσι καὶ καθαυτοῖς στοιχείοις γραφομέναις.

"For it is impossible to find more difficult and weighty subjects written in simpler and purer elements."

"He appears indeed to have been more fond of preserving to the science all its *accuracy* and *evidence*, than of advancing *paradoxes*."

"And although the pursuit of *general* and *easy* methods may have induced some moderns to make use of *exceptionable* principles, and the vast extent which the science has of late acquired, may have occasioned their proposing incomplete demonstrations, yet the *first essays* were deduced from a careful attention to his steps :—"C'étoit en observant de près la marche d'ARCHIMEDE, qu'il (M. De Roberval) étoit arrivé à cette sublime et merveilleuse science."—*Acad. Scient.* an. 1693.—This is generally acknowledged by the writers of that time."

51. "Hence it was with caution that the Doctrine of *Indivisibles* was at first employed in Geometry by *Cavalierius* ; therefore he subjoined more *unexceptionable* demonstrations to those he had deduced from his own principles : and the disputes which ensued (the first of any moment that were known between *geometricians*) justified his precautions."

52. "Geometricians of the first rank had recourse to this method of *limits* long ago, on several occasions, as a method of the strictest kind : M. de *Fermat*, in a letter to *Gassendi*, and M. *Huygens*, in his *Horologium Oscillatorium*, have employed it for completing the demonstrations of some theorems that were proposed by *Galileo*, and proved by him in a less accurate manner ; and Dr. *Barrow* has demonstrated by it a theorem concerning the *Tangents of Curve Lines*."

53. In Quarto Lemmate demonstrat Newtonus rationes limitum curvilinearum, ultimò esse easdem ac polygonorum inscriptorum.

Nam ut partes singulæ polygonorum inscriptorum ad singulas, ita (componendo) fit summa omnium ad summam omnium ; et proindè (per Lem. 3. Cor. 2. præced.) limes curvilineus ad limitem curvilineum. Q. E. D.

Cor. Hinc, si duæ cujuscunque generis quantitates, in eundem partium numerum utcumque dividantur ; et partes illæ, ubi numerus earum augetur et magnitudo diminuitur

diminuitur in infinitum *datam obtineant ad invicem rationem*, prima ad primam, secunda ad secundam, cæteræque suo ordine ad cæteras : erunt *tota* ad invicem *in eadem illâ datâ ratione*. Nam, si sumantur tota inter se ut partes, erit totum prius ad summam partium, et totum posterius ad summam partium, in ratione æqualitatis ; et proindè, per hypothesin, in ultima ratione partis ad partem.

Q. E. D.

54. Ope hujus *principii generalissimi*, tam GEOMETRIAM quàm ALGEBRAM permeantis, emendare licet demonstrationem *Tacquetianam*, *Prop. 1. Lib. 6. El.* Nempè “*triangula æqualia, &c. esse ad invicem in ultimâ ratione basium* ;” quæ deficit, ubi de *incommensurabilibus* agitur ; cum “*indiciu proportionis*” pro *Euclidiano* malè substitutum à *Tacquet*, *commensurabilibus* tantum restringitur.

Sumptis partibus similibus DEG et DG trianguli prioris DEF, et basis prioris DF ; si substituantur hæ partes in triangulum posterius ABC et basin posteriorem AC respectivè, quoties fieri potest ; ubi numerus partium augetur et magnitudo diminuitur in infinitum, ultimò *datam obtinebunt ad invicem rationem* : et proindè, erunt *tota* ad invicem *in eadem illâ datâ ratione* : hoc est, erit triangulum prius DEF ad posterius ABC, ut basis prior DF ad posteriorem AC.

Q. E. D.

55. Quàm iniquæ igitur sunt in *Newtonum* criminationes, quasi “*methodum recentem*,” “*veteribus ignotam*,” “*alienis principiis fundatam*, à doctrinâ *motus* translatis,” iisdemque “*minimè necessariis*” excogitasset !—Pudet equidè tales cavillationes unquam admisisse analyistas ex peritissimis *Landen*, *D'Alembert*, et *Censorem* illum *Professorium*, præ incuriâ et oscitantia saltè, si non ex invidia oriundas. Maximè autem pudet viri clarissimi *De La Grange* :—nonne *dormitat bonus*, ubi profitetur “*se traditurum functionum analyticarum theoriâ*, ab omni consideratione *limitum* immunem !”—(si fides habenda sit isti *Censori* apud THE MONTHLY REVIEW.)

Dic quibus in terris—

—et eris mihi magnus Apollo !

56. Atqui luce clariùs constat, ex his documentis, principia infirma et vacillantia *methodi differentialis*, quæ primò posuerat *Leibnitz*, ex notionibus suis vagis et incongruis de *infinitesimis*, (seu quantitibus infinitè parvis) gradatim et lentè purgâsse, et tacitè, discipulos suos ; atque ad *Newtonianam veritatem*, atque *rigorem Geometricum*, tandem et quasi proprio jure, perfecisse : nunc verò reclamare audent ingrati, debito suo honore *Newtonum*, successorem dignum veterum analyistarum, *Archimedis*, *Apollonii* et *Euclidis*, iniquissimè defraudantes !

57. Demonstrârat *Euclides*, *Prop. 20. Lib. 6. El.* quod “*Polygonorum similium latera omnia quæ sibi mutuo respondent sunt proportionalia*, et, 2. *Areae sunt in duplicatâ ratione laterum* :” hanc proprietatem insignem ad *circulum* transtulit

Archimedes; et ipse *Euclides* in *Prop. 2. Lib. 12. El.* Et generalitèr, extendit *Newtonus* ad *curvas quascunque*, in *Lemmate quinto*; eodem ratiocinio posito; idque, five *latera sint curvilinea*, five *rethilinea*.

58. In *Lemmate septimo*, demonstrat *Newtonus*, (per evanescentiam anguli contactus, *Lem. 6.*) quod *ultima ratio arcus quam minimi, chordæ et tangentis est ratio æqualitatis*; paritèrque *semiarculus, ordinatæ et abscissæ tangentis*; vel *chordæ semiarculus, ordinatæ, &c.* Et propterea hæc omnes lineæ in omni de *ratiociniis ultimis* argumentatione pro se invicem usurpari possunt.

59. In *Lemmate octavo*, quod *ultima forma triangulorum evanescentium*, (ubi ex rectis à centro quodam quasi vertice, ad basin productis, five hæc basis, sit vel arcus, vel chorda vel tangens, fiunt hæc triangula) *est similitudinis, et ultima ratio æqualitatis*. Et hinc *triangula illa* in omni de *ratiociniis ultimis* argumentatione pro se invicem usurpari possunt.

60. In *Lemmate nono*, demonstrat (per *Lem. quintum*), quod *area triangulorum curvilineorum similium* ex curvarum tangentibus seu secantibus, et *ordinatis* seu subtentis externis angulorum contactus factorum, *erunt ultimò ad invicem in duplicatâ ratione laterum*.

61. Elegantissimâ translatione factâ ad *Mechanicam Rationalem*, in *Lemmate decimo* demonstrat, quod *si corpus, urgente vi quâcunque finitâ, moveatur*; five, 1. *vis illa determinata et immutabilis sit*; five, 2. *eadem continuò augeatur vel continuò diminuat*; erunt *spatia ipso motûs initio in duplicatâ ratione temporum*.

Nam, si *tempora motûs* exponantur per *abscissas*, et *velocitates genitæ* per *ordinatas externas*, (seu subtensas angulorum contactus) *spatia*, hæc vi urgente, descripta ritè exponuntur per *areas triangulares curvilineas* hisce ordinatis descriptas; et proinde (per *Lem. 5 et 9*), erunt in duplicatâ ratione laterum correspondentium, h. e. *abscissarum seu temporum*.

Cor. 1. Et quoniam virium diversarum intensitates ritè exponuntur per *velocitates* quas generant, erunt *hæc vires*, ut *spatia*, ipso motûs initio descripta directè et *quadrata temporum* inversè; per *Prop. 23. Lib. 6. El.*

62. In *Lemmate undecimo* demonstrat *Newtonus*, quod *ordinata externa* (seu *subtensa evanescens anguli contactus*) in *curvis omnibus curvaturam finitam ad punctum contactus habentibus*, *est ultimò in ratione duplicatâ chordæ arcus contermini*.

Nam, ex naturâ *circulorum æquicurvorum*, circà has *areas triangulares* descriptorum, erunt *ordinatæ externæ*, seu *circulorum sagittæ*, his proportionales, ut *quadrata chordarum* arcuum conterminorum directè, et *diametri* circulorum inversè; sed, *ultima ratio diametrorum*, evanescente differentiâ, sit *ratio æqualitatis*; adeoque *ordinatæ externæ* erunt ut *quadrata chordarum*.

Cor. 1. Hinc, *sagittæ curvarum* quoque, quæ *chordas arcuum duplorum* bisecant, et ad datum centrum motûs convergunt, sunt etiâ ultimò ut *quadrata arcuum conterminorum*. Nam *sagittæ illæ* sunt ut *ordinate externæ*, et *arcus* cum *chordis* ultimò coincident, per *Lem. 7.*

Cor. 2. Ideoque *sagittæ illæ* sunt in *duplicatâ ratione temporum* quibus corpus datâ *velocitate* describit *arcus quam minimos*. Nam datâ *velocitate*, hi *arcus* sunt ut *tempora*; et proindè *quadrata arcuum* ut *quadrata temporum*.

63. Atque sunt hæc *principia mathematica generalissima*, quorum ope demonstrat *Newtonus* in sequentibus propositionibus, *Seçt. 11. rationes virium centripetarum* ad invicem; *legēsque* secundum quas corpora circâ commune centrum motûs revolvuntur, accelerantur aut retardantur pro diversis à centro distantijs. Vide § 88, 89 et 103.

De Fluxionum Elementis Primis.

64. Quo magis pateat hujusce Methodi, sive *Fluxionalis* sive *Differentialis*, evidentia et certitudo, visum est "*fundamentum totius methodi*" inspicere et perscrutari, quale illud struxit artifex ille egregius in *Lemmate secundo Lib. 2. Princip.* ulteriusque obiter exposuit, in *Analysi per Series Numerorum Infinitas*, atque in *Quadraturâ Curvarum*;—quæ tria opera "*chordam triplicem haud facile dirumpendam*" constituunt, vel viribus obrectatorum *Britannorum* vel *Gallorum*, vel aliorum quorumvis contrâ hanc doctrinam infaustè combinatorum.

65. Terminos fluxionum *technicos* scitè definit *Newtonus*, et expositione satis dilucidâ distinguit:

1. "*Genitam* voco, quantitatem omnem quæ ex lateribus vel terminis quibuscunque, in *ARITHMETICA* per multiplicationem, divisionem et extractionem radicum; in *GEOMETRIA*, per inventionem vel contentorum et laterum, vel extremarum et mediarum proportionalium, sine additione et subductione generantur: ejusmodi quantitates sunt *facti*, *quoti*, *radices*; *rectangula*, *quadrata*, *cubi*, *latera quadrata*, *latera cubica*, et similes. Has quantitates, ut *indeterminatas* et *instabiles*, et *quasi motu fluxûve perpetuo* crescentes vel decrescentes, hîc considero: et

2. "*Earum incrementa vel decrementa momentanea*, sub nomine *momentorum* intelligo; ita ut *Incrementa*, pro *momentis additiis* seu *affirmativis*; ac *Decrementa*, pro *subductiis* seu *negativis*, habeantur.

66. "Cave tamen intellexeris [sub nomine *momentorum*] *particulas finitas* : particulae finitae non sunt *momenta*, sed *quantitates ipsae* ex *momentis* genitae ; [per vocem *momenta*] intelligenda sunt *principia jamjam nascentia* finitarum magnitudinum. Neque enim spectatur in hoc lemmate *magnitudo* momentorum, sed *prima nascentium proportio*."

Idem quoque fusius exponit in *Analysi per Aequationes* ; doctissimo Colson ita Anglicè eam reddente :

"The *moments* of *flowing* quantities are their *indefinitely small* parts, by the accessions of which, in *indefinitely small portions* of time, they are continually increased."

3. "Now these quantities, which I consider as gradually and indefinitely increasing, I shall hereafter call *Fluents*, or flowing quantities."

67. 4. "Whereas only quantities of *the same kind* can be compared together, and also their *velocities* [or *rates*] of increase or decrease ; therefore, in what follows, I shall have no regard to *time*, formally considered ; but I shall suppose *some one* of the quantities proposed (being of the same kind) to be increased by an *equable* quantity, to which *the rest* may be referred, *as it were*, to *time* : and therefore, by way of *analogy*, it may not improperly receive the name of *Time*. When, therefore, the word *time* occurs in what follows, (which, for the sake of *perspicuity* and *distinction*, I have sometimes used,) by that word, I would not have it understood as if I meant time in its *formal* acceptation, but only that *other quantity*, by the equable increase or flux whereof time is *expounded* and *measured*."

68. 5. "Eodem recidit, si loco *momentorum* [ipforum,] usurpentur vel *velocitates* incrementorum ac decrementorum, (quas etiàm, "*motus mutationes*," et "*fluxiones* quantitatum," nominare licet,) vel *finite quavis* quantitates, velocitatibus hisce *proportionales*."

Et paritèr quoque, scribit *Newtonus* in *Quadraturâ Curvarum* :—" *Fluxiones* sunt quàm proximè ut fluentium *augmenta aequalibus temporibus quàm minimis genita* ; et, ut accuratè loquar, sunt in *primâ ratione* momentorum nascentium."

And again, in the *Analysis* :—"The *moments* of flowing quantities—are as the *velocities* [or *rates*] of their flowing or increasing ;"—"and the *velocities* [or *rates*] by which every fluent is increased, by it's generating motion [whatever that may be], I may call *Fluxions* ; or simply, *Velocities* or *Celerities*."

69. Ex his documentis undique congestis et intertextis, luculentissimè constat, quantâ peritiâ, sollicitudine et cautione, methodi suæ generalis fundamentum posuit *Newtonus*, in *definitionibus*, *distinctionibusque* accuratissimis ; "*perspicuitatis*" (ut ipse dicit) quàm maximè studiosus, in re longè difficillimâ et subti-

lissimâ. Abundè autem liquet, eum usurpâsse vocem “*velocities*,” (quæ tantas tragœdias excitavit), in latiore sensu, pro *rationibus* crescendi vel decrescendi—“*rates of increase or decrease*.”—Et inde sensu generalissimo definiri possunt *fluxiones*—*Mensuræ quævis rationum secundum quas quantitates fluentes quælibet ejusdem generis, quolibet in statu mutationis, utcumque variantur.*

70. Quidni autem eadem licentia *Newtono* concedatur, quam sibi sumebant *Analystæ veteres*? Cur non, æquo saltèm jure, *Newtono* fas fuerit, in rebus geometricis principium geometricum adhibere, et “*genitas*” quascunque quantitates, ex fluxu, seu “*quasi motu perpetuo*,” oriundas, considerare; quàm *Archimedi*, *Apolloni*, et *Euclidi*, concipiendi *lineam* quasi ex motu *puncti*, *superficiem*, quasi ex motu parallelo *lineæ*, et *solidum*, quasi ex motu parallelo *superficiæ*, respectivè genita? Idque, *illustrandi* gratiâ, nimirum, in rebus *metaphysicis* à sensibilibus valdè abstractis? Nonne GEOMETRIA ipsa, exiguos fines *definitionis nominis* aspernans, longè, imò longissimè, ultrà “*mensurationem terræ, sive agrorum*,”—“*land-surveying*”—in remotissimas cœlorum regiones, ultrà orbem *planetarum, cometarum* atque *astrorum*, ad ipsam *viam lactean* excurrit—fræni impatiens, “*nec audit habenas*?” Et quidni vocabulum “*fluxio*,” quoque, in sensu latissimo atque amplissimo, ad omnigenas *functiones*, MECHANICÆ RATIONALIS,—*vires, pondera, densitates, resistentias, reflexiones, refractiones, infractiones, aberrationes, &c.* seu quascunque quantitatuum mutationes, secundum magnitudines et directiones, ritè exponendas, extendatur et ampliatur?—De *definitionibus nominum* disputare, (non *κατὰ φύσιν* seu “*secundum rei ipsius naturam*,” sed *κατὰ θεον*, “*pro definitentis arbitrio*”—vel *instituto*)—pusilli est ingenii, *minutique philosophi*.—Eodem redeunt *Fluxiones NEWTONI* et *Functiones DE LA GRANGE*; ut ex definitione vocis posterioris, apud THE MONTHLY REVIEW, constabit:

71. “*A function of one or more quantities, is every expression in which these quantities enter after any manner whatever (combined, or not, with other quantities of given or variable values), while the quantities of the function may receive all possible values.*”

72. Eodem quoque recidit *Methodus Differentialis*, (definiente *analystâ* peritissimo, *Euler*, in suis *Institutionibus*) ac ipsa *Fluxionalis*:

“*Methodus determinandi rationem incrementorum evanescentium, quæ functiones quæcunque accipiunt; dum quantitati variabili cujus sunt functiones, incrementum evanesceus tribuitur.*”

Velut si quantitatis variabilis x , incrementum sit i ; ubi hæc quantitas sit $x + i$, fiet hujus quadratum, $x^2 + 2xi + i^2$. Erit igitur

$$\text{Increm. } x : \text{increm. } x^2 :: i : 2xi + i^2;$$

$$\text{vel, dividendo per } i, \text{ —} :: 1 : 2x + i.$$

Et proinde ratio 1 ad $2x$, exponet rationem incrementi lateris variabilis x ad incrementum quadrati variabilis x^2 , ipso in limite, ubi i evanescit, accuratè; antequàm evanescit, quàm proximè tantum. At limes rationis horum incrementorum evanescentium solummodò in *Calculo Differentiali*, *Eulero* iudice, spectatur.

73. Quis non videt analysta benè cordatus, ad amussim quadrare Methodum *Fluxionalem* cum *Methodo Differentiali*, tam *Euleri* quàm *De La Grange*?—et methodo utrique æquè competere considerationem temporis inter crescendum decrescendumve elapsi?—si non “*formalitè*” quidem, saltem per analogiam, sive per *accommodationem*, et illustrandi gratiâ?—et quale aut quantum intercedat discrimen (intelligo verum discrimen, non nòmini tenus) inter “*momenta*” *Newtoni*, et “*incrementa evanescentia*” *Euleri*, detegat, oro, analystarum vel perspicacissimorum curiosa felicitas, in sæcula sæculorum.

74. Non idcirco nisi imperitè objicientis est “*animadversio docti Torelli*,” quam commendat M. R. quasi—*inter exempla admodum perspicuè validæque argumentationis*” annumerandam:

“*Neque verò quidquam agunt, qui, infinitesimas quantitates rejicientes, momenta quædam earum loco GEOMETRIÆ inferunt; eaque ita vocant, quòd concipiuntur momento temporis finitæ magnitudinis fluxu oriri: Si enim momenta hæc vires suas fluendo explicant, non aliud sunt quàm ipsæ quas rejiciunt infinitesimæ quantitates; sin autem nituntur neque tamen ultrà tendunt, novum vocabulum frustra inducitur, dum id quod nihil est momentum appellatur.*”

At ruit objectio suis viribus eversa. Nam, 1. momentum non est nihil, ut ex definitione constat; nec, 2. est infinitesima quantitas, cum magnitudinis finitæ, seu determinatæ, (licet quàm minimæ) esse concipiatur; nam, ut accuratissimè loquitur *Newtonus*, est “*augmentum fluentis tempore quàm minimo genitum*”; neque, 3. si valida sit, minùs hostilitèr in Methodum *Differentialem*, quàm *Fluxionalem*, invehitur, iidem principiis innixas; totius analyseos sublimioris fundamentum æquè labefactando.

75. Nec minùs ruit altera ista objectio *D'Alembert* (f), in vocem “*Velocitatem*” animadvertentis, quippè quæ eodem redit in sensu *Algebraico*, ac “*ratio*.”

(f) “*To speak of D'Alembert merely as a mathematician,*” THE MONTHLY REVIEW seems rather to over-rate his merits:—However profound his researches, and extensive his labours “*to clear away the rubbish of jargon and mystery from science,*” his calculations and demonstrations are exceedingly rugged, prolix and intricate; far removed from the elegance, conciseness and arrangement of *Newton*, and even of *Euler* and *Clairaut*.—When *Landen* (himself a profound mathematician, and who first discovered the source of *Newton's* mistake, in determining the quantity of the precession of the equinoxes, the merit of which is attributed to *D'Alembert* by the M. R.) had examined *D'Alembert's* solution of the *Theory of Rotatory Motion*, differing from his own, he was unable to discover the source of that difference in the obscurity of his argument, until assisted by the perspicuity of *Euler's* manner; whose solution agreed in it's result with *D'Alembert's*.—And surely his objection to the Method of Fluxions shews that he was not deeply versed in the metaphysique of the science.

et in sensu *Geometrico*, legitimè pro *fluxione* usurpanda est, cum *velocitas* sit “*exponens rationis*” inter spatium et tempus intercedentis, ubi motus sit uniformis; et proinde *relationem* verè existentem inter veras quantitates commensurabiles in *Mechanicâ Rationali*, aptè, et “*naturalitèr* quidem,” (ipso fatente M. R. p. 498.) designat. Et secundùm *D'Alembert*—*Calculi Differentialis* in *Geometricis*, munus est—“*déterminer algebräiquement LA LIMITE d'un rapport de laquelle on à déjà l'expression en lignes; et à éгалer ces deux limites; ce qui fait trouver une des lignes qu'on cherche.*”

Atque illustrandi gratiâ, ductâ tangente *parabolæ*, cujus æquatio est $av = y^2$, primò exquirat *D'Alembert* valorem unius limitis algebräicè, $(\frac{a}{2y})$; et valorem alterius geometricè $(\frac{y}{s})$; quibus æquatis, deducit valorem *subtangents*, $(s) = \frac{2y^2}{a} = 2x$. Atque agnoscit paritèr, (ut, ex abundanti diluatur objectio *Torelliana*) quòd in formulâ $\frac{dv}{dz}$, secundùm notationem differentialem, seu $\frac{\dot{x}}{\dot{z}}$ secundùm fluxionalem), dx et dz (seu $\dot{x}$ et $\dot{z}$) non designare *quantitates infinitesimas*, seu infinitè parvas, sed in certo quodam statu *finite magnitudinis*; tales enim *symbolicè* tantùm usurpari pro formâ loquendi compendiosâ, calculis concinnandis accommodatâ, verùm nullatenùs *criticè*, seu pro usu *philosophico* terminorum technicorum.

76. Atque hoc pacto, ipse quoque *Censor*, p. 487, monet, quòd si *curvæ* cujusdam *axis* denotet *tempus* descriptionis; et *ordinatim applicatæ* quibusvis *curvæ* ipsius punctis, *spatia* (vel actu descripta vel quæ describenda forent *velocitate* exinde uniformitèr continuatâ); erit *velocitas* ubivis, ut *quotiens ordinatæ per subtangentem divisæ*, seu $v = \frac{o}{s}$; et proindè per hanc rationem verè exponitur seu mensuratur.

77. Haftenùs explicatis elementis primis *Methodi Fluxionum*, pro suâ subtilitate fusiùs; facilitati conceptûs consulens, et speculationibus *metaphysicis* faciem præbens, quantum res difficillima pateretur; et absolutis *vindictis* adversùs obtrectatores laudum *Newtonianarum*, tam exteros quàm nostrates; gratiam ut spero, promeriturus omnium cordatiorum analyistarum, qui quantum *Newtono* debent agnoscere haud gravantur; (*Anglorum* potissimùm, ingens illud decus BRITANNIÆ, atque *Analyseos* sublimioris principem et architectum, optimo jure admirantium): Superest ut ipsius *methodi* generalis *rudimenta* proponam; pressiùs atque concinniùs, et tamen uberiùs ac luculentiùs, quàm
alii

alii scriptores qui hactenus hoc facere aggressi sunt, hâc, nimirum, lege mihi propositâ et sedulò observatâ, ut vestigiis ipsius inventoris sagacissimi, quàm maximè potuerim, fidelitèr semper insisterem ;

*Te sequor, O MAGNÆ GENTIS DECUS ! inque tuis nunc
 Fixa pedum pono pressis vestigia signis ;
 Non ita certandi cupidus, quàm propter amorem,
 Quòd te imitari aveo : —
 E tenebris tantis tam clarum extollere lumen
 Qui primus potuisti, illustrans commoda vitæ. —
 Tu PATER ! ET RERUM INVENTOR ! —
 ————Tuisque ex, INCLUTE, chartis,
 Floriferis ut APES in saltibus omnia libant,
 Omnia NOS itidem depascimur aurea dicta,
 Aurea, perpetuâ semper dignissima vitâ.*

LUCRETIVS.

Nam æquum certè erit has laudes ad *Newtonum* transferre, VIRTUTIS VERÆ custodem, SAPIENTIÆ rigidum satellitem, quas de *Epicuro*, “*insanientis sapientie*” patre et “*patrono*” falsò prædicat *Lucretius* ; qui ipse, vitæ pertæsus, non ultrà quadragesimum ætatis annum provectus, necem sibi conscivit,

MISERÆ PHILOSOPHIÆ MISERUM SOLAMEN !

A N A.

1871
The first of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The second of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

The third of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The fourth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

The fifth of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The sixth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

The seventh of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The eighth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

The ninth of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The tenth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

The eleventh of the year was a very dry one, and the crops were much injured by the drought. The weather was very hot, and the crops were much injured by the drought.

The twelfth of the year was a very wet one, and the crops were much injured by the rain. The weather was very cold, and the crops were much injured by the rain.

ANALYSIS FLUXIONUM.

PARS SECUNDA.

DEFINITIONES.

§ 78. 1. *METHODUS Fluxionum* est quæ determinat quantitatum ejusdem generis, utcunque variabilium, rationes ad se invicem, quolibet in statu contemporaneo variationis seu mutationis, per *methodum primarum et ultimarum rationum* seu *limitum*.

2. *Rationes ultimæ* sunt *limites*, ad quos quantitatum sine fine decrefcentium rationes, 1, *semper appropinquant*; et, 2, quas *propiùs assequi* possunt quàm *pro datâ quâvis differentiâ*; 3, *nunquam verò transgredi*; 4, *nec priùs attingere*, quàm quantitantes ipsæ diminuuntur in infinitum. Et par est natura *rationum primarum*.

3. *Fluens* est quantitas ipsa variabilis, sed tamen finita, quemvis mutationis statum subiens, antequam evanescat, seu in infinitum augeatur.

4. *Genita* est ipsius fluentis magnitudo *certo* quodam variationis puncti, seu mutationis statu.

5. *Momentum* est fluentis augmentum aut decrementum *momentaneum*; id est, tempore quam minimo genitum. Estque *fluxioni* proportionale.

6. *Fluxio*, seu *Functio Analytica*, seu *Differentialis*, est *mensura quævis rationis secundum quam variatur fluens*, quolibet in statu mutationis.

7. Fluentis ipsius *Fluxio* appellatur *prima*, seu *primi ordinis*; hujus iterùm *fluxio* (si prima fluxio variabilis sit) *secunda*, seu *secundi ordinis*; (si secunda quoque sit variabilis,) hujus denuò *fluxio*, *tertia*, seu *tertii ordinis*; atque ita deinceps, donec ultimò perveniatur ad *fluxionem constantem*, seu mutatione uniformi procedentem.

8. *Methodus Fluxionum Directa*, quæ ex fluentibus datis invenit fluxiones; *Methodus Inversa*, vicissim, ex fluxionibus datis invenit fluentes.

9. *Æquatio Fluentialis* seu *Integrans*, quæ ex fluentibus solis constat; *Æquatio Fluxionalis* seu *Differentialis*, quæ fluxiones fluentibus immixtas continet.

AXIOMATA.

79. 1. *Quantitates*, ut et *Quantitatum Rationes*, quæ sunt *constantes* aut *datæ*, nullas fortiuntur fluxiones.

2. *Quantitatum* vel *Rationum* in *maximis* aut *minimis* versantium, vel in *limitibus* ipsis, evanescent fluxiones.

3. *Fluxio summæ* duarum pluriûmve fluentium æquatur *summæ fluxionum* singularum fluentium cum propriis signis; scilicet, affirmativis, si momenta pro incrementis habeantur; negativis, si pro decrementis.

4. *Quantitates* quæ *constantè*, seu *æqualitè*, seu *certâ et determinatâ quâdam ratione* fluunt, fluxiones tantum *primas* fortiuntur.

5. Fluxio cujuscvis fluentis sive puræ, sive cum cognitis quibuscunque quantitatibus immixtæ, est eadem. Sic, si A sit quantitas variabilis, et e quantitas data, quantitatis A et $A \pm e$ eadem erit fluxio a .

6. Eadem fluens *plures* fortiatur fluxiones; si modo sint in datâ ad invicem

ratione. Sic quantitatis $A^{\frac{m}{n}}$, cujus prima fluxio est $\frac{m}{n} a A^{\frac{m}{n}-1}$, secunda erit

$$\frac{m}{n} \cdot \frac{m}{n} - a^2 A^{\frac{m}{n}-2}; \text{ vel huic proportionalis, } \frac{\frac{m}{n} \cdot \frac{m}{n} - 1}{2} a^2 A^{\frac{m}{n}-2};$$

nempè, in ratione datâ 1 ad $\frac{1}{2}$.

Pauca insuper, *præparationis* loco, subungere libet de *variis formis notationi* quæ in calculo sive fluxionali sive differentiali adhiberi solent.

De Notatione Fluxionali.

80. *Fluentes* primò designabat *Newtonus*, in *Lem. II. Lib. II. Princip.* per literas majusculas A, B, C, &c. Et harum *momenta*, (seu *fluxiones* momentis proportionales), per minuscultas a, b, c, &c.; quæ relationes quantitatum ipsarum respectu magnitudinis benè exponunt. Postea verò, alteram illam methodum, jam usitatiorē, excogitavit, designandi fluentes per minuscultas z, y, x, &c. harumque fluxiones diversorum ordinum per puncta super-imposita, hoc modo :

$\dot{z}$, $\dot{y}$, $\dot{x}$	&c	- - -	<i>fluxiones primæ,</i>
$\ddot{z}$, $\ddot{y}$, $\ddot{x}$	&c	- - -	<i>fluxiones secundæ,</i>
$\dddot{z}$, $\dddot{y}$, $\dddot{x}$	&c	- - -	<i>fluxiones tertiæ,</i>
$\ddddot{z}$, $\ddddot{y}$, $\ddddot{x}$	&c	- - -	<i>fluxiones quartæ.</i>

81. Secundùm notationem differentialem, minùs elegantèr exponuntur differentiales uniuscujusque gradus,

dx (seu xi),	-	<i>differentialis prima,</i>
ddx ,	- -	<i>differentialis secunda,</i>
$ddd x$,	- - -	<i>differentialis tertia,</i>
$dddd x$,	- - -	<i>differentialis quarta.</i>

Secundùm notationem quam usurpat *De La Grange*, si curvæ cujusvis æquatio sit $y = f x$, erunt valores, *subnormalis*, *subtangētis* et *radii curvaturæ* yy' , $\frac{y}{y'}$, et $\frac{(1 + y'^2)^{\frac{3}{2}}}{y'}$, respectivè, quæ secundùm notationem Newtonianam prodeunt $\frac{yy'}{x}$, $\frac{y\dot{x}}{y}$, et $\frac{(\dot{x}^2 + y^2)^{\frac{3}{2}}}{xy}$.—*Censore* referente.

In tertiâ editione *Princip.* p. 486, designat quoque *Newtonus* differentias primas per b , $2b$; secundas, per c , $2c$; tertiâs, per d , $2d$, &c.

82. Ingenti autem Matheseos commodo, introduxit *Newtonus* in *Algebram Speciosam* notationem *indicum fractorum* et *negativorum*; quarum rationem breviter exponere in animo est, ne confundantur tyrones notatione inusitatâ; præsertim,

præsertim, cùm non defint auctores, qui usum indicum *fractionum* et *negativorum* pravâ confusione miscent.

Indices *integri* quantitatum A^2, A^3, A^4, A^m , indicant varias *potestates* seu dignitates, per *involutionem* resultantes. Indices *fractioni* $A^{\frac{1}{2}}, A^{\frac{1}{3}}, A^{\frac{1}{4}}, A^{\frac{1}{n}}$, varias itidem *radices*, per *evolutionem*; quales nempè referunt *surdæ* quantitates, secundum notationem vulgarem, $\sqrt{A}, \sqrt[3]{A}, \sqrt[4]{A}, \sqrt[n]{A}$. Ex utriusque autem combinatione, emergent $A^{\frac{2}{3}}, A^{\frac{3}{4}}, A^{\frac{5}{2}}, A^{\frac{m}{n}}$, designantes nimirum radices per denominatores expositas, potestatum per numeratores, $\sqrt[3]{A^2}, \sqrt[4]{A^3}, \sqrt{A^5}, \sqrt[n]{A^m}$. Indices verò *negativi*, $A^{-2}, A^{-3}, A^{-m}, A^{-\frac{2}{3}}, A^{-\frac{3}{4}}, A^{-\frac{m}{n}}$, indicant *recipros* affirmativorum, $\frac{1}{A^2}, \frac{1}{A^3}, \frac{1}{A^m}, \frac{1}{A^{\frac{2}{3}}}, \frac{1}{A^{\frac{3}{4}}}, \frac{1}{A^{\frac{m}{n}}}$.

Atque hoc pacto, *fractiones tolli*, et *æquationes concinnari* possunt, transponendo quantitates indicibus negativis insignitas, à numeratore in denominatorem; aut vice versâ.

$$\begin{aligned} \text{Sic } \frac{A^1}{A^2} &= \frac{1}{A^2} = A^{-2}; \text{ et } \frac{A^m}{A^n} = \frac{1}{A^{n-m}} = A^{m-n}; \text{ et, } A^{\frac{1}{2}-3} = \frac{1}{A^{3-\frac{1}{2}}} \\ &= \frac{1}{A^{2+\frac{1}{2}}}; \text{ et, } A^{\frac{m}{n}-3} = \frac{1}{A^{3-\frac{m}{n}}}, \text{ et, sic, } \frac{naA^{n-1}}{A^n} \times \frac{1}{A^n} = \frac{na}{A} \times \frac{1}{A^n} = \\ &= \frac{na}{A^{n+1}} = naA^{-n-1}. \end{aligned}$$

De Methodo Fluxionum Directâ.

83. Continentur canones, seu methodi hujusce regulæ, in theoremate quodam generalissimo, omnes casus comprehendente:

LEMMA II.

“Momentum genitæ ($A^m B^n$) æquatur momentis (i. e. summæ momentorum, scilicet ($a + b$) laterum singulorum generantium (A, B) in eorundem laterum indices dignitatum (m, n) et co-efficientia ($A^{m-1} B^n, B^{n-1} A^m$), continuè ductis.” Sen fluxio $A^m B^n = maA^{m-1} B^n + nbB^{n-1} A^m$.

N. B. Phrasi.

N. B. Phrafi Newtonianâ, "*Lateris cujusque generantis co-efficiens, est quantitas quæ eritur, applicando [totam quantitatem] genitam ad hoc latus.*" Sic lateris A co-efficiens est $\frac{A^m B^n}{A} = A^{m-1} B^n$; et lateris B, est $\frac{A^m B^n}{B} = B^{n-1} A^m$.

84. Auspicatur *Newtonus*, à casu simplicissimo.

Cas. 1. Fluxio rectanguli geniti (AB) æquatur fluxionibus laterum generantium (a et b) in ipsa latera (B et A) continuè ductis; seu fluxio AB = aB + bA.

Rectanguli cujusvis perpetuò fluentis, sint latera, vel æquabiliter vel pari inæquabilitate crescentia, per datam quandam temporis particulam, in medio statu mutationis, A et B. Sumantur horum semi-momenta antecedentia, pro decrementis, $-\frac{1}{2}a$ et $-\frac{1}{2}b$; consequentia autem, pro incrementis, $+\frac{1}{2}a$ et $+\frac{1}{2}b$. Laterum igitur variabilium prodibunt valores successivi, eorumque differentiarum correspondentes à mediis:

Val. A.	Diff.	Val. B.	Diff.
$A - \frac{1}{2}a$	} $+\frac{1}{2}a,$	$B - \frac{1}{2}b$	} $+\frac{1}{2}b.$
A		B	
$A + \frac{1}{2}a$	} $\frac{+\frac{1}{2}a}{+a}$	$B + \frac{1}{2}b$	} $\frac{+\frac{1}{2}b}{+b}.$

Horum valorum primi erunt minimi, et postremi, maximi; sed fluxiones laterum sunt eorum incrementis totis simultaneis, a et b, proportionales respectivè; et proinde jure exponentur per summas differentiarum, a + b.

Rectanguli quoque variabilis, prodibunt valores successivi, eorumque differentiarum, à medio valore, AB:

Val.	Diff.
$(A - \frac{1}{2}a) \times (B - \frac{1}{2}b) = AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$	} $+\frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab.$
$A \times B = AB$	
$(A + \frac{1}{2}a) \times (B + \frac{1}{2}b) = AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$	} $\frac{+\frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab}{+aB + bA}.$

(g) "*Quantitates differentis suis proportionales sunt continuè proportionales.*" Et conversim quantitatibus continuè proportionalium erunt differentiarum ipsis quantitatibus proportionales.

1. Sit A ad A - B ut B ad B - C, et C ad C - D, &c. Et convertendo fiet A ad B ut B ad C, et C ad D.

2. Sit A ad B ut B ad C, &c. Et iterum fiet A ad A - B ut B ad B - C.

Vide *Maclaurin's Fluxions*, § 159—160, ubi prolixè demonstratur et Theoriæ Fluxionum Logarithmorum applicatur.

Et horum valorum quoque, erit primus minimus, et postremus, maximus. Sed fluxio rectanguli incrementis laterum *totis a* et *b* geniti, erit incremento toti proportionalis; et proinde jure exponetur per summam differentiarum, $aB + bA$.

Si negas, fingatur fluxionem veram esse vel majorem vel minorem, quantitate quâvis assignabili *D*, seu, $aB + bA \pm D$. Quod si major foret, emergere *possit* dimidium novæ fluxionis, $\frac{aB + bA + D}{2}$ majus maximâ, (per constructionem) differentiarum, $\frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$; et proinde non ritè exprimeret augmentum rectanguli post medium statum *AB*: sin minor, emergere *possit* dimidium novæ fluxionis $\frac{aB + bA - D}{2}$, minus minimâ differentiarum, $+\frac{1}{2}aB + \frac{1}{2}bA - \frac{1}{4}ab$; et proinde non ritè exprimeret augmentum rectanguli ante medium statum *AB*. Q. E. D.

In numeris res patebit:

Sit $A = 20$, et $a = 2$; $B = 30$, et $b = 4$; erunt

	<i>Val.</i>	<i>Diff.</i>
$(A - \frac{1}{2}a) \times (B - \frac{1}{2}b) = 19 \times 28 = 532$	}	68
$A \times B = 20 \times 30 = 600$		
$(A + \frac{1}{2}a) \times (B + \frac{1}{2}b) = 21 \times 32 = 672$		
		140.

Sed $aB + bA = 60 + 80 = (140 =) 68 + 72$.

85. Demonstrationem totam brevissimè expedit *Newtonus*:

“Rectangulum quodvis *motu perpetuo* auctum *AB*, ubi de lateribus *A* et *B* deerant momentorum dimidia $\frac{1}{2}a$ et $\frac{1}{2}b$, fuit $A - \frac{1}{2}a$ in $B - \frac{1}{2}b$, seu $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$; et quàm primùm latera *A* et *B* alteris momentorum dimidiis aucta sunt, evadit $A + \frac{1}{2}a$ in $B + \frac{1}{2}b$, seu $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. De hoc rectangulo subducatur rectangulum prius, et manebit *excessus* $aB + bA$. — Igitur laterum incrementis *totis a* et *b*, generatur rectanguli incrementum $aB + bA$.” Q. E. D.

In hac demonstratione, *Newtonus*, neglecto medio valore rectanguli fluentis, nempe *AB*, simul ac semel, subducit minimum valorem de maximo; et inde, *excessus* designabit totum rectanguli augmentum *totis* momentis *a* et *b* generatum, (sive summam differentiarum minimi valoris à medio, et medii à maximo); sed argumentum ad absurdum omittit, quasi ex suis principiis in *Lemmate primo* fati manifestum. Vid. § 16. Not. (c).

86. Demonstrationis *Newtonianæ* vim atque subtilitatem minimè capiens “*Analysta*” (*Berkeley*,) validitatem ejus imperitè improbavit. Nec fati per-spexerunt vindices, *Colson* et *Maclaurin*: *Colson* enim argumentum ad absurdum, propositioni plenè, perfectè et rigidè demonstrandæ necessarium, omisit; *Maclaurin* verò vicissim, priorem demonstrationis partem, à *Colson* benè explicatam, paritèr

paritèr omisit; huic parti posteriori tantùm immorans, prolixius et obscurius, in demonstratione parallelâ fluxionis *quadrati*; unde, ordine inventoris lucido in pejus mutato, fluxionem *rectanguli* deducit; *non benè relata* interpretis *parmulâ*: sicut in *Parte Primâ* monuimus, § 16.

87. Analystæ recentiores quoque, Euler, (§ 72.) &c. solidam et rigidam demonstrandi viam *Newtonianam* deferentes, rem paulò alitèr instituunt, ex *methodo infinitesimalium*.

Laterum generantium sint postremi valores, $x + x'$ et $y + y'$; emerget postremus quoque valor rectanguli simultanei, his in se ductis, $xy + x'y + xy'$ + $x'y$. De hoc valore subducatur primus, xy , et manebit *excessus* $x'y + xy' + x'y$; nempe incrementis x et y generatus; cæterum terminus ultimus $x'y$, utpotè *infinitè parvus*, tutò *negligi* potest: et prondè incrementum ipsius rectanguli quàm proximè exponet $x'y + xy'$; sive substitutis, pro incrementis x et y , fluxionibus x et y , fluxionem rectanguli exponet $xy + xy'$. Q. E. D.

Vide *Hutton's Mathematical Dictionary*, voce *Fluxion*, p. 487; undè hanc demonstrationem *Newtonianæ* (ibidem malè omissæ) evidentiâ et elegantia longè cedentem, deprompsimus: Nam fluxionem rectanguli variabilis, quam in statu mutationis *primo*, xy , non nisi *quàm proximè* exponit $xy + xy'$; in statu *medio*, seu *limite* in ipso, AB, longè accuratius exponit $aB + bA$, secundùm constructionem *Newtonianam* artificiosiore.

88. Atque ex hujus demonstrationis subtilissimæ genio seu indole, *methodum limitum* indubitato respiciente, optimè forsàn explicare licet principium illud spinosissimum et vexatissimum, sed nobilissimum et generalissimum sanè, unde pendet omnis theoria *Newtoniana* virium centripetarum, *Princip. Lib. I. Prop. 1. Cor. 3. et 4.*

Cor. 3. “*Vires centripetæ in [mediis arcuum momentaneorum ABC et DEF] B et E, sunt ad invicem in ultimâ ratione diagonalium BV, EZ, [quæ convergunt ad centrum virium S] ubi arcus isti in infinitum diminuuntur.*”

Nam vires centripetæ, in his mediis locis B et E, utpote *acceleratrices*, propriè mensurantur per *velocitates* versùs commune centrum motûs cadendo acquisitas, æqualibus his temporibus quàm minimis; atque hæ iterùm *velocitates genitæ*, utpote temporum quàm minimorum decursu, ferè æquabiles, ob hæc tempora, per hypothefin, æqualia, erunt ut *spatia*, impulsibus vis centripetæ in locis B et E interim generata versùs commune centrum motûs S; sive *deflexiones* de extremis *c* et *f* tangentium per B et E ductarum; cæterum hæ deflexiones ferè (*b*) coincident cum *subtensis* angulorum contactûs *cC, fF, ubi arcus isti*

(*b*) Vide *La Caille, Elémens d'Astronomie*, seu Versionem Anglicanam à *Roberison*, § 157, et fig. 33. ubi gradus hie demonstrationis præcipuus, à *Newtono* omissus, et à commentatoribus malè neglectus, optimè suppletur: ibi enim notantur *hæ deflexiones vere* versùs centrum per QR, pF, quæ cum *subtensis* angulorum contactûs, radio vectori parallelis respectivè, ferè coincidunt.

ABC et DEF in infinitum diminuuntur; (nimirum recedendo utrinque ab extremis A et C, atque D et F, versùs media B et E, respectivè); itémque hæ subtensæ cC , fF ;—si resolvatur totus revolutionis motus per semiarculus BC et EF, (qui utique cum chordis suis et cum tangentibus suis Bc et Ef ultimò coincident, per Lem. 7.) binos in motus, unum, secundum directiones tangentium, et alterum, secundum directiones radiorum vectorum BS, et ES,—erunt parallelæ et æquales diagonalibus BV et EZ, quæ convergunt ad centrum virium et chordas AC, DF bisecant: ergò vires, &c. erunt ipsi diagonalibus proportionales. Q. E. D.

In hac demonstratione, (quam, brevissimè à Newtono absolutam, fusiùs explicuimus,) quis non videt quod “ubi arcus isti in infinitum diminuuntur,” nimirum recedendo utrinque, &c. semiarculus priores AB, DE antè media puncta B et E, decrementis $-\frac{1}{2}a$, $-\frac{1}{2}b$; et semiarculus posteriores BC, EF, incrementis $+\frac{1}{2}a$, $+\frac{1}{2}b$, respondent respectivè?

89. Cor. 4, “Vires illæ centripetæ, &c. sunt quoque proportionales sagittis illis quæ chordas bisecant arcuum quorumvis, æqualibus temporibus quam minimis descriptorum, et convergunt ad centrum virium.”

Nam hæ sagittæ sunt semisses diagonalium prædictarum, seu $\frac{1}{2}BV$, et $\frac{1}{2}EZ$; et proinde pro ipsis diagonalibus, (veris utique virium exponentibus,) compendii gratiâ, calculo cautè instituto, et servatâ ubique semissium proportionem, per totum computum, tutò substitui possunt. Alièr enim suboriri possunt errores in calculo minimè contemnendi. Vide Maclaurin's Fluxions, § 440; Simson's Fluxions, Vol. I. p. 237; La Lande, Astronomie, § 3392; et Milner, Philosoph. Transf. anno 1779, Part II. p. 521.—Vide plura, § 103, et antea § 62 et 63.

90. Quò magis eluceant evidentia, certitudo et ubertas primi hujus casûs, quem totius methodi generalis fundamentalem constituit Newtonus, placet insuper ex considerationibus Geometricis, abstractiùs et subtiliùs hætenus tradita, illustrare.

1. Lateribus generantibus, A et B, uniformitèr crescentibus, five in eâdem, five in datâ quâdam ratione ad se invicem; erit diagonalis rectanguli duplici motu geniti recta linea. Et complementa parallelogrammorum circâ diagonalem ritè exponent fluxiones laterum respectivè; adeoque horum aggregatum fluxionem ipsius rectanguli geniti. Sed in hoc casu simplicissimo, erunt ipsa complementa æqualia, per Prop. 43. Lib. I. Elem.; five $aB = bA$, adeoque areæ rectanguli fluxio $= 2aB$ vel $2aA$. Et si latera sint æqualia, erit fluxio quadrati $= 2aA$.

2. Lateribus difformitèr crescentibus, seu in diversâ ab invicem ratione; ex. gr. si momentum a sit constans, at momentum b variabile, erit diagonalis rectanguli geniti curva linea: quæ erit convexa à parte baseos seu abscissæ motu constanti descriptæ, si b crescat; concava autèm, si decrescat. In utròque casu, complementa illa quæ fluxiones laterum exponunt, erunt inæqualia.

Hinc forsàn *Sectiones Conicæ*, nomina sua propria acceperunt: *Hyperbola*, quasi motu ordinatæ ὑπερβαλλοντι, “*redundante*,” genita; *Ellipsis*, quasi motu ελλειποντι, “*deficiente*,” et *Parabola*, παραβαλλοντι, seu motum ad *constantem* seu æquabilem accedens, genita. Atque his definitionibus nominum affectiones singularum sectionum probè respondent.

3. Crescente uno latere et decresciente altero, five *constantè*, five *diversimodè*, ob momentum posterioris jam negativum, $-b$, rectanguli fluxio fiet residua, $aB - bA$.

Casu hoc primo, omnium fundamentali, jam rigidè demonstrato ex principiis purè *Algebraicis*, omni motus formalis seclusà consideratione; et fusiùs illustrato ex *Mechanicâ Rationali* atque *Geometriâ*, secundùm mentem ipsius inventoris; reliquos casus brevius expedire fas erit; quibusdam aliundè intertextis.

91. *Caf. 1. Cor.*—Hinc fluxio fractionis $\frac{A}{B}$ fiet $\frac{aB - bA}{B^2}$. Ponatur $\frac{A}{B} = Q$; eritque $A = BQ$. Adeòque per *Caf. 1.* $a = qB + bQ$; five $q = \frac{a}{B} - \frac{bQ}{B}$.

$$\frac{bQ}{B} = \frac{a}{B} - \frac{b}{B} \times Q = \frac{a}{B} - \frac{b}{B} \times \frac{A}{B} = \frac{a}{B} - \frac{bA}{B^2} = \frac{aB}{B^2} - \frac{bA}{B^2} = \frac{aB - bA}{B^2}.$$

 Q. E. D.

92. Fluxio contenti sub lateribus quocunque, A, B, C , &c. æquatur summæ ex fluxionibus singulorum in reliqua latera sigillatim continuè ductis. Sic fluxio ABC , &c. $= aBC$, &c. $+ bAC$, &c. $+ cAB$, &c.

Ponatur $AB = G$. Et per *Caf. 1.* erit fluxio ABC , seu $GC = gC + cG = (aB + bA) \times C + c \times AB = aBC + bAC + cAB$. Et par est ratio contenti sub lateribus quocunque.

93. *Caf. 3.* Fluxio dignitatis cujusvis affirmativæ (A^m) æquatur indici illius dignitatis (m), fluxioni lateris (a), ejusque co-efficienti (A^{m-1}), in se continuè ductis. Seu, fluxio $A^m = maA^{m-1}$.

Fluxio quadrati A^2 , seu $A \times A$, erit $aA + aA = 2aA$, per *Caf. 1.* Fluxio itèm cubi A^3 , seu $A \times A \times A$, erit quoque $aA^2 + aA^2 + aA^2 = 3aA^2$, per *Caf. 2.*

Et, eodem argumento, fluxio $A^m = maA^{m-1}$. Q. E. D.

Cor. Hinc fluxio $\frac{A^m}{m}$ erit aA^{m-1} .

Scholium. Proprietate curvarum elegantissimâ, si area quævis curvilinea super basi (AP) designetur per A^m , quolibet in statu mutationis; semper exponetur ordinata (PM) per maA^{m-1} . Vide *Maclaurin's Fluxions*, § 935, fig. 352.

94. *Cas. 4.* Fluxio dignitatis cujuscvis negativæ (A^{-m}) æquatur indici, in lateris fluxionem ductæ ($-ma$), et per indicem genitæ unitate auctum divisæ ($m+1$). Seu fluxio $A^{-m} = \frac{-ma}{A^{m+1}}$.

Quantitatis constantis $A \times \frac{1}{A} = 1$, erit fluxio nihil, per *Ax. 1.* Fiat $\frac{1}{A} = B$; eritque fluxio $AB = aB + bA = 0$. Adeoque b , seu fluxio $\frac{1}{A}$ vel A^{-1} $= \frac{-aB}{A} = \frac{-a}{A} \times \frac{1}{A} = \frac{-a}{A^2}$. Et generalitèr, quantitatis constantis $A^m \times \frac{1}{A^m} = 1$, fluxio erit nihil. Fiat $\frac{1}{A^m} = B$, eritque fluxio $A^m \times B = bA^m + maBA^{m-1} = 0$. Adeoque b , (seu fluxio $\frac{1}{A^m}$ vel A^{-m}) $= \frac{-maBA^{m-1}}{A^m} = \frac{-maB}{A^1} = \frac{-ma}{A^1} \times \frac{1}{A^m} = \frac{-ma}{A^{m+1}}$. Q. E. D.

95. *Cas. 5.* Fluxio cujuscunque dimensionis fractæ affirmativæ ($A^{\frac{m}{n}}$) æquatur indici illius dimensionis ($\frac{m}{n}$), lateris fluxioni (a), ejusque co-efficienti ($A^{\frac{m}{n}-1}$), in se continuè ductis. Seu fluxio $A^{\frac{m}{n}} = \frac{m}{n} a A^{\frac{m}{n}-1}$.

Quantitatis $A^{\frac{1}{2}} \times A^{\frac{1}{2}} = A$ fluxio erit a . Fiat $A^{\frac{1}{2}} = B$; adeoque fluxio B^2 (seu $A^{\frac{1}{2}} \times A^{\frac{1}{2}}$) erit $2bB = a$; et proinde b (seu fluxio $A^{\frac{1}{2}}$) $= \frac{a}{2B} = \frac{a}{2A^{\frac{1}{2}}}$. Et generalitèr: fiat $A^{\frac{m}{n}} = B$; eritque $A^m = B^n$; et proinde harum fluxiones erunt inter se æquales, nempè $maA^{m-1} = nbB^{n-1}$; et dividendo

videndo utrinque per $A^m = B^n$, erit $maA^{-1} = nbB^{-1}$; ideòque erit b (seu fluxio B, five $A^{\frac{m}{n}}$) $= \frac{maA^{-1}}{nB^{-1}} = \frac{maA^{-1}}{n \times A^{-\frac{m}{n}}} = \frac{maA^{\frac{m}{n}-1}}{n}$. Q. E. D.

96. N. B. Pro co-efficientis indice, $\frac{m}{n} - 1$, usurpat *Newtonus* indicem æquipollentem $\frac{m-n}{n}$ ($= \frac{m}{n} - \frac{n}{n}$) $= \frac{m}{n} - 1$, in demonstratione hujus casûs; unde prodit fluxio $A^{\frac{m}{n}} = \frac{m}{n} aA^{\frac{m-n}{n}}$. Eâdem nimirum notatione indicum hâc occasione usus, ac compluribus ante annis, nempe anno 1676, ubi primò præclarum suum *Theorema Binomium* in lucem protulit. Vid. ANALYS. ÆQUAT. § 57. Cæterum hoc non levem suppeditat conjecturam, Methodum Fluxionum, vel illo tempore, Newtono perspicuè innotuisse; ut posthâc plenius et irrefragabilius evincetur ex insigni analogiâ inter terminos successivos seriei binomiæ, atque fluxiones superiorum ordinum, intercedente; talem enim analogiam inter utramque methodum, utriusque inventorem effugisse, suspicari non fas est.

97. Caf. 6. Fluxio genitæ cujuscunque ex duobus pluribûsve factoribus, æquatur summæ ex fluxionibus uniuscujusque factoris in reliquam factorem vel reliquos factores, continuè inter se ductis. Sic fluxio $A^m B^n = maB^n A^{m-1} + nbA^m B^{n-1}$.

Ex præcedentibus constat; nempe, Caf. 1 et 3. Et par est ratio contenti sub pluribus factoribus; nam, per Caf. 2, erit fluxio $A^m B^n C^r = maB^n C^r A^{m-1} + nbA^m C^r B^{n-1} + rcA^m B^n C^{r-1}$.

Idem quoque obtinebit per Caf. 5, ubi indices factorum sunt fracti, $\frac{m}{n}$, $\frac{r}{s}$, &c. $A^{\frac{m}{n}} B^{\frac{r}{s}}$; cujus fluxio evadet $\frac{m}{n} a B^{\frac{r}{s}} A^{\frac{m}{n}-1} + \&c.$

Et hoc est *Theorema Fluxionale* nobilissimum et generalissimum, uno intuitu omnes casus comprehendens, ipsi *Theoremati Binomio*, elegantia, concinnitate et dignitatis præstantiâ vix, et ne vix quidem, cedens.

98. Conclusiones præcedentes nullo negotio ad notationem posteriorem accommodare licet.

$$\text{Fluxio rectanguli } xy = \dot{x}y + x\dot{y}.$$

$$\text{Fluxio contenti } - xyz = \dot{x}yz + x\dot{y}z + xy\dot{z}.$$

$$\text{Fluxio quoti } - \frac{x}{y} = \frac{\dot{x}y - x\dot{y}}{y^2}$$

$$\text{Fluxio quadrati } - x^2 = 2\dot{x}x.$$

$$\text{Fluxio dignitatis } x^m = m\dot{x}x^{m-1}.$$

$$\text{Fluxio dimensionis } x^{\frac{m}{n}} = \frac{m}{n} \dot{x} x^{\frac{m}{n}-1}$$

$$\begin{aligned} \text{Fluxio genitæ } - x^m y^n &= m\dot{x}x^{m-1}y^n + n\dot{y}y^{n-1}x^m, \text{ vel,} \\ &= (m\dot{x}y + n\dot{y}x) x^{m-1}y^{n-1}, \text{ vel,} \\ &= \left(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y}\right) x^m y^n. \end{aligned}$$

$$\begin{aligned} \text{—Itémque, } x^m y^n z^r &= m\dot{x}x^{m-1}y^n z^r + n\dot{y}y^{n-1}x^m z^r + r\dot{z}z^{r-1}x^m y^n, \\ \text{vel, } &= (m\dot{x}yz + n\dot{y}xz + r\dot{z}xy) x^{m-1}y^{n-1}z^{r-1}, \\ \text{vel, } &= \left(\frac{m\dot{x}}{x} + \frac{n\dot{y}}{y} + \frac{r\dot{z}}{z}\right) x^m y^n z^r. \end{aligned}$$

99. Corollaria quædam *Newtoniana*, à *Maclaurin* omiffa, hîc adjungere libet, ex præcedente theoriâ resultantia, eandémque optimè illustrantia, et in *Logarithmotechniâ* valdè utilia.

Corol. 1. Hinc, in continuè proportionalibus, si terminus unus datur, fluxiones terminorum reliquorum, erunt ut iidem termini multiplicati per numerum intervallorum inter ipsos et terminum datum.

Sunto A, B, C, D, E, F, continuè proportionales, et si *detur* terminus intermedius C, (cujus idcirco fluxio erit *nihil*,) exquirantur tùm valores reliquorum, ex notis proportionalium proprietatibus, tùm horum fluxiones, ex præcedenti theoriâ, itémque fluxionum rationes :

Proport.	Valor.	Flux.	1 Rat.	2 Rat.
A five	$\frac{C^3}{D^3}$	$C^3 \times \frac{-2d}{D^3}$	$-\frac{2C^3}{D^2}$	$-2A$
B	$\frac{C^2}{D}$	$C^2 \times \frac{-d}{D^2}$	$-\frac{C^2}{D}$	$-1B$
C	1	0	0	0
D	D	d	+D	+1D
E	$\frac{D^2}{C}$	$\frac{2dD}{C}$	$+\frac{2D^2}{C}$	$+2E$
F	$\frac{D^3}{C^2}$	$\frac{3dD^2}{C^2}$	$+\frac{3D^3}{C^2}$	$+3F$

In hac tabulâ, ex serie fluxionum elicitur prima series rationum $-\frac{2C^3}{D^2}$, $-\frac{C^3}{D}$, &c.; multiplicando nimirum singulas fluxiones correspondentes, per communem quantitatem $\frac{D}{d}$; secunda series autem, ex primâ, restituendo pro valoribus suis, proportionales ipsos, $-\frac{2C^3}{D^2} = -2A$, &c. Q. E. D.

100. Corol. 2. Et si in quatuor proportionalibus duæ mediæ dentur, fluxiones extremarum erunt ut eadem extremæ.

Nam, ob datum idcirco rectangulum sub extremis AD (utpote æquale BC), erit fluxio ejus $aD + dA = 0$; ideoque, $aD = -dA$; et proinde, $a : -d :: A : D$. Q. E. D.

Idem intelligendum est de lateribus rectanguli cujuscunque dati. Pari enim ratione, erit $b : -C :: B : C$. Cæterum in hujusmodi casibus, ob fluxiones negativas $-c$, $-d$, diversimodè variantur latera generantia, uno crescente, altero decrescente.

101. Corol. 3. Et si summa vel differentia duorum quadratorum detur, fluxiones laterum erunt reciproce ut latera.

Sint enim quadrata $A^2 \pm B^2 = 1$, seu cuivis quantitati datæ; erit fluxio $2aA \pm 2bB = 0$. Adeoque, $2aA = \mp 2bB$; et proinde, $a : \mp b :: B : A$. Q. E. D.

De Fluxionibus Superiorum Ordinum.

102. Si quantitates fluentes ratione constanti, æquabili seu datâ, simul augentur vel minuuntur, fluxiones primas tantum, seu primi ordinis, fortientur. Quod si harum fluxiones sint quoque variables, suas iterum fluxiones fortientur, secundas nempe, seu secundi ordinis; atque hæ iterum suas fortientur fluxiones seu tertias; atque ita deinceps, donec ultimò perveniatur ad quantitates constantèr seu æquabiliter fluentes.

103. Sic in theoriâ virium centripetarum, binis viribus conjunctis urgentibus, revolvuntur corpora in orbibus, seu trajectoriis quibuscunque, circa commune centrum virium: nempe, vi projectili, quæ constantèr et æquabiliter agit, secundum directiones tangentium singulis in orbitæ locis; et, vi centripetâ, perpetuo motu variabili, corpora de tangentibus versùs centrum deflectente; cum actio
vis

vis projectilis igitur est *constans per se* (nisi quatenus per alteram vim *obliquè* agentem, augetur vel minuitur), *primas* tantum pariet fluxiones; quarum exponentes erunt *spatia* perbrevia æqualibus temporibus quàm minimis percurfa, seu *arcus* momentanei cum *tangentibus* suis ferè coincidentes. At *vis centripeta*, cum accedendo versùs centrum, fortiùs, recedendo à centro, languidiùs operatur, actio erit variabilis, ejùsque exponentes in locis correspondentibus, non erunt *ipsa spatia*, *momentanea* cadendo versùs centrum percurfa, si sisteretur vis projectilis (cum ipsæ deflexiones erunt variabiles, pro diversis à centro distantis), sed *horum dupla*, seu spatia quæ velocitatibus *ultimis* cadendo acquisitis describerentur (si exindè constantèr velocitates forent uniformes), per eadem tempora quam minima. *Vis centripeta*, igitur, *fluxiones* suas *secundas* fortietur; quarum exponentes sunt *diagonales illæ*, suprâ memoratæ, quæ ad centrum commune virium convergunt: hæ enim sunt *duplæ sagittarum* illarum, quas corpora his momentis liberè cadendo, *actu* describerent. In *circulis* autem hæ sagittæ sunt ipsi *sinus versi*.—Atque hæ observationes, theoriæ illæ difficillimæ ac subtilissimæ virium *projectilium* et *centripetarum*, cui innititur omnis ASTRONOMIA NEWTONIANA, explicandæ et illustrandæ, haud leve adjumentum forsàn tyronibus suppeditabunt.

104. *Fluxiones superiorum ordinum* iisdem legibus inveniendæ sunt ac ipsæ *fluxiones primæ*. Sic generalitèr: genitæ cujuscunque $A^{\frac{m}{n}}$ erit

$$\text{Fluxio prima, per Caf. 5,} \quad - \quad - \quad - \quad - \quad \frac{m}{n} a A^{\frac{m}{n} - 1};$$

$$\text{Secunda,} \quad - \quad \left(\frac{m}{n} - 1\right) \times (a) \times \left(\frac{m}{n} a A^{\frac{m}{n} - 2}\right) = \frac{m}{n} \cdot \frac{m}{n} - 1 a^2 A^{\frac{m}{n} - 2};$$

$$\text{Tertia,} \quad - \quad \left(\frac{m}{n} - 2\right) \times (a) \times \left(\frac{m}{n} \cdot \frac{m}{n} - 1 a^2 A^{\frac{m}{n} - 3}\right) = \frac{m}{n} \cdot \frac{m}{n} - 1 \cdot \frac{m}{n} - 2 a^3 A^{\frac{m}{n} - 3}.$$

Et sic deinceps, eâdem servatâ lege derivationis. Ut, si cubi A^3 quærantur fluxiones: designante $\frac{m}{n}$, 3; erit *fluxio prima*, - - $3aA^2$;

$$\text{secunda,} \quad - \quad 6a^2A;$$

$$\text{tertia,} \quad - \quad 6a^3.$$

Fluxio autem *quarta* erit *nihil*; jam evanescente factore A.

Hinc, si genitæ index sit numerus integer, series erit finita; sin fractus, indefinita.

105. Combinando autem terminos inter operandum inventos, brevius exponere licet seriem fluxionalem, ponendo nimirum, $\frac{m}{n} = p$; $\frac{m}{n} \cdot \frac{m}{n} - 1 =$

q ; $q \times \frac{m}{n} - 2 = r$; erit $pa A^{\frac{m}{n} - 1}$, fluxio prima,

$qa^2 A^{\frac{m}{n} - 2}$, fluxio secunda,

$ra^3 A^{\frac{m}{n} - 3}$, fluxio tertia,

$sa^4 A^{\frac{m}{n} - 4}$, fluxio quarta.

Atque ulterius combinando, formam adhuc simpliciore indicet series; ponendo $pa = P$; $\frac{m}{n} - 1 \times aP = Q$; $\frac{m}{n} - 2 \times aQ = R$.

$\frac{m}{n} - 3 \times aR = S$, &c. erit jam series,

$PA^{\frac{m}{n} - 1} + QA^{\frac{m}{n} - 2} + RA^{\frac{m}{n} - 3} + SA^{\frac{m}{n} - 4} + \&c$; ipsi seriei binomiali, in simplicitate non inferior.

De Theoremate Binomio.

106. In *Appendice ad ANALYSIN ÆQUATIONUM*, p. 33, &c. dudum demonstratum est, "ex principiis Algebrae propriis," (ut ibi dictum,) ad mentem *Johannis Landen*, in tractatu suo *De Analyfi Residuâ*, theorema hoc longè celeberrimum, sub variis ejus formulis, § 57; (quales protulit *Newtonus*, in *Epistolis* cum *Leibnitz* communicandis, an. 1676,) atque sub formulâ usitatore, *Cor. 1.* qualem non semel adhibuit ipse *Newtonus*. Ex. gr. *Prop. 45. Lib. 1. Princip.*

ubi statuit radium vectorem $A^n = (T - X)^n = T^n - nXT^{n-1} + \frac{n^2 - n}{2} X^2$

T^{n-2} , &c. Atque iterum, in *Scholio, Prop. 93, ordinatam* $(A + O)^{\frac{m}{n}} = A^{\frac{m}{n}}$

$+ \frac{m}{n} OA^{\frac{m-n}{n}} + \frac{mm - mn}{2nn} O^2 A^{\frac{m-2n}{n}}$, &c.—Plenior et fusior explicationem

cationem Methodi *Landenianæ* inveniet lector horum studiosus, apud *Maseres*, in libro cui titulus *Scriptores Logarithmici*, Vol. II. p. 170, &c. et 176, &c.

107. Verùm hanc methodum perpendenti, atque iterùm ad examen revocanti; vix affirmare licet, ex principiis *merè Algebræ* eam prorsus peti: assumptio enim cui potissimùm innititur demonstratio, nempe, “quòd *salvâ æquatione variari potest y ad libitum*, donec fiat æqualis ipsi *x* in singulis dimensionibus fractionis $\frac{1+y}{1+x}$,” calculum *fluxionalem* planè redolet, qualem in demonstratione *tertii* casûs, § 93, antecedentis jam perspeximus, æquatis valoribus A, B, C, &c.

108. Admirandam fanè intercedere analogiam inter *Theorema Binomium* et *Theoriam Fluxionum*, liquidò constat ex collatione terminorum seriei resultantis, cum diversis fluxionum ordinibus suprà expositis, § 105.

Exponat enim A quantitatem quamvis generantem, et *a* hujus *momentum*, seu fluxionem, sive crescendi sive decrescendi; designabit igitur $A \pm a$ valorem

generantis, et $(A \pm a)^{\frac{m}{n}}$ *genitæ*, simul ac semel fluentium. Hujus autem resolutio dabit seriem, citissimè convergentem, cui apponitur series quoque fluxionum diverforum ordinum.

Series Binomia.	Series Fluxionum.
1. $A^{\frac{m}{n}}$ - - - - -	$A^{\frac{m}{n}}$ - - - - - <i>Fluens.</i>
2. $\pm \frac{m}{n} a A^{\frac{m}{n} - 1}$ - - - - -	$\frac{m}{n} a A^{\frac{m}{n} - 1}$ - - - - - 1 <i>Fluxio,</i>
3. $\frac{m}{n} \cdot \frac{m}{n} - 1 \times a^2 A^{\frac{m}{n} - 2}$ - - - - -	$\frac{m}{n} \cdot \frac{m}{n} - 1 a^2 A^{\frac{m}{n} - 2}$ - - - - - 2 <i>Fluxio,</i>
4. $\pm \frac{m}{n} \cdot \frac{m}{n} - 1 \cdot \frac{m}{n} - 2 \times a^3 A^{\frac{m}{n} - 3}$ - - - - -	$\frac{m}{n} \cdot \frac{m}{n} - 1 \cdot \frac{m}{n} - 2 a^3 A^{\frac{m}{n} - 3}$ - - - - - 3 <i>Fluxio.</i>
&c.	&c.

Hinc constat, quòd *primus* seriei terminus est *ipsa fluens*; *secundus* terminus, *ipsa prima fluxio*; reliqui autem terminis fluxionibus superiorum ordinum, *proportionales* respectivè, et proinde pro his usurpari possunt; per fluxionum definitionem, “*mensuræ quævis rationum*,” &c.

Nam

Nam tertius terminus est ad secundam fluxionem in ratione $\frac{1}{2}$ ad 1, seu 1 ad 2.

Quartus terminus ad tertiam fluxionem in ratione $\frac{1}{2} \times \frac{1}{3}$ ad 1, seu 1 ad $2 \times 3 = 6$.

Quintus terminus ad quartam fluxionem in ratione $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$ ad 1, seu 1 ad $2 \times 3 \times 4 = 24$.

Et generalitèr, terminus quilibet $m + 1$, ad fluxionem ordinis m proximè inferioris, in ratione $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \dots \frac{1}{m}$ ad 1, seu 1 ad $2 \times 3 \times 4 \times 5 \&c \times m$; hoc est, in ratione datâ.

109. Et hinc facillimè quidèr, et optimè forsàn, ex theoriâ fluxionum *germanâ* derivatur constructio seriei ex resolutione theorematìs binomii resultantis; nam quilibet seriei terminus $m + 1$ invenietur dividendo fluxionem ordinis m per $2 \times 3 \times 4 \dots m$. Sic terminus seriei *sextus* est æqualis fluxioni *quintæ*,

$1a^5 A^{\frac{m}{n} - 5}$, per 24×5 divisæ. Et vicissim, ex serie binomiâ inveniri possunt fluxionum ordines.

110. Simplicitate atque facilitate longè præstat hæc demonstratio, ex fluxionum *serie* petita, *Landenianæ* isti breviori, ex consideratione *primæ* fluxionis tantùm haustæ; ab eâ quam dudùm exposuit *Maclaurin*, § 748, vix discre-

panti; quippe quæ primo *assumit*, quantitatem binomiam $(1+x)^{\frac{m}{n}}$ designari posse per seriem indicibus integris, perpetuò crescentem, $1 + ax + bx^2 + cx^3 + dx^4$, &c.

Et harum quantitatum fluxionibus utrinque inventis, erit $\frac{m}{n} \times (1+x)^{\frac{m}{n} - 1} \times \dot{x} = a\dot{x} + 2bx\dot{x} + 3cx^2\dot{x} + 4dx^3\dot{x}$, &c. unde, dividendo omnes terminos

per $\dot{x}$, emergit æquatio finalis $\frac{m}{n} \times (1+x)^{\frac{m}{n} - 1} = a + 2bx + 3cx^2 + 4dx^3$, &c.; eadem prorsùs ad quam tanto circuitu tandem pervenit methodus *Landeniana*; vide *Analyfin Æquationum*, p. 38. Cujus iterùm resolutio, calculo fatìs operoso, dabit valores singulorum co-efficientium, b, c, d , &c.

Hinc manifestò liquet, *Analyfin Residuam* ipsam, cujus laudes tantopere prædicat *Landen*, vix discrepare in principiis suis, licet longè operosior ac tortuosior, à *Methodo Fluxionum*; ac proinde pro larvâ ejus jure censerî posse.

III. Ex dictis constat, *seriem binomiam* $(A \pm c)^{\frac{m}{n}}$ citiùs convergere quam *series fluxionum genitæ* $A^{\frac{m}{n}}$; ob decrementum terminorum prioris per denominatores successivos terminis posterioris subjiciendos, $1 \times 2 = 2$; $2 \times 3 = 6$; $6 \times 4 = 24$; &c.

$$\text{Sic, } (A + a)^3 = A^3 + 3aA^2 + 3a^2A + a^3.$$

$$\text{Fluxiones, } A^3, = 3aA^2 + 6a^2A + 6a^3.$$

De Methodo Inversa Fluxionum.

III.2. *Methodus Inversa*, seu *Calculus Integralis*, vicissim, ex fluxionibus datis invenire fluentes aggreditur. Estque “hoc molestissimum et omnium difficillimum problema,” *Newtono* judice: nam hætenùs desideratur methodus generalis eruendi fluentes directè, seu immediatè ex fluxionibus; ad particulares casus tantùm restringitur investigatio, eosdèmq; maximâ ex parte, cursum relegendo methodi directæ.

Auspicari libet à casibus simplicioribus.

III.3. *Cas. 1. Fluxionum simplicium unius tantùm dimensionis fluentes substitutione solâ inveniuntur.*

Sic fluxionis a , fluens est A ; fluxionis $\frac{m}{n} a$, $\frac{m}{n} A$; vel fluxionis $\dot{x}$, fluens x ; $\frac{m}{n} \dot{x}$, $\frac{m}{n} x$.

III.4. *Cas. 2. Fluxionum simplicium plurium dimensionum, fluentes resolutione regularum methodi directæ inveniuntur.*

Sic quoniam fluentis A^3 fluxio fuit $3aA^2$; dividendo priorem per posteriorem, prodit $\frac{A}{3a}$; quæ utique, multiplicando datam fluxionem, dabit fluentem vicissim.

cissim. Itémque fluxionis $3x^5$, multiplicando per $\frac{x}{6x}$, fluens erit $\frac{3x^6}{6} = \frac{1}{2}x^6$.

Et generalitèr, formulæ $\frac{m}{n} aA^{\frac{m}{n}-1}$, multiplicando per $\frac{A}{\frac{m}{n}a}$, eruetur fluens

quæsitæ $A^{\frac{m}{n}}$. Similitèr, formulæ aA^{m-1} erit fluens $\frac{A^m}{m}$, ut ex casibus 3 et 5 methodi directæ constât.

115. Methodum directam excogitavit *Pacassi* inveniendi fluentes, instituendo analogiam: *Ex fluxione (x) datâ, eliciatur nova quædam fluens (y); eritque hujus fluxio (y) ad fluxionem datam (x) ut fluens elicitæ (y) ad fluentem quæsitam (x):* et indè prodit $x = \frac{xy}{y}$.

Sit fluxio data $3aA^2$; substituendo A pro a , emergit nova fluens $3A^3$; inveniatur hujus fluxio, $9aA^2$; undè prodibit $\frac{3aA^2 \times 3A^3}{9aA^2} = A^3$, fluens quæsitæ. Atque hæc regula, licet præcedente operosior, in *fluentium correctione* per methodum inverſam inventarum, ufui esse potest.

116. *Fluxionis compositæ, unius tantùm dimensionis formulæ $xy \pm xy$, invenietur fluens, substituendo fluentes pro fluxionibus, et dimidiando aggregatum; nam $\frac{xy + xy}{2} = xy$.* Et paritèr, formulæ $\frac{xy - xy}{y^2}$ fluens erit $\frac{xy + xy}{2y^2} = \frac{x}{y}$.

117. Veruntamen sæpenuerò occurrunt quantitates fluxionales magis compositæ, quarum fluentes per regulas prædictas neutiquàm erui licet. Decursu autem computorum fluxionalium, complures *formulæ fluentiales*, formulis *fluxionalibus* magis compositis respondentes, inventæ sunt ab analyſtis celeberrimis, inter operandum; et potissimùm, à *Newtono, Cotes, Euler, Clairaut, Walmsley, Landen, Simpson, Waring, De La Grange, &c.*; quarum præcipuæ ad calcem capitis *de Fluxionibus Logarithmorum* rejiciuntur, § 132.

De Correctione Fluentium.

118. Cæterùm fluentes hoc pacto repertæ, five per regulas præcedentes, five per formulas, hæud rarò correctione quâdam ulteriore egent, quo ad

amiffim respondeant cafibus problematum fluxionalium particularibus inveniendis. Nam fluxionis $\dot{x}$ fluens effe poteft vel x , vel $x \pm c$, feu x constanti quâdam quantitate, c , aucta vel diminuta; adeò ut fluens jam inventa, conditionibus quibusdam, quæ problema aliquatenus reſtringunt, poterit forſan non accuratè reſpondere. Unde neceſſe erit ulterius fluentem veram eruere ex ipſius problematis conditionibus, relationem fluentium ad invicem in certo quodam temporis puncto, aut mutationis ſtatu, definiens ſeu limitans. Reducatur igitur generalis æquatio fluentialis ad iſtum ſtatum particularem, ſubſtituendo pro fluentibus ipſis cognitos, quos ſortiuntur, valores. Deinde ſubducatur æquatio reſultans ex æquatione generali, et reſidua utrinque æquationem correctam fluentium ex regulis inventarum, elicient.

Ut, ſi æquatio fluxionalis fit $\dot{z} = ax^3 \dot{x}$, æquatio fluentialis poteſt effe vel $z = ax^4$, vel $z = ax^4 \pm c$. Si ſimul evaneſcunt x et z , ſubſtituendo 0 pro x et z , fiet æquatio $0 = 0 + c$. Adeoque $c = 0$: quo in caſu, ax^4 veram fluentem exponet, evaneſcente c .

2. At, ſi, evaneſcente z , reſtat $x = b$, finitæ cuidam quantitati cognitæ; fiet æquatio fluentialis, in iſto caſu, $0 = ab^4 + c$: unde prodit $c = -ab^4$; et ſubſtituto hoc valore, prodit $z = ax^4 - ab^4$.

3. Fiat jam $z = d$, iſto in ſtatu ubi x fit æqualis b ; et his valoribus ſubſtitutis, emerget æquatio fluentialis $d = ab^4 + c$; adeoque erit $c = d - ab^4$; et, ſubſtituto iterum hoc valore pro c , prodit $z = ax^4 - ab^4 + d$, valor nimirum correctus fluentis quæſitæ.

4. Idem quoque brevius expedire licet, ſimul et ſemel ſubſtituendo valores quovis contemporaneos quantitatum x et z . Nam ſi ab æquatione fluentiali $\dot{z} = ax^3$, ſubducatur novus valor $d = ab^4$, fiet reſiduum $z - d = ax^4 - ab^4$; adeoque $z = ax^4 - ab^4 + d$, ut antea: aut igitur evaneſcentibus b et d , aut æquatis $+d$ et $-ab^4$, fiet ipſa fluens quæſita ax^4 .

De Fluxionibus Diverſas Fluentes admittentibus.

119. Nullâ re magis arguitur imperfectio *Methodi Inverſæ*, quam quòd eadem æquatio fluxionalis ſeu differentialis, aliquando duas ſolutiones, omninò ab invicem diverſas, fortiatur; quod primò detexit *Clairaut*, *Mem. Acad.* 1734, et poſteà confirmarunt *Euler*, *D'Alembert*, *De La Place*, *De La Grange*, et *Le Gendre*. Vide M. R. Append. Vol. 28, p. 537.

Huiusmodi eſt æquatio differentialis $x dy dx - dy^2 = y dx^2 - dy dx$, ſive ſecundùm notationem fluxionalem, $x \dot{x} \dot{y} - \dot{y}^2 = y \dot{x}^2 - \dot{y} \dot{x}$. Huic enim

enim solvendæ æquæ respondent vel æquatio fluentialis $4y = x^2 + 2x + 1$, vel $2ax - 2x = -4y + 1 - a^2$.

Similiter quoque æquationem differentialem $ady^2 + xdy^2 - ydydx = xdydx - ydx^2$ æquæ solvent, vel $\frac{x}{\sqrt{y}} - \sqrt{y} = \sqrt{4a}$, vel $b^2y - 2bx + 2by = -4a$.

Supereſt ut breviter proponere aggrediar applicationem Fluxionum ad *Logarithmotechniam*, planius ut ſpero, et plenius, quam haſtenſ; ob ingentem Logarithmorum utilitatem in Calculis Trigonometricis, &c. abbreviandis; ſdque libentiùs, quòd inter celeberrimorum *Scriptorum Logarithmicorum* congeſta opera locum habere permittatur hæc *Analysis Fluxionum*, mole quidem parva, ſed quæ tamèn, pro rerum difficultate, ſubtilitate, varietate, et pondere, (quæ fortaffe fuſius explicari debuiffent,) quantumvis impar, tanto honore non prorsùs indigna videatur.

De Fluxionibus Logarithmorum.

120. *Logarithmi*, quaſi λογων αριθμοι, “*Rationum numeri exponentes*” ſunt numeri arithmetice proportionales, numeris geometricè proportionalibus reſpondentes, ſeu ſecundùm definitionem vulgatam, “*Numerorum proportionalium comites æquidifferentes* :” ut in ſeriebus inſequentibus Logarithmorum vulgarium :

Numeri	{	&c. 10^3 , 10^2 , 10^1 , 10^0 , 10^{-1} , 10^{-2} , 10^{-3} , &c.						
		vel 1000 , 100 , 10 , 1 , $\cdot 1$, $\cdot 01$, $\cdot 001$.						
Logarithmi	{	3,	2,	1,	0,	-1,	-2,	-3.

De utilitate inſigni Logarithmorum in omni genere Computorum Arithmeti-
corum, Trigonometricorum, abbreviando operationes *Multiplicationis*, *Divisionis*, *Involutionis* et *Evolutionis*, hoc loco monuiſſe tantùm ſufficiat : adeat harum rerum ſtudioſus librum antea memoratum, cui titulus *Scriptores Logarithmici*, à viro cl. *Maſeres* congeſtum et editum.

121. In omni ſyſtemate Logarithmorum, ſtatuitur logarithmus unitatis eſſe nihil, aut 0. In omni ſyſtemate igitur, *ſumma logarithmorum cujuſvis numeri logiſtici, ejusque reciproci*, cum ſint æquales et contrariis ſignis affecti, fiet 0, ſeu evaneſcet.

In

In Systemate Logistico vulgari, seu *Briggiano*, sumitur *unitas* pro logarithmo numeri 10; at in Systemate Hyperbolico, seu *Neperiano*, *unitas* prodit logarithmus numeri 2.7182818, &c.

122. In quovis autem systemate, numerus qui crescit vel decrescit eadem ratione ac logarithmus ejus, fit quasi *modulus* istius systematis: eritque *modulus unius systematis ad modulum alterius, ut logarithmus dati numeri in uno ad logarithmum ejusdem numeri in altero.*

Sic in systemate vulgari, logarithmus numeri 10 est 1; at in hyperbolico, logarithmus 10 est 2.3025850, &c. In hyperbolico, item modulum est 1; undè prodit modulus systematis vulgaris $= \frac{1.0000000}{2.3025850} = 0.4342944$.

Eritque *ratio modularis*, in systemate hyperbolico 1 ad 1, five ratio æqualitatis; at in vulgari, 0.4342944 ad 1, nempe ratio data.

123. *Fluxiones logarithmorum, fluxionibus ordinis secundi ascribendæ sunt; cum ipsi logarithmi fluxionibus primis respondent: ut ex definitionibus constat.*

Methodus hæc inveniendi propositionibus insequentibus fundatur.

124. *Prop. 1. Quantitatum logisticarum fluxiones sunt ipsis quantitibus proportionales.*

Nam quoniam quantitates ipsæ proportionem *geometricam* crescunt vel decrescunt, erunt differentiæ earum ipsis quantitibus proportionales; per conversam *Lem. 1. Lib. II. PRINCIP.* Et proinde, fluxiones, quæ variantur ut hæ differentiæ, erunt quoque ipsis quantitibus proportionales. Q. E. D.

125. *Prop. 2. Fluxio quantitatis logisticae est ad fluxionem logarithmi ejus, ut ipsa quantitas ad modulum systematis logistici.*

Per propositionem præcedentem est n , fluxio quantitatis N , ad m , fluxionem quantitatis M , ut N ad M . Designet M *modulum*, et quoniam m in hoc casu est constans, substitui potest pro fluxione logarithmi N quoque, cum hujus logarithmus est in datâ quâdam ratione ad modulum. Erit igitur n , fluxio ipsius N ad l , fluxionem logarithmi ejus, ut N ad M . Q. E. D.

126. *Prop. 3. In quovis systemate logarithmorum, fluxio logarithmi cujuscvis dati æquatur fluxioni ipsius quantitatis logisticae, in modulum ductæ, et per ipsam quantitatem divisæ.*

Nam (per præced.) est $n : l :: N : M$; et invertendo, $l : n :: M : N$, five $l = \frac{nM}{N}$. Q. E. D.

127. *Prop.*

127. Prop. 4. In systemate hyperbolico, fluxio logarithmi cujusvis dati æquatur fluxioni ipsius quantitatis logistica, per ipsam quantitatem divisæ.

Nam in hoc systemate, $M = 1$, et proinde, $l = \frac{n}{N}$. Q. E. D.

Idem quoque geometricè illustrare licet ex naturâ arearum *hyperbolæ rectangulæ*, inter curvam, asymptotas et ordinatas interclusarum. Designantibus enim x asymptoti abscissam; y , ordinatam; et m , quadratum constans, contentum utique asymptotis et parallelis à vertice ductis, seu modulum; (per naturam hyperbolæ,) erit $xy = m$; adeoque, $y = \frac{m}{x}$; et multiplicando utrinque per $\dot{x}$, erit $\dot{x}y = \frac{\dot{x}m}{x}$, sive ob $m = 1$, $= \frac{\dot{x}}{x}$; sed $\dot{x}y$ est fluxio logarithmi dati; (per naturam hyperbolæ et logarithmorum communem). Ergo, &c. Q. E. D.

Hinc manifestum est quòd *hyperbolæ rectangulæ* cujus axis est asymptotos, est curva logistica, seu curva ad designandum logarithmos, et eos quasi ante oculos ponendum, accommodata: crescentibus enim abscissis in progressionem *Arithmeticâ*, decrescunt ordinatæ in progressionem *Geometricâ*.

De Fluxionibus Quantitatum Exponentialium.

128. Quantitates exponentiales sunt aliarum quantitatum potestates quarum indices sunt variabiles; sicut vel e^x , vel y^x ; quarum prima est potestas quantitatis constantis e variabili indice x designata, altera verò est potestas quantitatis variabilis y , variabili indice x paritèr designata.

Cas. 1. Fiat $e^x = z$. Erítque $\log. z = \log. e, \times x$; adeoque per præced. $\frac{\dot{z}}{z} = \log. e, \times \dot{x}$; et proinde, $\dot{z} = \log. e \times \dot{x}z = \log. e \times \dot{x}e^x$. Sive fluxio quantitatis exponentialis (e^x) æquatur ipsi quantitati (e^x), in indicis fluxionem ($\dot{x}$), et lateris logarithmum ($\log. e$) continuè ductæ. Q. E. D.

Cas. 2. Fiat $y^x = z$. Erítque $\log. z = \log. y \times x$; adeoque $\frac{\dot{z}}{z} = \log. y \times \dot{x}$; et proinde, $\dot{z} = \frac{\dot{y}}{y} \times \dot{x}z = \frac{\dot{y}}{y} \times \dot{x}y^x = \dot{y} \times \dot{x}y^{x-1}$. Sive fluxio quantitatis exponentialis variabilis (y^x) æquatur fluxioni lateris ($\dot{y}$) in fluxionem indicis ($\dot{x}$) et co-efficienti (y^{x-1}) continuè ductæ. Q. E. D.

N. B. In

N. B. In demonstratione hujus casûs, ex *Hutton's Mathematical Dictionary* excerptâ, occurrit *erratum* preli haud contemnendum, + pro =, bis. Art. *Fluxions*, p. 489.

De Logarithmis Imaginariis.

129. Logarithmi quantitatum impossibilium, ut $\sqrt{-a^2} = a\sqrt{-1}$, appellantur *Imagarii*; quales sunt fluentes fluxionum quarundam impossibilium, sicut $\frac{\dot{x}}{x\sqrt{-1}}$. Nam quoniam fluxionis $\frac{\dot{x}}{x}$ fluens est logarithmus *verus* quantitatis realis x ; pari ratione, erit fluxionis $\frac{\dot{x}}{x\sqrt{-1}}$ fluens, quoque logarithmus *imagi-*
narius quantitatis impossibilis $x\sqrt{-1}$.

Veruntamen ubi tales quantitates in problematum solutionibus occurrunt, facile transmutari possunt in *arcus circulares*, aut *sectores veros*, ex *hyperbolicis impossibilibus*. Vide MACLAURIN, *Of the Analogy betwixt Circular Arcs and Logarithms; Fluxions*, § 762:—PLAYFAIR, *On the Arithmetic of Impossible Quantities; Phil. Trans.* an. 1778, p. 318; or, *Analys. Æquat.* § 223, et *Scriptores Logarithmicos*, tomum secundum, in paginis 381, 382, &c - - - ad 585, ubi methodus quam invenit celeberrimus Leibnitzius anno 1676 pro extensione duarum regularum à Cardano olim traditarum de radicibus æquationum cubicarum formæ $x^3 + qx = r$ in omnibus earum casibus, et formæ $x^3 - qx = r$ in primo earum casu, seu cum $\frac{rr}{4}$ est major $\frac{q^3}{27}$, ad secundum casum formæ secundæ, in quo $\frac{rr}{4}$ est minor ipsâ $\frac{q^3}{27}$ (qui *casus irreducibilis* à scriptoribus Algebrae appellatur,) fusè et accuratè explicatur.

130. Analogiam inter *Methodum Fluxionum* et *Logarithmorum*, ex recentioribus primus detexisse videtur sagacissimus Kepler; qui logarithmos ritè definivit "*mensuras rationum*," sive "*numeros ratiuncularum inter rationem numeri alicujus ad unitatem intercedentium*." Cæterùm antiquis penitus latuisse hanc analogiam, haud facile persuadeat, vel ex solo opere celeberrimo Apollonii, *De Sectione Rationis*; quâcum ad amussim quadrat hæc expositio Kepleriana; atque, non injuriâ, Apollonio forsân debuit editor ejus, sagacissimus Halley, methodum suam egregiam, sed paulò reconditiorem exquirendi et construendi logarithmos, sive "*numeros rationum exponentes*," (optimâ nominis definitione usus), per varias combinationes et *sectiones rationum*, series logísticas citissimè convergentes instituendo; qui omnium primus, ut fertur, *problema inversum* extudit: *dato logarithmo invenire numerum ejus logisticum*.

131. Methodum *Halleianam*, celeberrimam at perdifficilem quidè, acutiùs exposuit *Jones*, in *Synopsi Palmariorum Matheseos*; plenissimè autem et luculentissimè pervestigavit analysta ex peritissimis *Maseres*, in magno suo repositoio *Scriptorum Logarithmicorum*, tomo primo in paginis 235, 236, &c - - - 383, et tomo secundo in paginis 76, 77, &c - - - 152.

132. Accedit ultimò TABULA præcipuarum FORMULARUM FLUXIONALIUM et FLUENTIALIUM hætenùs reperiatarum, ex Cl. *Hutton's Mathematical Dictionary* excerpta.

Formule Fluxionum atque Fluentium.

Fluxionum.	Fluentium.
I. $x^{n-1} \dot{x}$ - - - - -	$\frac{1}{n} x^n$.
II. $(a \pm x^n)^{m-1} x^{n-1} \dot{x}$ - - - - -	$\pm \frac{1}{mn} (a \pm x^n)^m$.
III. $\frac{x^{mn-1} \dot{x}}{(a \pm x^n)^{m+1}}$ - - - - -	$\frac{1}{mna} \times \frac{x^{mn}}{(a \pm x^n)^m}$.
IV. $\frac{(a \pm x^n)^{m-1} \dot{x}}{x^{mn+1}}$ - - - - -	$\frac{-1}{mna} \times \frac{(a \pm x^n)^m}{x^{mn}}$.
V. $m x^{m-1} y^n \dot{x} + n y^{n-1} x^m \dot{y}$ } vel $(m \dot{x} y + n x \dot{y}) x^{m-1} y^{n-1}$ } vel $(\frac{m \dot{x}}{x} + \frac{n \dot{y}}{y}) x^m y^n$ }	$x^m y^n$.
VI. $m x^{m-1} y^n z^r \dot{x} + n x^m y^{n-1} z^r \dot{y} + r x^m y^n z^{r-1} \dot{z}$ } vel $(m \dot{x} y z + n x \dot{y} z + r x y \dot{z}) x^{m-1} y^{n-1} z^{r-1}$ } vel $(\frac{m \dot{x}}{x} + \frac{n \dot{y}}{y} + \frac{r \dot{z}}{z}) x^m y^n z^r$ }	$x^m y^n z^r$.

Fluxionum.	Fluentium.
VII. $\frac{\dot{x}}{x}$	Log. x .
VIII. $\frac{x^{n-1}\dot{x}}{a \pm x^n}$	$\pm \frac{1}{n}$ log. of $a \pm x^n$.
IX. $\frac{x^{-1}\dot{x}}{a \pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a \pm x^n}$.
X. $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a-x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a} + \sqrt{x^n}}{\sqrt{a} - \sqrt{x^n}}$.
XI. $\frac{x^{\frac{1}{2}n-1}\dot{x}}{a-x^n}$	$\begin{cases} \frac{2}{n\sqrt{a}} \times \text{arc. tang. } \frac{\sqrt{x^n}}{a}, \text{ vel} \\ \frac{1}{n\sqrt{a}} \times \text{arc. cofin. } \frac{a-x^n}{a+x^n}. \end{cases}$
XII. $\frac{x^{\frac{1}{2}n-1}\dot{x}}{(\pm a + x^n)^{\frac{1}{2}}}$	$\frac{2}{n}$ log. $\sqrt{x^n} + \sqrt{\pm a + x^n}$.
XIII. $\frac{x^{\frac{1}{2}n-1}\dot{x}}{(a-x^n)^{\frac{1}{2}}}$	$\begin{cases} \frac{2}{n} \times \text{arc. fin. } \frac{\sqrt{x^n}}{a}, \text{ vel} \\ \frac{1}{n} \times \text{arc. verf. fin. } \frac{2x^n}{a}. \end{cases}$
XIV. $\frac{x^{-1}\dot{x}}{(a \pm x^n)^{\frac{1}{2}}}$	$\frac{1}{n\sqrt{a}}$ log. $\pm \frac{\sqrt{a \mp x^n} \pm \sqrt{a}}{\sqrt{a \pm x^n} + \sqrt{a}}$.
XV. $\frac{x^{-1}\dot{x}}{(-a+x^n)^{\frac{1}{2}}}$	$\begin{cases} \frac{2}{n\sqrt{a}} \times \text{arc. sec. } \sqrt{\frac{x^n}{a}}, \text{ vel} \\ \frac{1}{n\sqrt{a}} \times \text{arc. cofin. } \frac{2a-x^n}{x^n}. \end{cases}$

Fluxionum.

Fluentium.

XVI. $(dx - x^2)^{\frac{1}{2}} \times \dot{x} = - \frac{1}{2} \text{ circ. fegm. ad diam. } d \text{ et verf. fin. } x.$

XVII. $c^{nx} \dot{x} = \frac{c^{nx}}{n \log. c}.$

XVIII. $xy^x \log. y + xy^{x-1} \dot{y} = y.$

XIX. $(a + x^n)^m x^{rn-1} \dot{x} = \frac{(a + x^n)^{m+1} x^{rn-n}}{n} \times \left(\frac{1}{1} - \frac{r-1 \cdot a}{s-1 \cdot x^n} + \frac{r-1 \cdot r-2 \cdot a^2}{s-1 \cdot s-2 \cdot x^{2n}}, \&c. \right)$

De Formularum Applicatione.

133. Methodus applicationis hujusmodi est.

Conferatur fluxio proposita cum formulis; nempe, partes correspondentes cum correspondentibus fluxionis tabularis quæcum consentit; et hisce valoribus substitutis in formulam fluentis generalem, invenietur fluens quæsitæ.

Ex. gr. 1. *Invenire fluentem fluxionis* $3x^{\frac{5}{3}} \dot{x}$. *Consentit hæc cum primâ formulâ. Adeoque, propter* $n - 1 = \frac{5}{3}$, *erit* $n = \frac{5}{3} + 1 = \frac{8}{3}$; *eritque* $\frac{1}{n} x^n = \frac{3}{8} x^{\frac{8}{3}}$. *Et proinde fluens quæsitæ, multiplicando per 3, co-efficientem fluxionis datæ, erit* $3 \times \frac{3}{8} x^{\frac{8}{3}} = \frac{9}{8} x^{\frac{8}{3}}$.

2. *Invenire fluentem fluxionis* $\frac{x^2 \dot{x}}{1+x^3}$. *Consentit hæc cum octavâ formulâ.*

Nam prodit $\frac{x^{-1} \dot{x}}{a+x^n} = \frac{x^2 \dot{x}}{1+x^3}$, *itémque* $n = 3$, *et* $a = 1$; *quibus substitutis in formulam fluentis generalem, prodit fluens quæsitæ* $\frac{1}{3} \log. \frac{1}{1+x^3}$.

X 2

3. *In-*

3. Invenire fluentem quantitatis $5 \dot{x} x^2 \sqrt{c^3 - x^3}$ seu $5 \dot{x} x^2 (c^3 - x^3)^{\frac{1}{2}}$. Quadrat hæc cum formulâ secundâ. Erit igitur $a = x^3$, $m - 1 = \frac{1}{2}$, seu $m = \frac{3}{2}$, et $n = 3$; quibus substitutis, prodit fluens quæsitâ, $5 \times \frac{1}{\frac{3}{2} \times 3} (c^3 - x^3)^{\frac{3}{2}}$
 $= \frac{10}{9} (c^3 - x^3)^{\frac{3}{2}}$.

134. Ubi quantitas fluxionalis proposita nullis ex his formulis finitis respondeat, inveniri potest ejus fluens quàm proximè, *resolvendo propositam in seriem infinitam; vel per divisionem, vel per theorema binomium; et inveniendo singulorum terminorum fluentes; quorum summa ad libitum continuatorum, dabit fluentem quæsitam, proximè aut quàm proximè.*

4. Invenire fluentem $\frac{1-x}{1+x-x^2} \dot{x}$. Dividendo numeratorem per denominatorem, prodit $\dot{x} - 2x\dot{x} + 3x^2\dot{x} - 5x^3\dot{x} + 8x^4\dot{x}$, &c. Et inventis singulorum fluentibus, prodit summa $x - x^2 + x^3 - \frac{5}{3}x^4 + \frac{8}{5}x^5$, &c. fluens quæsitâ.

De correctione fluentium per regulas aut per formulas inventarum, vide suprâ, § 118.

APPENDIX I.

DE ANALYSI ANTIQUÂ.

Antiquam exquirite Matrem. VIRG.

APPENDIX I.

DE ANALYSE ANTIQVA

APPENDIX I.

DE ANALYSI ANTIQUÂ.

§ 135. ANALYSIN Antiquam, qualem exercebant clarissimi Geometrarum veteres, *Euclides, Archimedes, Apollonius*, &c. jamdudum laudavimus; optimam sanè, judice in hisce versatissimo, *Halley*, ANALYS. ÆQUAT. § 128. Et quàm curiosâ felicitate eandem excoluit et auxit *Newtonus*, ex operibus ejus philosophicis abundè constat. Placet igitur, hoc tempore, ubi labefactantur PHILOSOPHIÆ NATURALIS vel ipsa PRINCIPIA MATHEMATICA, et juventuti nostræ horum studiosæ potiùs commendantur, (utpotè majoris evidentiae, et rigoris geometrici,) “*functiones quædam analyticae*,” “*fluxiones hypotheticae*,” è *Scholâ Gallicâ* novatrice profluentes, eximiam illam *Præfationem Pappi Alexandrini*, ex Editione Halleianâ *Apollonii de Sectione Rationis*, hoc loco edere ab integro.

136. “*Locus de Resolutione* inscriptus, *Hermodore* fili, ut paucis dicam, propria quædam est materia in eorum usum designata qui, perceptis communibus elementis, in GEOMETRIA facultatem sibi desiderant investigandi solutiones problematum, et in hunc finem solummodò utilis. Traditur autem à tribus viris, *Euclide*, nempe, Elementorum scriptore, *Apollonio Pergæo*, et *Aristæo* seniore.

1. “*Procedit verò, per Methodum Resolutionis et Compositionis. Resolutio* Quid sit
Analysis, five
Resolutio. *(ανάλυσις)* autem est *methodus quæ à quæsito quasi jam concesso, per ea quæ deinde consequuntur ad conclusionem aliquam cujus ope compositio fit, (ἐπὶ τι ὁμολογημένον ἐν εὐθεσίᾳ) perducamur.* In resolutione enim quòd quæritur ut jam factum supponentes, ex quo antecedente hoc consequatur, expendimus; iterumque quodnam fuerit hujus antecedens; atque ita deinceps; usque, dum in hunc modum regredientes, in aliquid jam cognitum, locoque principii habitum, incidamus. Atque hic processus *Analysis* vocatur, quasi dicas *(ανάλυσις λυσις)* *Inversa Solutio.*

“ E. con.

Quid sit
Synthesis, five
Compositio.

“E contrario autem, in *compositione* (ἐν συνθεσεί,) cognitum illud in *resolutione* ultimo loco acquisitum, ut jam factum præmittentes, et quæ ibi consequentia erant, hinc ut antecedentia, naturali ordine disponentes, atque inter se conferentes, tandem ad *constructionem* quæsitæ pervenimus. Hoc autem vocamus *Synthesin*.

“Duplex autem est *analyseos* genus: vel enim est veri indagatrix, diciturque *Theoretica*, vel propositi investigatrix, ac *Problematica* vocatur. In *theoretico* autem genere, quod quæritur, reverà ita se habere supponentes; ac deinde per ea quæ consequuntur, quasi vera sint, (ut sunt ex *hypothesi*,) argumentantes, ad evidentem aliquam conclusionem (ἐπὶ τι ὁμολογούμενον) procedimus. Jam si conclusio illa vera sit, vera est quoque *propositio* de quâ quæritur; ac demonstratio reciprocè respondet *analyysi*: si verò in falsam conclusionem incidamus, falsum quoque erit de quo quæritur.

“In *problematico* verò genere, quod proponitur, ut jam cognitum sistentes, per ea quæ exinde consequuntur tanquam vera, perducimur ad conclusionem aliquam. Quod si conclusio illa *possibilis* sit ac *πρωσις* (quod mathematici *datum* appellant) *possibile* quoque erit quod proponitur: Et hinc quoque demonstratio reciprocè respondet *analyysi*: si verò incidamus in conclusionem *impossibilem*, erit etiâ *problema impossibile*.

“*Diorismus* autem (διορισμός), five *determinatio*, est quâ discernitur quibus conditionibus, et quot modis, problema effici potest. Atque hæc de *resolutione* et *compositione* dicta sunt.”

137. —“Verum perpendendum est (notat *Halley*,) aliud esse problema *aliquatiter* resolutum dare, quod modis variis plerumque fieri potest; aliud, *methodo elegantissimâ* id efficere: *Analysi* brevissimâ et finem perspicuâ; *Synthesi* concinnâ et minimè operosâ. Hoc veteres præstitisse, argumento est *Apelonii Liber De Sectione Rationis*.”

138. —“Ex his, credo, manifestum est (monet quoque *Newtonus*, *De Resolutione Quæstionum Geometricarum*,) quid sibi velint *Geometrae*, cum jubent, ‘*putes factum esse quod quæris*.’ Nullo enim inter cognitæ et incognitæ quantitates habito discrimine, *quælibet* ad ineundum calculum *assumere* potes, quasi omnes ex præviâ solutione essent notæ; et non ampliùs de *solutione problematis* sed de *probatione solutionis* ageretur.” —“Convenit ut fingas quæstionem de *ejusmodi datis et quæsitis* propositam esse per quas arbitreris te posse ad æquationem *facillimè* pervenire.” —“Cum varios ordines quibus termini quæstionis evolvi possint perspexeris, *E syntheticis quælibet adhibe* assumendo lineas tanquam *datas*, à quibus ad alias *facillimus videtur progressus*, et ad ipsas *vicissim difficillimus*.” —“Manuducit *analysis* ad *compositionem*: sed *compositio* non priùs verè fit, quam liberatur ab omni *analyysi*. Infit *compositioni* vel minimum *analyseos*, et *compositionem* veram nondum attectus es. *Compositio* in se perfecta est, et à mixturâ *speculationum analyticarum* abhorret.”

Hæc

Hæc *Newtonus*, in "LIBRO" illo verè aureo, "DE COMPOSITIONE ET RESOLUTIONE," quem concinnavit, ut firmior "*staret super vias antiquas*" *Juventus Britannica*; et quo melius obviam iret prævæ isti novitati ex Scholâ *Cartesianâ* grassanti, quâ *Cartesius*, "mixturis speculationum *analyticarum*" nimis indulgendo, evidentiam et certitudinem *Geometriæ*, primus, obscurare et labefactare cœpit; et proinde, quasi per antithesin operi *Cartesiano*, cui nomen erat, per abusum, "GEOMETRIA:" Librum suum, *Newtonus*, titulo magis idoneo, ARITHMETICÆ UNIVERSALIS inscripsit.—Atque ad hunc librum, mole parvum, at pondere et subtilitate materiæ eximium ("cui forsàn secundum vix inveniatur, nisi opus analyticum *Algebra Claviusiana*") interpretandum atque illustrandum, jamdudum edebam ANALYS. ÆQUATIONUM, anno 1784. Vide *Prefat.* illius operis.

139. Methodum *Analyticam* et *Syntheticam* non solum in *Mathesi*, sed etiã in *Mechanicâ Rationali*, optimè exponit *Newtonus*, sub finem *Opticorum*.

"As in *Mathematics*, so in *Natural Philosophy*, the investigation of *difficult* things by the Method of *Analysis*, ought ever to precede the Method of *Composition*."

"*This Analysis* consists in making *experiments* and *observations*, and in drawing *general conclusions* from them by *Induction*; and admitting of no *objections* against the conclusions, but such as are taken from *experiments* or other *certain truths*: for *hypotheses* are not to be regarded in EXPERIMENTAL PHILOSOPHY. And although the arguing from experiments and observations be *no demonstration* of general conclusions, yet it is the *best way* of arguing which the nature of things admits of, and may be looked-upon as so much the stronger, by how much the induction is more general: and if *no exception* occur from phenomena, the conclusion may be pronounced generally. But if, at any time afterwards, any exception shall occur from *experiments*, it may then begin to be pronounced with *such exceptions as occur*."

"By this way of *Analysis*, we may proceed from *Compounds* to *Ingredients*, and from *Motions* to the *Forces* producing them; and, in general, from *Effects* to their *Causes*; and from *particular* causes to *more general* ones, till the argument end in the most general:—this is the Method of *Analysis*."

"And the *Synthesis* consists in *assuming* the *Causes* discovered and established, as *Principles*; and by them *explaining* the *phenomena* proceeding from them, and *proving* the explanations."

And in the Preface to his PRINCIPIA, he gives the following elegant summary of this statement:—

"*Omnis enim PHILOSOPHIÆ difficultas in eo versari videtur, ut à phænomenis motuum investigemus Vires Naturæ; deinde, ab his viribus, demonstremus phænomena reliqua.*"

140. And he furnishes the following instances of his application of this method, in those two immortal works, his *OPTICS* and *PRINCIPIA*:

"In the two first books of these *OPTICS*, I proceeded, by *this Analysis*, to discover and prove the *original differences* of the *RAYs OF LIGHT*, in respect of *Refrangibility*, *Reflexibility*, and *Colour*, &c.—And these discoveries being *proved*, may be assumed, in the *Method of Composition*, for *explaining the phænomena* arising from them: an instance of which I gave in the end of the first book, [in the solution of the colours and figure of the *Rainbow*, from the refraction and reflexion of the Sun's rays in drops of falling rain.]

"In libro autem tertio [*PRINCIPIORUM*] exemplum hujus rei proposuimus per explicationem *SYSTEMATIS MUNDANI*: Ibi enim, ex *phænomenis cælestibus*, per propositiones in libris prioribus *mathematicè* demonstratas, derivantur *vires gravitatis*, quibus corpora ad Solem et Planetas singulos tendunt: deinde, ex *his viribus*, per propositiones etiàm *Mathematicas*, deducuntur motus *Planetarum*, *Cometarum*, *Lunæ* et *Maris*."

141. And he expressly states that he adopted this method from the Ancients:—

"Cum *VETERES Mechanicam* [*Rationalem*, quæ per demonstrationes accuratè procedit,] (uti auctor est *Pappus*,) in *rerum naturalium* investigatione maximi fecerint; et *RECENTIORES*, missis *formis substantialibus*, et *qualitatibus occultis*, *phænomena NATURÆ* ad *leges mathematicas* revocare aggressi sint: *visum est in hoc tractatu MATHESIN excolere, quatenus ea ad PHILOSOPHIAM spectat.*"

How highly *NEWTON* admired the *Ancient Geometricians*, we learn from *Dr. PEMBERTON*:—

"He often censured the handling of *Geometrical* subjects by *Algebräick* calculations; and his Book of Algebra he called by the name of *Aritmetica Universalis*, in opposition to the injudicious title of *Geometry*, which *DES CARTES* had given to the Treatise in which he shews how the Geometrician may assist his invention by such kind of computations. He frequently praised *SLUSIUS*, *BARROW*, and *HUYGENS* for not being influenced by the *false taste* which then began to prevail: he used to recommend the laudable attempt of *HUGO D'OMERIQUE* to restore the *Ancient Analysis*; and very much esteemed *APOLLONIUS's* book *De Sectione Rationis*, for giving us a clearer notion of that Analysis than we had before. *Dr. BARROW* may be esteemed as having shewn a compass of invention equal, if not superiour, to any of the Moderns, our Author only excepted: but

NEWTON

NEWTON particularly recommended HUYGENS's style and manner; he thought him *the most elegant of any mathematical writer of modern times, and the truest imitator of the Ancients*. Of their taste and mode of demonstration our Author always professed himself a great admirer; and even censured himself for not following them yet more closely than he did: and spoke with regret of his mistake, at the beginning of his Mathematical Studies, in applying himself to DES CARTES and other *Algebräick* writers, before he had considered *the Elements* of EUCLID with that attention which so excellent a writer deserves."

Absolutis hæcenus VINDICIIS NEWTONIANIS, quatenus *Elementa Metaphysica* et *Principia Mathematica* respiciunt, (quibus fundantur opera ejus eximia Analytica et Philosophica) adversus objectiones *Gallorum, Germanorum, Anglorumque*, horum evidentiam et certitudinem, aut malignè, aut imperitè, aut oscitantè, infectantium; superest *altera* et *acrior cura*, ut repellatur crimen multo gravius, "*facinusque majoris abollæ*," quasi PHILOSOPHIÆ atque THEOLOGIÆ cultor ille, omnium modestissimus et sanissimus, fuisset et ipse fautor dogmatis pravissimi atque turpissimi quod *Materialismum* vocant, Naturam *Mentis Humanæ* merè mechanicam statuentis, et "*divinæ particulam auræ*," (etiàm ab antiquissimis agnitæ,) extinguere gestientis; ipsum NUMEN SUMMUM—OPTIMUM, SAPIENTISSIMUM, MAXIMUM et POTENTISSIMUM omnium, quasi *Animam Mundi, Ætheream*, pro nefas! absurdè et impiè fingentis, et deliramenta "*insanientis sapientiæ*," iterùm recoquentis, per Philosophiam pravam et Mythologiam *Leibnitzianam*, gravissimòsque errores inde prognatos; hoc *sæculo rationis* (ut dicitur) infaulto, *Revelationem* aspernante.

APPENDIX II.

DE ÆTHERE VIBRATORIO,—DE MODO SENTIENDI,

ET

DE ENTE SUPREMO.

“AUCTORIBUS UTI OPTIMIS *in omnibus causis, et debet et solet valere plurimum:*
et primum quidem, OMNI ANTIQUITATE; quæ, quo proprius aberat ab ortu et
divinâ progenie, hoc melius ea, fortasse, quæ erant vera, cernebat.” CIC.

Οὐκ ἔγνω ὁ Κοσμος, δια τῆς σοφίας, ΤΟΝ ΘΕΟΝ.

“Non novit Mundus, Sapientiæ [humanæ] ope, DEUM.”

PAUL.

APPENDIX II.

DE ÆTHERE VIBRATORIO.

“ *C*E grand homme (NEWTON) voyoit à travers d'un voile, ce qu'un autre ne distingue qu'à peine avec un Microscope !” DAN. BERNOULLI *.

142. Nuper, cum perlegerim opus eximium, perutile, tempestivum, à viris bonis ubique laudatum,—“ *Proofs of a Conspiracy against all the Religions and Governments of Europe, &c.* by John Robison, A. M. Professor of Natural Philosophy, &c. Edinburgh,”—incidi in censuram haud levem Patris Philosophiæ Britannicæ, p. 483.

“Were it possible for the departed soul of *Newton* to feel pain, he would surely recollect with regret that *unhappy hour*, when, provoked by Dr. *Hooke*'s charge of plagiarism, he first threw out his *whim* of a *Vibrating Æther*; to shew what might be made of an *Hypothesis*: (for Sir *Isaac Newton* must be allowed to have paved the way for much of the *Atomical Philosophy* of the Moderns.) *Newton*'s *Æther* is assumed as a *fac totum* by every *precipitate sciolist*, who, in despite of logic, and in contradiction to all principles of mechanics, gives us *theories* of *muscular motion*, of *animal sensation*, and even of *intelligence* and *volition*, by the *undulations of ætherial fluids*.

“Not one of a hundred of these *theorists* can go through the fundamental theorem of all this doctrine, (the 47th *Prop.* of the *Second Book of the PRINCIPIA*); and not one of a thousand know that *Newton*'s investigation is *inconclusive*:—yet they talk of the effects and modifications of those undulations as familiarly as if they could demonstrate the propositions in *Euclid*'s *Elements*.”

Talis viri ictus, PHYSICAM excolentis, et NEWTONUM quasi præceptorem adamare profitentis, acerbius lædunt: Antistiti ipsi scientiarum venerando,

* Vide Note (c) posthac.

annon exclamare fas sit, jamdudum hujusmodi crimina auscultanti :—" Si id *Cinerem* aut *Manes* credis curare sepultos (a)."

—" Pol, me occidistis AMICI!—
Obe jam satis est!—Et TU, BRUTE!!!—

Telum imbellè jactis; imò, tuum in cerebrum facillè retorquendum, vibranti medium transverberans ietu."

143. Plusquàm viginti retrò annis, Disquisitionem edebam, anno 1778, *Dublin, De Sonis et De Modificationibus Atmosphære*, secundum Theoriam *Newtonianam*,—primitias Studiorum meorum Philosophicorum,—dum, apud Academicos *Dublinienses*, *Socius* degerem. Et, cum ista disquisitio forsàn nondum ad manus pervenerit Professoris *Robison*, me operam vel illi viro clarissimo deque Republicâ optimè merenti, haud ingratam impensurum spero, si exinde pauca excerpserim, hanc rem spectantia.

Primo igitur in limine, monendus est, *Newtonum*, haudquaquam "*opinionum*" seu *hypotheseon* vanarum studiosum, nec novum neque inauditum "*commentum*" de "*aethere quodam vibratorio*" finxisse; sed vetustum omninò, atque à philosophis sanioribus receptum: § 42.

Zeno, audiendi facultatem ità exponit, *Laert. Lib. 7. § 158*.—*Ακρὴν δὲ, τὴ μεταξὺ τῆς τῆ φωνῆς καὶ τῆ ἀκοῆς, αἶρος πλεθρομένη σφαιροειδῶς, εἴτα κυματῶμεν, καὶ ταῖς ἀκοαῖς προσπιπλοντος ὥς κυματῶται τὸ ἐν τῇ δεξαμένῃ ὕδρῳ, κατὰ κυκλῶς, ὑπὸ τῆ ἐμβληθέντος λίθου.*

"*Auditus* fit, ubi aer inter loquentem audientemque interpositus, pulsatur *spharicè*; deinde undulatur, auribusque impingit: sicuti undulatur *aqua* in cisternâ, secundum circulos, injecto lapide."

(a) Quantum à disputationibus controversiisque abhorruit mite *Newtoni* ingenium, constabit ex *Epistolâ* ejus ad *Oldenburgh*, Oâ. 24, 1676.

—"Five years ago, (1671,) when, at the persuasion of my friends, I had resolved to publish a Treatise of the *Refraction of Light and Colours*, which I had then by me; I began again to turn my thoughts to these *Serieses* [for determining the Areas of Curves]; and wrote likewise a Treatise upon them, that I might publish both together.—But having wrote you a letter upon the *Reflecting Telescope*, in which I explained briefly my notions on the *Nature of Light*, something unexpected fell out, which made me imagine it concerned me to write to you speedily about printing that Letter. Immediately upon the back of this, I met with frequent interruptions from several persons' letters, taken up with objections, and other things, which frightened me entirely from executing my design; and made me find fault with myself for my imprudence—that, by catching at a shadow, I had been so far deprived of the peace and quiet of my own mind; which is a thing of more substantial worth."

At last *Newton* found out an expedient that rid him of much of this troublesome and teasing correspondence: when objections were stated against the accuracy or validity of his *Optical* experiments, he simply replied—"I doubt the fact;"—"repeat the experiment,"—instead of entering into any discussion or vindication about his own.

Diogenes

Diogenis atque *Platonis* placita quoque, sic refert *Plutarchus*, *De Placit. Philos.* Lib. 4. cap. 16.—*Διογενής, τὸ ἐν τῇ κεφαλῇ αἶρος ὑπὸ τῆς ὄψεως τυπόμενον καὶ κινούμενον· Πλάτων, καὶ οἱ ἀπ' αὐτοῦ, πλεῖσθαι τὸν ἐν τῇ κεφαλῇ αἶρα, τῶν δὲ ἀνακλασθαι εἰς τὰ ἡγεμονικά, καὶ γίνεσθαι τῆς ἀκοῆς τὴν αἰσθησιν.*

“*Diogeni* [placet hoc effici] aëre intrâ caput à voce pulsato atque commoto. *Platoni*, ejúsque sequacibus, aërem intrâ caput pulsari, deinde ad partes imperatorias reflecti; atque hoc pacto sensum auditûs fieri.”

Lucretius quoque phænomena *Lucis* et *Sonorum* disertè confert: *Lib. 4.*

“Quod superest, non est mirandum quâ ratione,
Quæ loca per, nequeunt oculi res cernere apertas;
Hæc loca per, voces veniant aureisque lacessant.
Conloquimur clausis foribus, quod sæpe videmus:
Nimirum, quia vox per flexa foramina rerum
Incolumis transire potest; simulacra renutant;
Perseinduntur enim, nisi recta foramina tranant,
Qualia sunt vitri, species quâ travolat omnis.

Præterea, partes in cunctas dividitur vox;
Ex aliis aliæ quoniam gignuntur, ubi una
Diffiluit semel in multas exorta: (quasi ignis
Sæpe solet scintilla suos se spargere in ignes);
Ergo replentur loca vocibus, abdita retrò,
Omnia quæ circum fuerint, sonitûque cientur:
At simulacra viis directis omnia tendunt,
Ut sunt missa semel; quapropter cernere nemo
Se suprâ potis est, at voces accipere extrâ.”

Atqui hîc habes quasi seminarium totius *Doctrinæ Sonorum*, rationalis et experimentalis, in difficillimâ parte difficillimi operis, nempe, in *Sectione octavâ*, *Lib. 2. Princip.* “*De Motu per fluida propagato* expositæ et demonstratæ.”

144. Haudquaquàm patitur hujus instituti ratio, de *Doctrinâ subtilissimâ* Pulsuum et Vibrationum Aëris, seu *Medii Ætherei*, jam differere, quæ vel mathematicè doctos non parùm morantur; accuratiùs autem exponitur fusiùsque illustratur tota hæc doctrina, in *Disquisitione prædictâ De Sonis*.

Unam saltèm observationem inde depromere non pigebit, p. 139, *de veritate propositionis illius fundamentalis* 47, à Professore *Robison* denegatæ:

“*Le Saur* et *Jacquier* veritatem propositionis deducunt ex vibrationibus partium *chordæ elasticæ*, quæ peraguntur viribus in ratione distantiarum à medio vibrationum loco: sed singulæ aëris particulæ motum chordæ mutantur successivè: ergo, instar chordæ ipsius, moventur viribus, &c.—At hæret deductio.

nisi ostendere possent, *omnia corpora sonitum edentia legem chordæ elasticæ in tremoribus suis servare*, [*aëremque similiter agitari*].

“*Rowning, Nat. Phil. Part 2. p. 48, veritatem propositionis immediatè deducit ex lege repulsivâ partium aëris*:—Nam particula quævis, pressione partium à tergo, versùs particulas antecedentes propelletur viribus quæ sunt ut intervalla centrorum reciproce: atque ideò, ut distantia hujus particulæ à primo suo loco directè.”

Ex hâc lege *vis elasticæ seu repulsivæ*, (quæ à *Newtono* mathematicè demonstratur, *Prop. 23. Lib. 2. Princip. et experimentis confirmatur, § 4, De Sonis*), elicitur conclusio, *Quod singulæ fluidi elastici tremuli particule, motu reciproco brevissimo eunt et redeunt, accelerantur semper et retardantur in ratione distantiarum à medio vibrationum loco*: hoc est, *pro lege penduli in cyclo-eide oscillantis*.

145. Demonstratio *Newtoniana*, quâ instituitur analogia hæc elegantissima inter *vibrationes aëris et oscillationes penduli*, longè subtilissima, est prorsus analytica, ex ipsis fontibus *ANALYSIS ANTIQUÆ* hausta:

“*Fingamus medium [elasticum] tali motu [vibratorio] à causâ quâcunque cieri; et videamus quid inde consequatur, &c.*”

Propositionis autem 47 ipsa conclusio, “*Aërem, pulsibus per medium elasticum propagatis, cieri seu agitari pro lege oscillantis penduli*,” abundè confirmatur experimentis de variis sonorum generibus, tormentorum dislosionibus, &c. factis ad investigandam *velocitatem sonorum*, (seu rationem secundum quam rectè progrediuntur), in *ITALIA*, observantibus Academicis *Florentinis*; in *ANGLIA*, observantibus *Newtono, Derbamo, &c.*; in *AMERICA AUSTRALI*, observantibus Academicis *Gallis atque Hispanis*; et nupèr etiàm in *AFRICA*, propè ostium *Nili*, victoriâ illâ insigni navali apud Sinum *Aboukir*, Aug. 1, 1798, à classe Anglicâ sub imperio ducis magnanimi *Horatii Nelson*, de classe Reipublicæ Gallicæ reportatâ; ubi fragor navis *L'Orient*, explosæ, *Rosettam* pertingebat, justo post fulgorem visum intervallo temporis, ratione habitâ distantia et fluxus uniformis sonorum (b).

(b) Si observationibus *M. Poussielgue*, Gallorûmque adstantium, tempore illo trepidationis bellicæ iniquo et insausito, fides est adhibenda, fragor navis *L'Orient*, in sinu *Aboukeriano* explosione combustæ, stationem suam apud *Rosettam* pertingebat tempore *duorum minutorum primorum* post fulgorem visum. At progagatur sonus mediâ velocitate 1142 pedum circitè, tempore unius minuti secundi, et proinde 13 milliarius Anglicorum tempore unius minuti primi. Distabant igitur *Galli* 26 milliaria Anglica à loco explosionis, ad minimum, recto cursu.

Augenda est enim hæc distantia, ob *calores asperos* istius regionis, in ratione 1142, seu potiùs pedum Gallicorum 1070 ad 1101; sicut pedum Gallicorum 1101, singulis minutis secundis, determinabatur velocitas sonorum ad insulam *Cayenne*; (intervallo 20230 hixapedarum (*toises*), seu milliarius Anglicorum 25 ferè, geometricè mensurato à *Condamine* Academicisque *Gallis*; atque adeò accuratè ut, ex quinque experimentis factis per tormentorum explosiones, quatuor vix uno semi-secundo tempore ab invicem discrepârunt. Distantia igitur, hâc ratione aucta, correctior emerget milliarius 27 ferè. Atque iterum forsàn augenda est, ob ingentem fragorem navis explosæ, intensissimos tremores, eodèque forsàn principio latissimos, excitantis. Academicorum *Anglorum* certè æquum foret rem determinare, geometricè mensurando distantiam stationis apud *Rosettam*, à nave *L'Orient*; ad persiciendam aut corrigendam *Theoriam Newtonianam*, experimento adeò illustri, minimè negligendo.

Consensus igitur theoriæ cum experimentis fidelitè institutis ubicunque regionum satè confirmat *Analyfin Newtonianam* ritè institutam fuisse; nam *veritas conclusionis infert veritatem præmissorum*.—Vide *De Sonis*, § 47; ubi datur Tabula Observationum *Florentinorum*, &c.; necnon *Append.* p. 71; ubi demonstrationes harum propositionum valdè difficillium, 47 et 49, plenius exhibentur, atque notis illustrantur.

146. Nec est cur de *Æthere Vibratorio* cavillationes excitentur:—vox *æther* à Newtono usurpatur pro *aëre purissimo*, vaporibus sordibúsque heterogeneis defæcato:—“hi enim, (cum sint alterius elateris et alterius toni,) vix, et ne vix quidè, participant motum *aëris veri* quo soni propagantur.” *Princip.* p. 373.

Hunc *ætherem*, seu *medium æthereum*, ingenti elatere (seu elasticitate) esse præditum, docet *Newtonus*, *Optics*, p. 325, his verbis:

“That the *elastic force* of this medium is exceeding great, may be gathered from the swiftness of it's vibrations; *sounds* move about 1140 English feet in a second of time, and in 7 or 8 minutes of time they move about 100 English miles: *Light* moves from the Sun to us in about 7 or 8 minutes of time; which distance (supposing the Sun's horizontal parallax to be about 12'',) is about 70 millions of English miles; and by consequence, above 700,000 times swifter than sounds: and therefore the elastic force of this medium must be above 490,000 million of times greater than the elastic force of the air [at the earth's surface]”—in proportion to their respective rarities (*c*).

147. The astonishing *rarefaction* of which the air is capable is remarked, *Optics*, p. 341, from experiment and from computation:

“Mr. *Boyle* has shewn that air may be rarefied about 10,000 times in vessels of glass; and the heavens are much emptier of air than any *vacuum* we can make below. For, since the air is compressed by the weight of the incumbent atmosphere, and the density of the air is proportional (or it's rarity reciprocally proportional) to the force compressing it, it follows, by computation, [*Princip. Lib. 2. Prop. 22. Cor. et De Sonis*, § 14.] that, at the height of about $7\frac{1}{2}$ English miles from the earth, the air is 4 times rarer than at the surface of the earth; and at the heights of $22\frac{1}{2}$, 30, or 38 miles, it is respectively 64, 256, or 1024 times rarer, or thereabouts; and at the height of 76, 152, 228 miles, it

(*c*) *Newton*, *Principia*, p. 405, estimates the Sun's horizontal parallax, lower, at $10\frac{1}{2}$ ''.—And the calculations made since the Transit of Venus in 1761, reduce it further to 8'',6; which will enlarge the Sun's distance from us to upwards of 95 millions of miles; and consequently raise the proportion of the elastic force of the ætherial medium to upwards of 900,000 million times greater than that of our air.

The calculation depends on the velocities of the pulses (or propagation of the vibrations) of elastic fluids being in a compound subduplicate ratio of their elasticities and rarities, *Princip.* II. 49. *Cor. 2.*

is about a million, a billion, or a trillion, times rarer; and so on:" inasmuch that, in the *Principia*, p. 512, *Newton* computes, that "a globe of our air one inch in diameter, at the elevation of one semidiameter of the earth (about 4000 miles), would, by it's expansive power, fill all the planetary regions, far beyond the orbit of *Saturn*."

148. Non temère igitur conjicit *Newtonus*, *Princip.* p. 530—"De spiritu quodam subtilissimo corpora crassa pervadente, et in iisdem latente; cujus vi et actionibus particulæ corporum ad minimas distantias se mutuò attrahant; et contiguæ factæ, c hærent; et corpora electrica agunt ad distantias majores, tam repellendo quàm attrahendo corpuscula vicina; et lux emittitur, reflectitur, refringitur, inflectitur et corpora calefacit; et sensatio omnis excitatur, et membra animalium ad voluntatem moventur: vibrationibus scilicet hujus spiritûs, per solida nervorum capillamenta ab externis sensuum organis ad cerebrum, et à cerebro in musculos, propagatis."

149. Fufius hæc exponit, *Optical Queries*:

"Is not animal motion performed by the power of this [*ætherial*] medium, excited in the brain by the power of the will, and propagated from thence through the solid, pellucid, and uniform, capillamenta [or minute fibres] of the nerves into the muscles for contracting and dilating them?"—Vide § 195.

[N. B.] "I suppose that the capillamenta of the nerves are each of them solid and uniform, that the vibrating motion of the ætherial medium may be propagated along them from one end to the other uniformly and without interruption (for obstructions in the nerves create palfies): and that they may be sufficiently uniform, I suppose them to be pellucid when viewed singly, though the reflexions in their cylindrical surfaces may make the whole nerve, composed of many capillamenta, appear opaque and white."

Cautissimè igitur distinguit *Newtonus* his locis, inter mechanicas operationes fluidi hujusce ætherei seu "*spiritûs subtilissimi*"—et mentem ipsam seu voluntatem, quæ hæc sentit, et quodam modo nobis prorsus incognito, dirigit.

150. Et quàm sollicitè distinguit quoque inter modificationes corporum, et sensationes quas in mente excitant! in *Optics*, p. 108.

"If at any time I speak of light and rays as coloured or endued with colours, I would be understood to speak not philosophically and properly, but grossly, and according to such conceptions as vulgar people, in seeing all these experiments, would be apt to frame. For the rays, to speak properly, are not coloured: in them there is nothing else but a certain power and disposition to stir up a sensation of this or that colour: for as sound in a bell or musical string, or other sounding body, is nothing but a trembling motion; and in the air is nothing but

but that motion *propagated* from the object; and in the *sensorium* 'tis a *sense* of that motion under the form of *sound*: so COLOURS in the *object* are nothing but a *disposition* to *reflect* this or that sort of rays more copiously than the rest; in the *rays*, they are nothing but their *dispositions* to *propagate* this or that motion into the *sensorium*; and in the *sensorium*, they are sensations of those motions under the forms of *colours*."

DE SENSORIO ET DE VI SENTIENDI.

151. QUANTAS tragedias excitavit hæc vox *technica sensorium*! Reclamant *Leibnitziani*: "*Newtonum* hîc habes reum consistentem, et vim sentiendi omnem ex causis *mechanicis* solventem!"

Sed ipsum audiamus, de industriâ definientem hanc vocem, "*the place of sensation*."

"Is not the *sensory* of animals that *place* to which the sensitive substance is present, and into which the sensible species of things are carried through the nerves and brain; that there they might be *perceived* by their immediate presence to *that substance*?"

And this distinction between the *organs of sensation* and the *sensitive substance* itself was long ago noticed by *Socrates*, in the *Phædo* of *Plato*, thus ridiculing the *Mechanical Philosophists* of his day:

"It is true indeed, that without *bones* and *nerves*, and such other *organs* as I possess, it would be impossible to *act* as I thought fit; but to assert that I do what I do by their *agency*, and not from *choice* of the best, would be downright absurdity, and a mode of argument truly ridiculous."—Thus nobly stating his decided choice, rather to remain, and submit to the unjust sentence of the *Athenian* Judges, than to save his life by escaping to *Megara*, and seeking an asylum among the *Bæotians*.

But was this preference merely the result of *vibratiuncula* of the brain?

De industriâ itêm monet *Newtonus*, ne perperâm intelligatur *conjectura* sua sagacissima de modo sentiendi;—se nullatenus *theoriam* proponere: *Princip. fine*.

"Sed hæc *paucis* exponi non possunt; neque adest *sufficiens copia* *experimentorum* quibus *leges* actionum hujus *spiritûs* [*ætherei*] accuratè determinari et monstrari debent."

152. Con-

152. Conjecturæ *Newtonianæ* de actionibus hujusce spiritûs, seu fluidi electrici, ad stipulari fas sit, recitando phænomena quædam, quæ mihi ipsi obtigerunt abhinc circiter decennio; necnon viro clarissimo *Maclaurin*, moribundo; quæ satis confirmare videntur existentiam talis spiritûs in ipsis cerebri penetralibus ætæ latentis, et functionibus animi rectoris peragendis apprimè forsàn necessarij; intimo quodam et incomprehensibili nexu inter volitiones et actiones subsistente, mediantibus organis corporeis. Nam, secundum *Locke*:

“It may seem probable, that the constitution of the body does sometimes influence the memory; since we sometimes find a disease quite strip the mind of all it's ideas, and the flames of a fever in a few days calcine all those images to dust and confusion, which seemed to be as lasting, as if graved in marble.”—*Essay*, Book II. c. 10.—See also § 195.

153. Æstate anni 1789, ipse in febrem biliosam incidi, ex diutinis laboribus, assidujs studiis, nimisq; curis contractam. Et quùm febrîens insomnia miserè laborarem, medicus quidam ex peritioribus, mihiq; amicus, *laudani* liquidi potionem administrabat, ad reducendum somnum. At spem prorsùs fefellit: licèt enim potio bis terve repeteretur, manebat insomnium; et, quod notatu dignissimum est, unaquæque vice, decem circiter minutis temporis post potionem haustam, phænomenon singulare occurrebat: quasi scintillæ ignis affatim effluerunt ex oculis meis, velut ictu percitis. Me ipso igitur deprecante, intermissa est potio.—Exinde autem valetudine ægerrimâ afflictus sum; affectionibus nervosis, et capitis lento dolore potissimum, per triennium; obstructis quasi viribus animi;—inutilis mihi, inutilis propinquis et amicis, inutilis reipublicæ, *ετιωσιον αρχθς αρρης*.

Tandem in plenitudine temporis atque mœroris, ubi cernere erat indicia hæud obscura *paralyseos* jamjam imminentijs, opitulatus est MEDICUS SUMMUS—*Ἡ γὰρ ΔΥΝΑΜΙΣ αὐτῆς ἐν ΑΣΘΕΝΕΙΑ ΤΕΛΕΙΕΤΑΙ*.—Mihi enim suggestit ILLE DEUS, ut ex desidiâ ferè continuâ me extricarem, et ambulando strenuè me exercerem ad sudorem usque, si quâ ratione cieretur defluxio à cerebro, quod onere quasi incumbente valdè erat oppressum; et longè ultrà expectationem fuit successus: nam in fine hebdomadis unius tali exercitio occupatæ, sex aut septem horis per diem, defluxio humoris lymphatici, seu aquei, subito erupit, cum admodum incalescerem ambulando, et quasi à vertice ipso onus incumbens sublatum est, humore per glandes maxillares abundè stillante in os; et in fine hebdomadis alterius, gradatim exsiccata defluxione, ad perfectam integrâmq; valetudinem restitutus sum; et meliorem quidem quam per viginti antè annos. Diffugère febres; redii ad me-ipsam, redii ad studia diù intermissa, redditus sum propinquis amicisque, redditus patriæ—Iterum mihi concessa est, per misericordiam divinam, *Mens sana in corpore sano*.

DEO OPTIMO MAXIMO LAUS!

Luc. xvii. 18.

154. Re-

154. Recolebam febriciens isto tempore, hujusmodi phænomenon, vel eidem valdè simile, viro clarissimo *Maclaurin* obtigisse moribundo; et inde erat quod averfatus atque deprecatus sum, quasi infaustum et malè ominatum foret, positionem illam *laudani*, quæ id manifestò excitaverat; vaporibus forsan *laudani*, (ut postea suspicatus sum), à stomacho calido dissolutis, (qui, utpote valdè biliosus, actionem *laudani* tunc respuebat,) et inde ascendentibus in cerebrum; et ignem *electricum* ibi latentem expellentibus per nervos opticos; vaporibus ipsis in aquam postea conversis.

Casus *Maclaurini* ità describitur à biographo optimo:

—“His behaviour, during this tedious and painful distemper—[a *dropsy of the belly*, for which he had been thrice tapped]—was such as became a PHILOSOPHER and a CHRISTIAN, calm, chearful and resigned: his *senses* and *judgment* remaining in their full vigour, till within a few hours of his death. Then, for the first time, his *Amanuensis*, to whom he was dictating the last Chapter of the following work, [An Account of Sir *Isaac Newton's* Philosophical Discoveries,] (in which he proves the wisdom, the power, goodness, and other attributes of the DEITY,) observed some hesitation or repetition: no *pulse* could then be felt in any part of his body; and his hands and feet were already cold. Notwithstanding this extremely weak condition, he sat in his chair, and spoke to his friend Dr. *Munro*, with his usual serenity and strength of reason; desiring the Doctor to account for a *phænomenon* which he then observed in himself; *flashes of fire* seemed to dart from his eyes; while, in the mean time, his sight was failing, so that he scarce could distinguish one object from another.—In a little time after this conversation, he desired to be laid upon his bed; where, on Saturday the 14th of June, 1746, aged 48 years and four months, he had an *easy passage* (*) from this world to that state of bliss, which he had the most elevated ideas of, and which he most ardently longed to possess.”

“Let ME die the death of THE RIGHTEOUS;
And let MY last end be like HIS!!”

(*) —Μετὰ βελύχην ἐκ τῆς σάρατος εἰς τὴν ζωὴν. *John*, v. 24.

“*Migravit ex morte in vitam.*”

“Neque enim assentior iis qui hæc nuper differere cœperunt,” [Lucretius, &c.] “Cum corporibus animos simul interire, et omnia morte deleri.” *Cicero de Amicitia.*

“M. Video te aliè spectare et velle in calum migrare.
A. Spero fore ut contingat id nobis.” *Cicero, Tuscul. I.*

Maclaurin's Last Words.

155. THE concluding, unfinished paragraph of that sublime Discourse is too curious in the *History of Man*, too precious to the *Christian Philosopher*, to be withheld on this occasion:—

—“AS MAN is undoubtedly the *chief being* upon this globe; and *this globe* may be no less considerable in the most valuable respects than any other in the *solar system*; and *this system*, for aught we know, not inferior to any in the UNIVERSAL SYSTEM: So, if we suppose *Man* to *perish*, without ever arriving at a more competent knowledge of NATURE, than the *very imperfect* one he attains in his *present* state, by *analogy* (*d*) (or parity of reason), we might conclude, that the *like desires* would be frustrated in the *inhabitants* of all the other *planets* and *systems*; and that the *beautiful scheme* of NATURE would never be unfolded, but in an exceedingly imperfect manner, to *any* of them:—This, therefore, naturally leads us to consider our *present* state,—as only *the dawn* or *beginning of our existence*; and as a state of *preparation* or *probation*, for farther advancement: which appears to have been the opinion of the *most judicious Philosophers* of old. [See *Heb. xi. 13—16.* and Note (*e*).]

“And

(*d*) It was a maxim, recorded of *Hermes*, and ascribed to the primitive *Chaldean* or *Persian Magi* (the established priesthood of the *Babylonish* and *Persian Empire*)—

Συμπᾶν εἶναι τὰ ἀνω τοῖς κάτω.

“That what passes in the heavens above is analogous to what passes on earth below.”

But this sage maxim, like many others, derived from *patriarchal* wisdom, in process of time was much abused by succeeding corrupt *poets* and degenerate *philosophers*, receding from the native purity and simplicity of Revelation, and impiously attributing *human* passions to *celestial* beings. See § 169.

(*e*) The following curious and valuable anecdote we owe to Professor *Robison*, p. 236:—

“*Daniel Bernoulli*, the most elegant mathematician, the only philosopher and the most worthy man of that celebrated family, said to a gentleman (*Dr. Stasbling*), who repeated it to me, that *when* reading some of those *wonderful guesses* of *Sir Isaac Newton*, the subsequent demonstration of which has been the chief source of fame to his most celebrated commentators,—his *mind* has sometimes been so overpowered by *thrilling emotions*, that he has wished that moment to be his last; and that it was *this* which gave him the *clearst conception of the happiness of heaven*.”

And should not these “*thrilling emotions*,” to which, I am persuaded, Professor *Robison* is no stranger, induce him to treat the Doctrine of a “*Vibrating Æther*” with more respect, so widely removed from “a *whim* made out of an *hypothesis*,” by *Newton*’s ingenuity?—Should he not rather rank it among the most profound of his *wonderful guesses*,—approaching very nearly to the dignity of a *theory*, founded on such *experiments* as the preceding?—How otherwise, are we to account for such *thrilling emotions* of ecstatic delight? such *convulsive horrors* of agonizing fear? such strange effects, as the change of the colour of the hair of a condemned criminal *from black to white* in the course of a single night? such fatal and deadly disruptions of the *material vehicle*, as are produced by sudden

and

“ And whoever attentively considers THE CONSTITUTION OF HUMAN NATURE, particularly the *desires* and *passions* of men, (which are *greatly superior* to their present objects,) will easily be persuaded, that MAN was *designed* for *higher* views than those of this life: these, the AUTHOR OF NATURE may have in reserve; to be opened-up to us at *proper periods* of time, and after *due preparation*: surely it is in HIS power to grant us a much greater *improvement* of the faculties we already possess; or even to endow us with *new* faculties (of which at this time we have no idea) for penetrating farther into the SCHEME OF NATURE, and approaching nearer to HIMSELF—THE FIRST AND SUPREME CAUSE:—We know not how far it was proper or necessary, that we should not be let into knowledge *at once*, but should advance *gradually*; that, by comparing new objects or new discoveries with what was known to us before, our improvement might be more compleat and regular: or how far it may be necessary or advantageous that INTELLECTUAL BEINGS should pass through a kind of *infancy of knowledge*; for new knowledge does not consist so much in our having access to a new object, as in comparing it with others already known; observing it's *relations* to them (or discerning what it has in

and overpowering transports of joy or grief?—These are notorious *effects*: and how they are to be accounted for, upon attentive consideration of the *constitution of human nature*, composed of a *mind* operating upon it's receptacle or *sensorium*, otherwise than by some inconceivable *stimulus* of the nervous system, conveyed through *muscular vibrations*, or some *impulse* analogous thereto, I leave to his ingenuity to suggest.

2. To the taste and piety of Professor Robison (whom I hail as a *Brother*, notwithstanding our disagreement in one instance,) we are also indebted for circulating in his deservedly popular work, *Mason's Sacred Ode*, drawn from the purest sources of the *Pierian Spring*:

“ Think not THE MUSE whose *sober* voice you hear,
Contracts with *bigot* frown her fullen brow;
Casts round RELIGION's orb the mists of *fear*,
Or shades with horror, what with *smiles* should glow:

No.—SHE would warm you with *seraphic fire*;—
Heirs as you are of HEAVEN's eternal day;—
Would bid you *boldly* to THAT HEAVEN *aspire*,
Not sink and slumber in yon cells of clay!—

Is this the *bigot's* rant?—Away, ye *vain*!
Your doubts, your fears, in *gloomy dulness* sleep:
Go!—*soothe* your souls in sickness, death, or pain,
With the *sad solace* of eternal sleep!

Yet know, *vain sceptic*! know, th' ALMIGHTY MIND,
Who breath'd on Man a portion of HIS fire;
Bade his *free soul*—by earth nor time confin'd—
To HEAVEN; to IMMORTALITY; aspire!

Nor shall this *pile of hope*, HIS BOUNTY rear'd,
By *vain Philosophy* be e'er destroyed;
ETERNITY, by ALL or *hoped* or *fear'd*,
Shall be by ALL or *suffer'd* or *enjoy'd*.”

common with them), and wherein their *disparity* consists. Thus, our knowledge is vastly greater than *the sum* of what all it's objects separately could afford; and when a *new* object comes within our reach, the addition to our knowledge is the greater, the more we already know: so that it increases, not *as* the new objects increase, but in a *much higher proportion*." * * *

156. And was the exalted *mind*, that reasoned thus powerfully, and triumphed over *bodily* infirmity in the very jaws of *death*, *material*?—Was all this merely the result of corporeal *mechanism*? Was the profound process of this noble argument for the survival and improvement of the soul; the grand *climax* at the beginning, rising from this globe to the milky way; such clear-sighted intelligence of the master principles and lofty affections, and aspiring views of *human nature*, in the sequel; such subtle and abstruse analysis of the doctrine of *combinations* and of the *fluxionary* calculus, applied at it's unfinished close; such humble, yet astonishing penetration and insight into the mysterious ways of Providence, in this life—"Looking thro' NATURE up to Nature's GOD:"—while the quivering lamp of light, thus darting the brightest flashes of celestial fire, was expiring in the socket.—Was all this admirable assemblage, of the pious and the profound, the sublime and the beautiful, the argumentative and the eloquent, no other, ye *Hartleys*! ye *Priestleys*! ye *Condorcets*! ye *Godwins*! or ye *Darwins*! than mere animal mechanism? wrought and effected, wholly and solely, by *vibratiunculae* or minute tremors of the *pulpy* substance of the brain!!! (f)

"O curvæ in terras animæ et cœlestium inanes!
Quid juvat hoc, templis nostros immittere mores;
Et Bona Dis ex hac sceleratâ ducere pulpâ?"

PERSIUS.

D'Alem-

(f) The sublime Philosophy of HOLY WRIT, and that alone, has *authoritatively* decided the mysterious and momentous questions of the *immateriality* of the *Soul*, of it's natural *mortality*, and of it's *reunion* with an *incorruptible body* after death; recording the *Origin* and the *End* of the *Human Race*:—

1. "THE LORD OF GODS formed *the Man*, *dust* of the ground,
And breathed into his nostrils, *breath of life*;
So *Man* became a *living soul*." Gen. ii. 7.
2. "*Dust* thou art, and unto *dust* shalt thou return." Gen. iii. 19.
3. "All *Flesh* shall perish together;
And *Man* shall turn again unto *dust*." Job xxxiv. 15.
4. "I know that my DELIVERER IS LIVING;
And at the *last* [day] will arise [in judgment] upon *dust*:
Yet, after this *skin* be mangled,
Even in my *flesh* shall I perceive GOD;
Whom I shall perceive, *on my side*,
And mine eyes shall see [HIM] *not estranged*." Job xix. 25.

5. "The

D'Alembert's Mournful Reflections.

157. As a terrific contrast to the death of the *Christian Philosopher*, I shall propose the dreadful gloom that overhung the latter days of a *philosophizing infidel*, furnished by his correspondence with the King of *Prussia*, another of the original conspirators against Christianity in the train of *Voltaire*.—*D'Alembert*—at once the glory and disgrace of the French *Academy of Sciences*—“*deceiving*” others, by his perfidious sophistry, but not fully “*deceived*” himself—so as to conquer and blunt the stings of remorse, and the terrors of a guilty conscience, anticipating “the just reward” of those fiends in human form, who,

5. “The *dust* shall return to the *earth* as it was;
But the *spirit* shall return to God who gave it.” *Ecclef. xii. 7.*
6. “JESUS CHRIST—“*illustrated life and incorruption.*” *2 Tim. i. 10.*
“Himself—“*the first fruits of the sleepers.*” *1 Cor. xv. 20.*
7. “Lo! I tell you a *mystery* :
We shall not all *sleep*, but we shall all be *changed*;
In a moment, in the twinkling of an eye,
At the *last trumpet.*”——
8. “For this *corruptible* (body) must put on *incorruption*;
And this *mortal* (spirit) must put on *immortality.*”
9. “O *Death*! where is thy *sting* [in the *body*] ?
O *Hades*! where is thy *victory* [over the *soul*] ?
10. “The *sting* of *Death* is *Sin*——
But thanks be to God, who giveth us the *victory*,
Through our Lord JESUS CHRIST.” *1 Cor. xv. 51—57.*
11. “I am the *First* and the *Last* and the *Living*,
And I became *dead*; and lo! I am *living*
For ever and ever. Amen.
And I hold the keys of *Hades* and of *Death.*” *Rev. i. 17.*
12. “I say unto you, *my friends* :
Be not afraid of them who kill the *body*,
But are not able to kill the *soul*;
But be afraid rather of HIM, who is able
To destroy both *soul* and *body* in *hell.*” *Matt. x. 28.*
13. “Who will transform the *body* of an *humiliation*,
To become the resemblance of the *body* of his *glory*;
According to the energy of his ability,
Even to *subject* all things unto himself.” *Phil. iii. 21.*

who, after zealously labouring through life to undermine the *hopes* of others, too fatally succeed at length in blasting their own!

In one of his last letters, published by the ingenious, the unprincipled, the execrable, and the wretched, *Condorcet*; who beset *D'Alembert* when expiring, and boasted, that, if he had not, "*D'Alembert would have flinched also*,"—like their grand master *Voltaire*—in all his dying horrors and agonies, "*worse than the furies of Orestes*"—the peevish and gloomy philosopher thus complains to his royal correspondent, *Frederick*:

—"Study *sometimes* engages me, and conversation *sometimes* entertains me; but I am soon fatigued with either; and am no sooner left to myself than my *uneasy reflections* recur, and my *solitude* again frightens and freezes me. In this condition, I resemble a man who sees before him a long and dreary desert, which he *must* pass; and at the end of that melancholy prospect, the *abyss* of destruction open to receive him: without finding, at the brink of that hideous chasm, a single person that will be *afflicted* with his downfall, or that will even remember his existence, when he has sunk into *endless perdition!!!*"

From SUCH Philosophy, GOOD LORD! deliver us.

158. How utterly false, then, and unfounded are the visionary speculations of the votaries of "*a mistress that never grows grey!*" (if we listen to the illi-

14. "For *flesh* and *blood* [i. e. an *earthly* body] is not able
To inherit God's kingdom; neither shall *corruption*
Inherit *incorruption*." 1 Cor. xv. 50.

15. "The mass of the *sleepers*, [though] *dust* of the earth,
Shall awake; some to everlasting life,
But some to shame and everlasting contempt:
When the *sages* shall shine, as the *brightness*
Of the *firmament*; and the *justified* of the many,
As the *stars* for evermore." Dan. xii. 2.

16. "As one star differeth from another star
In glory." 1 Cor. xv. 41.

17. "In my FATHER's house are many mansions." John xiv. 2.

Such are the awful, the animating, the ennobling doctrines of *Patriarchal* and *Evangelical* WISDOM—revealed, "at sundry times and in divers manners," to his *rational* creatures, by the FATHER OF LIGHTS, through 'THE SON OF HIS LOVE'—THE ORACLE or Expounder of his Decrees, under the *Patriarchal* and *Christian Economy*. Nor can this important selection of strictly harmonizing *Texts* be deemed irrelevant or unseasonable by the *Mathematician* or the *Philosopher*, who would wish zealously to stem that Torrent of *Infidelity*, which is the disgrace of a boasted AGE OF REASON:

Discite, O miseri! et causas cognoscite rerum:
Quid sumus, et quidnam victuri gignimur; Ordo
Quis datus; ————— quem Te DEUS esse
Jussit. —————

PERSIUS.

terate

terate Paine's eulogy on SCIENCE.)—And how wretched, how delusive the happiness she can afford, detached from RELIGION—that true mistress of the wise and good—whose humble handmaid SCIENCE ought to be.

—“*To be happy in old age,*” says Paine, “it is necessary that we accustom ourselves to objects that *can* accompany the mind *all the way through life*; and that we take the rest a good in their day. The mere man of *pleasure* is miserable in old age; and the mere drudge in *business* is but little better: whereas *Natural Philosophy, Mathematical and Mechanical Sciences*, are a continual source of *tranquil pleasure*. And, in spite of the gloomy dogmas of *Priests* and of *Superstition*, the study of these things is the study of the *True Theology*; it teaches man to *know* and *admire* the CREATOR: for the *principles of Science* are in the *Creation*, are *unchangeable* and of *divine origin*.”

159. Alas! what a wretched happiness did Science procure D'Alembert, who, at the close of his days, thus found to his cost, that he had been all his life long embracing a cloud for a goddess!—And that “the study of *these things* is not indeed the study of the *True Theology*,” we learn from the monstrous errors of ancient and modern *Sophists*; which led them to *forget* and to *vilify* the CREATOR: “The *principles of Science* are *not* indeed in the *Creation*;”—they lie much deeper than the *visible* world; and are only to be learned from REVEALED WISDOM and OMNISCIENCE:—whose *Sovereign Will* is the *Law* of the Universe; *changeable* at pleasure, though in HIMSELF “*unchangeable*.” James, i. 17.

160. These principles, the lowly and unassuming spirit of *Primeval* and *Patriarchal Science* endeavoured cautiously to trace and develope, amidst “the *obscurity* that prevails in the midst of things;”—humbly and devoutly thanking the FATHER OF LIGHTS for those faint *glimpses* of ETERNAL TRUTHS, which, from time to time, and from season to season, He vouchsafes to reveal to those truly Philosophic Students, who, in “an *honest and good heart*,” *hear, seek, embrace, and keep the Word of God*, whether communicated by *Revelation*, or thence unfolded by *Reason*: and earnestly and diligently, and devoutly, strive to *bring forth fruit with patience*.

“*Cælum quippè ac Terra, ac Mare, omnisque Creatura quæ videri atque intelligi potest, ad hanc præcipuè disposita est humani generis utilitatem, ut Natura Rationalis de contemplatione tot specierum, de experimentis tot bonorum, de perceptione tot munerum, ad cultum et dilectionem sui imbueretur AUCTORIS; implente omnia SPIRITU DEI, “in quo vivimus, movemur, et sumus.”—*Adhibita semper est universis hominibus *quædam* supernæ mensura doctrinæ, quæ [licet] *parcioris occultiorisque gratiæ*, sufficit tamèn (sicùt DOMINUS judicavit), quibus iam ad remedium, omnibus ad *testimonium*.—Ait Plotinus: “*Si mundi vocem audiremus, nihil aliud eum dicere quam DEUS ME FECIT; non Cretenfis Jupiter aut Arcas Mercurius, sed DEUS ille ΑΓΝΩΣΤΟΣ [nescibilis]*, de quo Paulus ad Athenienses.”

De Vocatione Gentilium.

Vide posthàc, § 205, Not. (p).

Scriptori

Scriptori huic egregio, verissimè de Dei cognitione, quatenus ex rationibus *physicis* eam colligere fas sit, suffragatur sana *Newtoni* sententia: *Princip.* p. 529.

“De Deo utique ex *phænomenis* differere ad PHILOSOPHIAM NATURALEM *pertinet.*”

161. Placet igitur, quoniam omnis disquisitio de *causis* et *effectis* naturalibus ad agnitionem CAUSÆ PRIMARIÆ perducere debet, “quæ *minimè materialis* est,” *Newtono* iudice—VINDICTIAS quasdam THEOLOGICAS *Sapientum Priscorum*, atque *Newtoni* horum cultoris, subjungere.

De Philosophiâ Atomicâ Priscorum Chaldaeorum atque Hebræorum.

162. Nec minùs injuriosè invehitur *Robison* in *Philosophiam Atomicam*, qualem excoluit *Newtonus*; quæ minimè confundenda est cum *Epicuræâ* istâ nupèr redi-vivâ, Rationibus Metaphysicis atque Mechanicis rerum naturalium causas tribuente, et CAUSÆ PRIMARIÆ oblitâ. Hanc enim jure perstringit ipse *Newtonus* in *Optics*, p. 343, his verbis:

“And for rejecting such a *medium* (as is totally dense or full of matter), we have the authority of the oldest and most celebrated Philosophers of *Greece* and *Phœnicia*, who made a *vacuum* and *atoms* and the *gravity* of atoms the first principles of their philosophy: tacitly attributing gravity to *some other cause* than dense matter.”

“Later Philosophers banish the consideration of *such a cause* out of *Natural Philosophy*; feigning *hypotheses* for explaining all things *mechanically*, and referring other causes to Metaphysics: whereas the main business of *Natural Philosophy* is to argue from *phænomena*, without feigning hypotheses, and to deduce *causes* from effects, till we come to the VERY FIRST CAUSE, which certainly is not *material.*”

Hæc breviter effata et delibata tantum, fusiùs exponere, atque ex philosophiâ primâ confirmare, jam libet:

163. Priscorum Chaldaeorum atque Hebræorum, Philosophia physica erat *Atomica*, *Cosmogoniæ Mosaicæ* consentanea. Hanc invexerunt in Thraciam, *Orpheus*, ille “*Sacer interprèsque Deorum,*” qui undecim generationes antè Trojæ excidium floruit, teste *Suidâ*, seu annis $366 + 1184 = 1550$ circitèr antè Æram Christi Vulgarem; in Ioniam et Græciam, *Thales*, “*physicorum princeps,*” *Pherocydis* Syri discipulus; in Italiam, *Pythagoras*.—Lucida fidera omnes.

Quanto

Quanto fuerunt in honore Sapientes *Chaldaei* et *Hebraei*, apud *Græcos* vetustiores, discimus ex responso *Oraculi Pythici*, quibusdam sciscitantibus, *Ex gentibus quinam omnium sapientissimi?*

Μῆνοι Καλδαῖοι σοφὴν λαχόν, ἡδ' αὖ Ἑβραῖοι,
 Αυτογενήτων Ἀναΐα σεβάζομενοι ΘΕΟΝ ΑΥΤΟΝ.

“Soli *Chaldaei* sapientiam fortiti sunt atque *Hebraei*,
Per-se-genitum Regem venerantes *DEUM IPSUM*.”

164. Præclarissimum extat testimonium in fragmentis *Versuum Orphicorum*, ab *Onomacrito* congestis, de Decalogo Sacro, apud Montem *Sinai* promulgato, annis 1649 circitèr ante *Ær. Christ.* (secundùm emendationem nostram *Sacræ Chronologiæ*, posthàc forsàn edendam).

Ἀρχὴν ΑΥΤΟΣ ἔχων, ἅμα καὶ Μέσον, ἡδὲ Τελευτήν·
 Ὡς λογῶν Ἀρχαίων· ὡς Ὑδρογενὴς διέλαξεν,
 Ἐκ ΘΕΟΘΕΝ γνωμαῖσι λαβὼν κατὰ διπλᾶ καὶ Θέσμον.

“*Principium* IPSE habens, simul et *Medium*, atque *Finem* :
 Ut sermo *Priscorum* ; ut *Aqua* genitus præcepit,
 E *DEO* sententiis quum accepisset *duplicem Legem*.”

Præclara hæc descriptio *DEI IPSIUS* optimè consentit cum *Scripturis Sacris*.
Isa. xli. 4. Rev. i. 8. &c. &c.

“*HIM First, HIM Last, HIM Midst, and without End.*” *Milton.*

Vox Ὑδρῶ, antiquitùs pro Ὑδωρ, “*aqua*.”—*Hesychius*. Per Ὑδρογενὴς igitur elegantèr expressit *Orpheus* vocem *Hebræam Mosch* (vel *Moses*), quæ significat “*ex aquâ ereptum*.”—*Exod. ii. 10.* Et per διπλᾶ καὶ Θέσμον, “*duplicem Decalogi Tabellam*”—“*digito DEI descriptam*.” *Exod. xxxii. 26. et xxxiv. 1. (g).* Atque hoc *Orphei* testimonium expressissè videtur *Juvenalis, Sat. xiv. 102.*

“Tradidit arcano quodcunque volumine *Moses*.”

165. Alterum

(g) Hujusce eximii *Fragmenti Orphici*, ab *Onomacrito* conservati, auctoritatem infectatur doctissimus *Cudworth*, ὁ πλατωνίζων, sed hypercriticè potius, ut videtur, *Intellectual System*, p. 300.

“There seem to be some *Orphic Verses* supposititious, as well as there were *Sibylline*, &c.—Cæterum cum hisce probè consentiunt versus *Orphici* quos ipse *Cudworth* pro genuinis agnoscit, et ex *Proclo* citat, p. 301.

ΖΕΥΣ πρῶτος γενέλο, ΖΕΥΣ ὕστατος ἀργικεραυνός,
 ΖΕΥΣ κεφαλῇ, ΖΕΥΣ μεσση, ΔΙΟΣ δ' ἐκ παντὶ τέτυκται.

“*Jovis primus* exillebat, *Jovis ultimus* altinonans,
Jovis caput, *Jovis medius*, à *Jove omnia fabricantur*.”

Et

165. Alterum, quoque, nec minùs insigne testimonium de *Chaldæorum* divinatione profert *Orpheus*.

Ου γὰρ κεν τις ἰδοὶ θνητῶν μεροπίων ΚΡΑΙΝΟΝΤΑ,
Εἰ μὴ μνηογενὴς τις, ἀπορῶς Φυλᾷ ᾠθεύῃ
Χαλδαίων· Ἰδρὶς γὰρ ἐστὶν Ἀστροῖο πορείας.

“Nullatenùs enì quiscquam mortalium articulatè loquentium viderit *REGEM*,
Nisi *dilectus* quidam antiquo ab origine gentis
Chaldæorum; peritus enì erat *astri exortùs*.”

Referunt hoc vulgò interpretes ad *Abrahamum*, “quem omnis vetustas Orientis summum astrologum fuisse crediderit.” *Scaliger* de *Emend. Temp.* Not. in *Fragment.* p. 48; et *Cudworth*, p. 300. Sed manifestò respicit vatem *Chaldæum*, *Balaam*, *Mosis* co-ævum, “qui apud *Peth-Ur* (seu Hebræicè *Beth-Ur*, “Domus Lucis,”) juxtà fluvium (*Tigrim*) degebat,” (*Numb.* xxii. 5.) ; et qui insigne illud vaticinium cecinit de *Shilo*, *Jacobi* progenie, (*Gen.* xlix. 10.) seu de *Messia* “Rege,” *Num.* xxiv. 17.

“*Video ILLUM, sed non nunc ;*
Cerno ILLUM, sed non propè ;
Prodibit ASTRUM ex Jacobo,
Exorietur SCEPTRUM ex Israël.”

Cujùsque posterì, *Μαγοὶ ἀπο ἀνατολῶν*, “*Magi Orientales*,”—actū “*viderunt astrum ejus in exortu suo*,” (ἐν τῇ ἀνατολῇ,) 1600 annis postea, è longinquo conspicati, *nocte* forsàn *natali* Christi. Confer *Matt.* ii. 1—10, et *Luc.* ii. 9—15.

166. Priscos *Chaldæos* atque *Hebræos* vacuum et atomas admisisse in cosmogoniam suam, colligere licet ex *Virgilio*, mythologiæ antiquæ observantissimo,

Et versum secundum, levi variatione, ex ipso *Plutarcho* quoque citat, p. 305.

ZETE ἀρχή, ZETE μεσσα, ΔΙΟΣ δ' ἐκ παντὶα πελονταί.
“*Jovis principium, Jovis medius, à JOVE omnia fiunt.*”

Ubi pro κεφαλή citantis *Procli*, *Plutarchus* vocem synonymam ἀρχή citat; utramque proculdubio à voce *Hebræa*, שׂר, vel ראשׁ ambigüe “*caput*” vel “*principium*” designante. *Gen.* i. 1.—Vide quoque Not. (n).

Necnon ex *Timotheo* chronographo quasi *Orphicæ Theologiæ* foret; p. 305.

Διὰ τῆς ΘΕΟΤΗΤΟΣ παντὰ ἐγενετο,
Καὶ ΑΤΤΟΣ ἐστὶ παντὰ.

“*Per DEITATEM omnia facta sunt,*
Et IPSE est Omnia.”

Veruntamen fatendum est quòd *Orpheus* ipse falsa veris immiscuit. Refert enim *Justinus Martyr*: Πολυθεοτης πατηρ και πρωτος διδασκαλος—quòd “*Polytheismi pater atque præceptor primus fuit*” *Orpheus*. Vide *Cudworth*, *Int. Syst.* p. 298.

ubi *vinosum Silenum*—"Custodem famulumque Dei (*Bacchi*) alumni," et proinde ex Patriarchis forsàn vetustissimum, (instar *Noah*, vino oppressi, *Gen.* ix. 21—24.) irà vaticinantem inducit, *Eclog.* vi. 31.

"Námque canebat, uti MAGNUM per INANE coacta
Semina *terrarumque*, *animarumque*, *marisque* fuissent,
Et *liquidi* simul *ignis*: ut his exordia primis
Omnia; et ipse tener Mundi *concreverit orbis*."

167. Paritérque *Euripides*, *Anaxagoræ* discipulus, traditionem vetustissimam refert, in fragmento *Menalippes*, *Barnes Eurip.* p. 481, citato à *Diodoro Siculo*, I.

Οὐκ ἐμὸς δ' μὺθος, ἀλλ' ἐμῆς μητρὸς παρὰ
'Ὡς οὐρανὸν τε γαῖα τ' ἦν μορφή μίαι·
Ἐπεὶ δ' ἐχωρίσθησαν ἀλλήλων δίχα,
Τίησι πάντα, κ' ἀνέδωκαν εἰς Φαῶς,
Δένδρα, πτηνὰ, θήρας, ἔς τ' ἄλμυρ θρεφεί,
Γένος τε θνητῶν.

"Nec meus hic sermo, sed quem præcepit Mater :—
Ut cælum et terra erant forma una ;
Ubi verò à se invicem sejuncta sunt,
Pariunt omnia, edideruntque in lucem,
Arbores, aves, feras, [piscisque] quos alit mare,
Genusque mortalium."

Manifestò desumuntur hæc *mythica* ex *Cosmogoniâ Mosaicâ*; de statu terræ *chaotico*, et generatione *vegetabilium*, *animalium* atque *hominum*, ordine *Mosaico* differentia; cæterum in hoc peccant quod SUMMI OPIFICIS vim creatricem vix, et ne vix quidem, respiciunt, sed productionem rerum *spontaneam* innuunt.

168. Atqui hîsce excerptis, seminarium habes *Philosophiæ Atomicae* qualem excolebant Sapientes Scholæ *Chaldae primæ* ac *Hebrææ*, antequàm eam pessunderunt atque inquinaverunt pravè philosophantes, *Magi*, *Brachmanni*, &c. Orientales; atque *Stoici*, *Epicuræi*, &c. Occidentales, ingenti veræ Theologiæ atque Philosophiæ detrimento.

169. Et egregiè distinguit *Plutarchus*, inter "valdè antiquos theologos atque poëtas"—quales fuerunt *Orpheus* ejusque discipuli, qui "Jovem principium, Jovem medium, à Jove omnia fiunt"—præceperunt, et "hîsce juniores, qui *Physici* sunt appellati"—quales post *Thaletem* exstiterunt *Anaximander*, *Anaximenes*, &c. Scholæ *Ionice*; atque *Itali* isti degeneres Scholæ *Pythagoreæ*, *Leucippus*, *Democritus* et *Epicurus*, qui, "ab hoc principio honesto atque divino temerè palantes, causam
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omnium primariam in corporibus, corporeisque affectionibus, pulsibus, mutationibus (commixtionibusque posuerunt." — Undè Φυσικοί et Ἀθεοί ("Naturalists" and "Atheists") indifferentè, et quasi per nomina ejusdem ferè significationis, vulgò sunt appellati. *Cicero*. p. 305.

170. Etiàm *Anaxagoras* ipse, appellatus Νῆς, "Mens," κατ' ἐξοχήν, qui priscam notionem MENTIS SUPREMÆ revocavit, "Ejus opificis in mundo condendo, rationibusque physicis, nullatenus est usus, sed ad causas rerum merè mechanicas, aëres, ætheres, et aquas minimè idoneas, aliæque prorsus absurda, confugit," ita querente *Socrate*, in *Phædone*.

171. Hinc ansam vituperandi arripiens *Aristoteles*, totam *Physicam Atomicam* insectabatur præceptis, ex rationibus suis *Metaphysicis*, multa somnians de quinto genere elementorum è quo fiant mentes et astra, de fugâ vacui, de pleno universali, de æternitate mundi, de formis substantialibus (seu εὐτελεχείαις). Inaniter autem atque arrogantè gloriatur—"Se videre, quod paucis annis magna accessio facta esset; brevi tempore, *Philosophiam* planè absolutam fore."—*Cicero*, *Tuscul.* I. 10; III. 28. *Acad.* I. 4.

172. *Aristotelis* factum jactantiàmque secuti, *Cartesiani* et *Leibnitziani*, ingenti scientiarum detrimento, *Physicorum* Græcorum *Mechanicam* irrationalem et phantasticam recolebant; *vortices*, *monadas*, &c. temerè fingentes: donec modestum *Newtoni* ingenium priscas revocabat artes; atque à *Logomachiis* averfus, ad Disciplinam Scholæ *Ionicae*, *Antiquæ Hebrææ* atque *Chaldaæ*, ferò tandèm redibat, *Physicam Atomicam* instaurabat, eam fundamentis solidis atque inconcussis *Inductionis* atque *Experimentorum* stabiliendo.

—"Hypotheses non fingo: Quicquid enim ex *phenomenis* non deducitur, Hypothesis vocanda est. Et Hypotheses, seu *Metaphysicæ*, seu *Physicæ*, seu *Qualitatum occultarum*, seu *Mechanicæ*, in PHILOSOPHIA EXPERIMENTALI locum non habent." *Principia*, pag. ult.

173. Quàm injuriosè igitur invehitur Professor *Robison* in *Philosophiam Atomicam*, qualem excolebant prisci *Chaldaei* et *Hebraei*, eorumque Discipuli; et qualem instaurabat *Newtonus*, sobriam, sanam, principiis *Mathematicis* reconditionibus et subtilioribus forsàn quam pro captu tyronum, imò *Mathematicè* doctorum, (etiàm *Professorum* forsàn), stabilitam atque demonstratam. *Logicam* certè parùm sapit, ex abusu alicujus rei, artis vel scientiæ, usum detrectare; et *Mechanismum* adedò irrationalem, absurdum et impium Recentiorum *Philosophitarum*, ex *Mechanicâ Rationali Newtonianâ* deducere.

De Æthere Antiquo, Deo Fiſto Gentilium.

174. Vocem ipſam *Æther* mutuatus eſt *Newtonus* ex Philoſophiâ Priſcâ, ita referente *Cicerone*. N. D. II. 33.

“Et cùm quatuor ſint genera corporum, *viciffitudine* eorum mundi continuata natura eſt. Nam ex *terrâ*, aqua; ex aquâ oritur aër; ex aëre, *æther*: Dein retrorsùm viciffim, ex *æthere*, aër; ex aëre, aqua; ex aquâ, *terra infima*, &c.”

Exemplum hîc habes *Compoſitionis* et *Reſolutionis* antiquiſſimæ, ſecundùm quas “mundi partium conjunctio continetur”—ex mente Phyſicorum priſcorum, ſanum, Cantui *Sileni* atque *Cosmogoniæ Moſaicæ* accommodatiſſimum: ubi *æther* manifeſtò reſpondet “*liquido igni*,” quarto nimirum elemento, (hodiè, *fluido elettrico*) ſubtiliſſimo, omnium corporum poros et interſtitia liberrimè permeanti, quod inter potentiſſimos naturæ agentes jure recenſetur; et cujus identitatem cum *igne cœleſti (lightning)* detexit et confirmavit, noſtro tandèm ævo, curioſa ſedulitas ſagaciſſimi Naturæ indagatoris, *Benjamini Franklin*.

Vox *αιθηρ* quoque, ſi compoſitionem ejus reſpicias, derivatur ex *αιδω*, “*uro*,” et *αἴρ*, “*aër*,” ſeu *atmoſphæra* (quæ forſan ex *אור aur*, “*lux*,” ultimò deducenda eſt). Et quidèm ab *Hefychio* definitur *Αἰθήρ*, ὁ ὑπὲρ τὰ νεφῆ τοπῶς, “*regio ſuprà nubes*,”—atque ſignificatione alterâ ſynonymâ, *ἐμπυρις*, “*inflammationem*.”

175. Pro ſymbolo (*b*) certè “*DEI lucem inacceſſibilem habitantis*” primitus, Patriarcharum ſæculis, *Job*, *Abraham* atque “*Chaldæorum progenitorum, Aur*,
(i. e.

(*b*) Exordium Elegiæ à reverendo viro et poetâ docto et diſerto, *Thomâ Maurice* ſcriptæ in obitum celeberrimi et optimi viri, *Gulielmi Jones*, Indorum Apoſtoli, Chriſtianique Philoſophi, pulchrè et ornate deſcribit patrias ſedes *Chaldæorum*, et primævam diſciplinam *Magorum*.

“Where the dark cliffs of rugged *Taurus* riſe,
From age to age by blaſting lightnings torn,
In glory burſting from the illumin’d ſkies,
Fair *Science* pour’d her firſt auspicious morn.

The hoary *Partbian* Seers, who watch’d by night
Th’ eternal Fire in *Mithra*’s myſtic cave,
(*Emblem* ſublime of that PRIMÆVAL LIGHT
Which to yon ſtarry orbs their luſtre gave,)

Exulting, ſaw it’s gradual ſplendours break,
And ſwept, ſymphonious, all their warbling lyres,
Mid *Scythia*’s frozen gloomis THE MUSES wake,
While happier *India* glows with all their fires.”

(i. e. "*Lux*,") incolentium," *Gen.* xi. 28. usurpatus est Αἰθερ, seu "*Aër ille igneus*."

176. *Theologiae Chaldaeae* praeclarissimum extat specimen, apud *Suidam*, de *Orpheo* ita differentem :

Εφ' ἧς δὲ, ὅτι ΦΩΣ, ῥήξαν τον Αἰθερα, ἐφωτίσε την Γην, καὶ πάσαν Κτίσιν· Ἐκεῖνο εἰπὼν "ΤΟ ΦΩΣ" τὸ ὙΠΕΡΤΑΤΟΝ πάντων, τὸ "ΑΠΡΟΣΙΤΤΟΝ," τὸ πάντα ΠΕΡΙΕΧΟΝ· Ὅπερ ὠνομάσατο ΒΟΥΛΗΝ, ΦΩΣ, ΖΩΗΝ.

Τὰς τὰ τρία Ὀνομαζα ΜΙΑΝ ΔΥΝΑΜΙΝ ἀπεφηνάτο καὶ ἘΝ ΚΡΑΤΟΣ τὴ ΔΗΜΙΟΥΡΓΟΥ πάντων ΘΕΟΥ, τὴ πάντα ἐκ τὴ μὴ ὄλος παραγαγοντος εἰς τὸ εἶναι, ὁράτα τε καὶ αἰσράτα.

"Fatus est *Orpheus*, LUCEM, rupto *Aethere*, Tellurem illustrasse, omnemque Creationem : Illam dicens LUCEM, supremam omnium, "*inaccessibilem*," omnia continentem ; quam nominavit CONSILIUM, LUCEM, VITAM."

"Hæc tria nomina declarant UNAM POTENTIAM, UNUMQUE DOMINIUM OPIFICIS omnium DEI, qui adduxit omnia ex non existente in existentiam, visibilia pariter et invisibilia."

177. Mirus fanè et ferè incredibilis consensus hic observandus cum Libris Evangelicis — Nam "*DEUS est LUX*," 1 *John*, i. 5. "*Lucem habitans inaccessibleem*," 1 *Tim.* vi. 16. Et *JESUS CHRISTUS est "LUX Mundi*," *John* viii. 12. Ut et "*SAPIENTIA DEI*," *Luk.* vii. 35. et *Matt.* xi. 19. — itémque "*VITA*," *John* i. 4. et xiv. 16. Et declaravit *CHRISTUS*, Εγὼ καὶ Πάτερ ἘΝ ΕΣΜΕΝ. — *Orpheo* nimirum optimè interpretante — ἘΝ ΚΡΑΤΟΣ — "*PATER et EGO unum sumus [DOMINIUM]*," *John* x. 30.

Et *Milton*, adhuc sublimius et eruditius, varias sententias veterum Philosophorum exponit, in celeberrimâ illâ *LUCIS* invocatione, P. L. III. sub initio :

"Hail, HOLY LIGHT ! Offspring of Heaven, First-born,
Or of th' eternal co-eternal Beam ;
May I express thee unblam'd ! Since "GOD IS LIGHT,"
And never but in "*unapproach'd light*
Dwelt" from Eternity, dwelt then in Thee,
Bright Effluence of bright ESSENCE INCREATE :
Or hear'st thou, rather, pure *Ætherial Stream*,
Whose fountain who shall tell ? Before the Sun,
Before the Heavens thou wert, and at the voice
Of GOD, as with a mantle, didst invest
The rising *World of Waters*, dark and deep,
Won from the void and formless Infinite."

Notandum quòd per *Mithram* intelligebant *Paribi Magi* antiquissimi non *Solem* (ut malè posteri), sed *Sole* superiore ; quem appellarunt, Ὁ κρυφίος Θεός, "*Deus absconditus*," — Παντων Ποιητής καὶ Πάτερ, "*Omnium Opifex et Pater*." *Cudworth*, p. 286. Quibuscum optimè consentit *Job* xxiii. 9. *Judg.* xiii. 18. *Isa.* ix. 5. *Isa.* xlv. 15. Vide § 179.

178. Sana

178. Sana et sublimis *Noachidarum* Theologia pessundari et corrumpi incœpit circâ quintam à Diluvio generationem; et potissimum à *Nimrod, Cusbitarum* principe, qui cultum astrorum primus introduxit; etiâ proverbio celebratus, ut “*Ingens ille venator, coram Domino;*” qui post mortem ejus, à subditis superstitionis, translatus est in astrum *Orionis* cum canibus (qui *Sirius* et *Canicula* sunt appellati), quasi in cœlis etiâ nûm venaturus *Ursam Majorem* ejûsque Catulos, Circulo *Arctico* commorantem, et motu diurno revolventem; quam vivus insectabatur in Montibus *Affyriæ, Ararat (i)*: Nam hic bellator acerrimus quoque, cum ab Oriente migraturus sit in sedem ab oraculo suis destinatam, substituit rebellis “in amœnâ terrâ *Sbinaar*, ubi principium dominationis suæ posuit *Babel*, seu *Babylonem*; et inde egressus, invasit *Affyriam* (sedem *Noachidarum* post Diluvium, et *Sbemitæ Peleg*, in distributione divinâ terrarum, allocatam), et *Niniven* condidit.” Confer *Gen. x. 8—12.* et *xi. 1—8.* et *Josephum*.—Colebatur hic *Rebellis* sub nomine *Bel* (à *Heb. בל Baal*, “Magister,”) ab *Affyriis*, undè *Græcorum Belus*; cujus nomen proprium forsân erat *Nin*—(i. e. נן “Filius,”) (*k*)—utpotè clarissimus filiorum *Chus*; et quidem appellatur ipsa *Ninive* ab *Herodoto*, passim, נין ניו, “Civitas *Nini*.”—Fuit autem *Pronepos Chus*, per intermedios *Raamah*

et

(i) Hujusmodi venationem pulchrè describit *Homerus, Il. xviii. 485.*

Ἀρκτον δ' ἦν καὶ Ἀμαξάν επικλησὶν καλεσθῆναι
Ἢ τ' αὐτο σφραγεται καὶ τ' Ὠριωνα δοκευεῖ.

“*Ursamque*, quam cognomine *Plausfri* appellant [*Ægyptii*],
Quæ ibidè sefe vertit, observâtque *Orionem*”

—quasi illum et canes venaticos metuens.

Atque iterum *Orionem*, in *Infernis*, similiter occupatum introducit, *Odys. xi. 571.*

Τὸν δὲ μετ' Ὠριωνα πελωρίον εἰσενόησα
Θηρὰς ὅμῃ εἰλευνῖα κατ' ἀσφοδελὸν λειμῶνα,
Τὸς αὐτὸς κατέπεφνεν ἐν οἰπολοῖσιν ὀρεσσίν,
Χερσὶν ἐχὼν ῥοπαλὸν παγχάλκεον αἰὲν ἀαγές.

“Postea, *Orionem ingentem* asperi,
Feras unâ cogentem per *alphodelum* pratum,
Quas *Ipsè* occiderat in montibus desertis;
Manibus tenentem clavam prorsus æneam, semper *infractam*.”

Annon facillè derivari potest *Ωριων* à *Chaldæo Uriah* (“Lux Domini”), quæ in *Versione Græcâ Septuagint.* in casu obliquo, redditur *Ουριαν*, 2 *Sam. xi. 6.*?

(k) *Ingentis* hujusce tyranni nomen proprium fuisse נן *Nin*, “filius,” cui per odium et ludibrium cognomen indiderunt *Sbemitæ* subjugati, נמרוד *Nimrod*, “Archirebellem,” confirmatur ex *Trogo Pompeio*, seu *Justini Lib. I. initio*.

“Primus omnium *Ninus*, Rex *Affyriorum*, veterem et quasi *avitum* gentibus morem, [i. e. *Regimen Patriarchum*,] novâ imperii cupiditate mutavit. Hic primus intulit bella *finitimis*, et rudes adhuc ad resistendum populos, terminos usque *Libyæ*, perdomuit.”—“Domitis igitur proximis, cum accessione virium fortior ad alias transiret; et proxima quæque victoria instrumentum sequentis esset, totius *Orientis* populos subegit. Postremum illi bellum cum *Zoracstre* Rege *Babrianorum* fuit (qui primus dicitur

et *Sheba*, adeoque *Pelegi* co-ævus, quintus nimirum à *Noah*, non tertius, ut hætenus à Chronologis malè supponitur.

179. Ingruenti huic idololatriæ fortiter se opposuit *Job*, septimus vel octavus à *Noah*. (Vide *Abulfaragi*, p. 13.)

“ Si aspicerem *lucem* * [*Solis*] ubi splendebat,
 Aut *Lunam* gloriâ incedentem;
 Si cor meum in occulto seductum esset,
 Aut manus mea ab ore oscularetur;
 Etiam hoc foret *crimen judiciale*,
 Ita enim mentitus essem DEO SUPREMO.” *Job xxxi. 26.*

180. Sed, *damnosa quid non imminuit dies?*—Sublimem et sinceram Patriarcharum *Chaldæorum* atque *Hebræorum* Theologiam—“ SPIRITUM SUPREMUM in spiritu atque veritate ritè venerantium,” tandem corrumpit et inundavit *Sabianismus*, seu pravissima idololatria astrorum; “*fingebant enim idololatræ*,” ut benè notat *Newtonus*, *Princip.* p. 529.—“*Solem, Lunam et Astra, Animas Hominum*, et alias *Mundi partes*, esse *partes DEI SUMMI*; et ideò *colendas*, sed falsò.—Atque ex cunctis Philosophis Mythologisque *Gentilibus*, eheu!

“ *Apparent rari nantes in gurgite vasto*,”

qui de DEO ritè sentiebant.

181. Operam Philosophis nostris *Britannis* hæud ingratham me impensurum spero, si cursim ex *Græcis Romanisque* Mythologis, miram et plorandam sanè cæcutiam atque vecordiam paucis exemplis illustrem, de *Deo materiali* per vocem *Αἰθερ*, vel *Æther*, ab utrisque intellecto, turpissimâ Priscæ Theologiæ depravatione.

Tragædorum Græcorum antiquissimus, *Æschylus*, *Ætherem*, esse *Deum* apertè posuit.

dicitur *Artes Magicas* [seu disciplinas *Magorum*] invenisse, et mundi principia, siderumque motus spectasse.) Hoc occiso, et ipse decessit; relicto impubere adhuc filio *Ninya* et uxore *Semiramide*—

Hic, *Justinus*, more *Græciæ* mendacis, vera fallis immiscuit. *Nimrodum* probè refert, finitimos invadentem, atque interficientem ultimò, paulò antè obitum, *Zoroastrem*, (forsàn *Peleg* ipsum, in *Montibus Ararat*,) *Archimagum*, peritissimūque *Astronomum*. Sed errore planè ridiculo, ob incitiam Linguarum Orientalium, *Trogus* transfudit urbem *Niveu* à *Nimrod* conditam, seu *Niniven*, in fluvium “*Ninyam*!”—Et de *Semiramide* fabulatur! Vera enim *Semiramis*, quinque tantum generationibus, teste *Herodoto*, antecessit *Nitocrem*, insignem Reginam, quæ *Babyloni* exstruendo atque ornando ultimam manum imposuit; illa autem videtur fuisse magni *Nebuchadnezzari* uxor, de quo *Daniel*, v. 10.

Secundùm emendationem nostram *Chronologiæ Sacræ*, incidit distributio terrarum habitabilium inter Colonias *Noachidarum*, “in diebus *Peleg*,” (i. e. “*Divisionis*,”) ubi floruit ille *Ninus* Archirebellis, anno 541 circiter, post diluvium; seu anno 2615 antè Æram Christianam: floruit autem illa *Semiramis* quæ mœnia *Babylonis* construxit, 166 annis saltè antè obitum *Nebuchadnezzar*, sive 166 + 561 = 727 antè Æram Christianam. *Semiramis* proinde pro uxore *Nabonassari*, (cujus æra incept 747 annis antè Christ.) jure haberi potest.

ΖΕΥΣ ἔστιν Αἰθήρ, Ζεὺς δὲ Γῆ, Ζεὺς δ' Οὐρανός, Ζεὺς τοι Πάντα.

"Jovis est Æther; Jovis item Terra, et Jovis Cælum, Jovis denique Omnia."

182. Audi quoque Hippocratis commentum, *De Princip.* § 1.

Δοκεῖ δὲ μοι ὁ καλεσµένος Θέρµον, ἀθανάτων τε εἶναι καὶ νοεῖν πάντα, καὶ ὄρῃν καὶ ἀκρῖν καὶ εἶδεναι πάντα, καὶ τὰ ὄντα καὶ τὰ ἐσόμενα· τῆς δ' ἐν, το πλείστον, ὅτε ἐλαρχθῇ πάντα. ἐξέχωρησεν ἐς τῶν ἀναλίσκων περιφορῇ καὶ ὀνοµῆναι μοι αὐτοῖς δοκεῖσιν οἱ παλαιοὶ Αἰθέρα.

Ἡ δὲ ἑτέρα μοῖρα, ἣ καλεῖται αὐτῇ, καλεῖται μὲν Γῆ, ψυχρὸν καὶ ξηρὸν καὶ πολὺ κινεῖν καὶ ἐν τῇ ἐν ἡ δὲ πολλὰ τε θερµα. Ἡ δὲ τρίτη μοῖρα, ἣ καὶ τε Ἡερός χωρίον εἰληφῇ, θερµὸν τι καὶ ὑγρὸν εἶναι. Ἡ δὲ τετάρτη, ἣ τε ἐγγύς τῃ Γῇ, ὑγρότατον τε καὶ παχύτατον.

"Mihī quidē videtur Id quod CALIDUM vocamus (Gallicè CALORIQUE) esse immortale; atque omnia percipere et videre et audire, omniāque scire tam præsentia quam futura. Hoc igitur, maximā ex parte, omnibus perturbatis [chaotice], excessit in orbem altissimum; idque nominasse mihi videntur Prisci Ætherem."

"Secunda pars, eaque inferior, vocatur TERRA. Estque frigidum et siccum et permobile; in hac quoque inest calidi multum. Tertia autem pars, quæ aëris regionem fortiebatur, est quoddam calidum simul et humidum. Pars verò quarta, terræ proxima, est humidissimum simul et densissimum."

183. Et convenienter Virgilius, *Æn.* VI. 724.

"Principio Cælum et Terras, Campōque liquentes,
Lucentēque globum Lunæ Titaniāque Astra,
SPIRITUS intūs alit; totāque, infusa per artus,
MENS agitat molem, et magno se corpore miscet."

Altiùs hīc sonant "Spiritus et Mens;"—sed sensu prorsus materiali, intelligenda; ut ex sequentibus patebit, *Georg.* II. 340.

"Cum primum lucem pecudes hausere, virūque
Ferreæ progenies duris caput extulit arvis,
Immissæque Feræ sylvis, et Sidera cælo."

184. Rationem reddente Ovidio, *Metam.* I. 75.

"Neu regio foret ulla suis animantibus orba,
Astra tenent Cælestē Solum, Formæque Deorum,
Terra Feras cepit."———

185. Plenius

185. Pleniùs autèm et luculentiùs, *Virgilio*, *Georg.* IV. 219.

“ His quidam signis, atque hæc exempla secuti,
 Esse *apibus* partem DIVINÆ MENTIS, et HAUSTUS
 ÆTHEREOS, dixere; DEUM nāmque ire per omnes
Terrâsque, tractûsque *Maris*, Cælûmque profundum.
 Hînc, *Pecudes*, *Armenta*, *Viros*, genus omne *Ferarum*,
 Quemque sibi tenues *nascentem* arcessere vitas:
 Scilicet hûc reddi deinde, ac resoluta referri
 Omnia; nec *Morti esse locum*; sed *Vivâ* volare
Sideris in numerum, atque alto succedere *Cælo*.”

186. Et apertissimè quidem, *Georg.* II. 323.

“ Tûm PATER OMNIPOTENS, fœcundis imbris ÆTHER,
Conjugis in gremium lætæ descendit, et omnes
 Magnus alit, magno commixtus corpore, *fœtus*.”

Hîc procûl dubio, “ *Pater Omnipotens*” et *Æther*, pro synonymis habentur à *Virgilio*; sicut à cæteris passim Poetis *Latinis*,—JUPITER et ÆTHER.

187. Paritèr *Horatius*, *Od.* I. xxii. 19. et I. i. 25.

“ Quod latus mundi, *Nebulæ*, MALUSQUE
 JUPITER urget.”

“ ———Manet sub JOVE FRIGIDO
 Venator.”———

188. Paritèrque *Lucanus*:

“ JUPITER est quodcunque *vides*, quocunque *moveris*.”

189. Atque identitatem *Mentis Divinæ* et *Humanae* apertissimè tradiderunt Philolophi, referente *Cicerone*:—“ Sive *anima* sive *ignis* sit animus (*humanus*) eum jurarem esse *divinum*.”—“ Et quidem si *Deus* aut *anima* aut *ignis* est, idem est animus hominis:”—“ Ergò *animus*, ut ego dico, *divinus* est; ut *Euripides* audet dicere, *Deus*.”

Et ex hominum motu libero argumentum fabricârunt pro animi æternitate: ut docet quoque *Cicero*:—“ Quòd *se ipsum moveat*, quis est qui hanc naturam *animis* esse tributam neget? *Inanimum* enim est omne quod *pulsu* agitur *externo*; quod autèm est *animal*, id *motu* cietur *interiore* et *suo*.—Sentit igitur animus *se moveri*; quod cum sentit, illud sentit, *se vi suâ*, non alienâ, *moveri*; nec *accidere posse* ut ipse unquam à se deferatur; ex quo efficitur *æternitas*.” (Vid. § 193.) —“ EX DIVINITATE *animos* haustos habemus.”

190. Talia

190. Talia sunt dogmata "*Insanientis Sapientiae*"—CREATOREM cum creatis, homines cum bestiis, animantia cum inanimatis, confundentis; SPIRITUI SUPREMO, existentiam *materialem* impiè tribuentis, *immortalitatem* innatam animorum hominum pecudumque falsò prædicantis; DEUM OPTIMUM MAXIMUM, (cujus est "*dare et adimere*," vitam commodare et vitam perdere, creare atque annihilare, solo sui ipsius numine, et liberrimo arbitrio, nullo cogente *Fato* seu *Necessitate cæcâ*,) ingrâtè prætermittentis et turpiter dehonestantis!

191. Optimè igitur et acutissimè *Paulus* de Philosophis Gentilibus:—*Rom. i. 22. (l).*

Φασκοντες ειναι ΣΟΦΟΙ, ΕΜΩΡΑΝΘΗΣΑΝ.

"Autumantes se esse SAPIENTES, facti sunt STULTI."

(l) Quantum autem insipuerunt, et quàm miserè cæcutierunt, vel ævo *Augusti* literarum culturâ celebrato, Philosophi *Romani*, uno alteròve exemplo ostendere libet:

Gravem illam, sublimem et magnificam, DEI SUMMI descriptionem *Horatianam* quis non miratur humaniorum litterarum scientiâ vel levitèr imbutus? *Od. III. 4, 42.*

— "Scimus ut impios
Titanas, immanemque turmam
 Fulmine sustulerit caduco,
 Qui *terram inertem*, qui *mare* temperat
Ventosum; et *urbes*, regnâque *tristia*,
Divosque, mortalesque turbas,
Imperio regit UNUS æquo."

Petuntur hæc sanè ex *Philosophiâ primâ*, et manifestò respiciunt *Sodomi* atque *Gomorreæ* excidium, *Gen. xix. 24.*; aut *Gigantes* forsàn, *Gen. vi. 4.* Diluvio peremptos. Sed quàm indigna proximè sequuntur?

"Magnum illa terrorem intulerat Jovi,
 Fidens *Juventus* horrida brachiis,
Fratresque tendentes opaco
Pelion imposuisse *Olympo*!"

Quàm incongrua, quàm absurda, quàm impia cadit conclusio?—Quinque mortales juvenes nempè "*terrorem intulisse* DEO SUMMO," ne cælum scandant ipsùmque à folio detrudant!!!—Nónne in hoc casu Poetarum *Ethicorum* princeps (ut fertur *Horatius*), factus est stultissimus?

Et quis feret vel ipsum *Ciceronem* pravissimè philosophantem de *perjurio*, quasi à NUMINE SUMMO prorsus neglecto atque impunito? *De Officiis, III. § 28.*

"Quid est igitur, dixerit quis, in *jurejurando*? Num iratum timemus JOVEM? At hoc quidem COMMUNE EST OMNIUM PHILOSOPHORUM (non eorum modò, qui *Deum nihil habere ipsum negotii*, et nihil exhibere alteri [ut *Epicureorum*]; sed eorum etiàm qui, *Deum semper agere aliquid et moliri* volunt [ut *Stoicorum*, *Peripateticorum*, &c.]) NUNQUAM NEC IRASCI DEUM NEC NOCERE."

Nónne *Sophistæ* tales, (sicut *Cicero Epicureos* incusat),

"DEUM nomine ponunt, re tollunt?"

Atque eò stultitiæ et vecordiae progressi sunt, ut NUMEN SUMMUM, passim, per vocem TO ΘΕΙΟΝ designarunt, cujus significatio communis apud Veteres, fuit "*Sulphur Sacrum*," Anglicè "*Lightning*."

Sic Lucas Casum Sodomii referens, xvii. 29. ex *Genesi*, xix. 24.

Εβρεξε πυρ και θειον, (Κυριου) απ' ουρανε.

"Depluit (DOMINUS) ignem et sulphur de cœlo."

Et phrasin classicam, Θεις πληρο, exponit *Hesychius*, τς κεραυνις πυρος λεγει' οδμην γαρ εχει ο κεραυνος θειε,—"Ignem fulmineum vult; habet enim fulmen sulphuris odorem." Appellatur autem Θειον—το εκ Θεε αφιγμενον, utpote "ex DEO proventum."

192. Notatu dignissimum est quòd unico tantum Evangelii loco, usurpatur TO ΘΕΙΟΝ in sensu philosophico, pro DIVINITATE seu NUMINE SUMMO, idque à Paulo, hallucinationem Stoicorum atque Epicureorum Athenis perstringente:—

"Minimè reputare debemus, auro vel argento, vel lapidi, sculpturae nimirum artis solertiae que humanæ (TO ΘΕΙΟΝ) DIVINITATEM esse similem."—*Acts* xvii. 29.

Omnibus aliis in locis vox Θειον in sensu communi adhibetur, pro sulphure sacro.

193. Quante acumine refellit Paulus dogmata Aristotelica de motu necessario, seu *mechanico*:—

Τις η της κινησεως αρχη τη ψυχη; δηλον δε ωσπερ εν τω ολω ΘΕΟΣ, και των εκεινω. Κινει γαρ πως τα παντα το εν ημιν ΘΕΙΟΝ.—'Εν τι των αδυνατων το υπαρχειν ψυχη κινησιν.

"Quid est principium motus animo? Patet fanè, sicut in toto DEUS, ita quoque Omne in Illo: Movet enim quodammodo omnia in nobis DIVINITAS."—

"Ex impossibilibus unum est quoddam, animo competere motum."

Contrà asserit Apostolus, motum hominum voluntarium, et tamèn à DEO oriundum:—Εν ΑΥΤΩ (ΘΕΩ) γαρ ζωμεν και κινεμεθα και εσμεν—"In ILLO (DEO) enim vivimus et nosmet movemus et sumus."—Nam κινεμεθα vox media est.—Quà ratione autem nosmet sponte movemus, mortalibus scire non fas est; à DEO hæc facultas conceditur, sed more nobis prorsus incognito, et nunquam forsàn cognoscendo. Vide § 189.

194. Sufficiant hæc de *Æthere*, seu De Deo Sulphureo et Materiali Sophistarum Veterum, hoc nostro sæculo rationis (ut dicitur) infausto, eheu! renascentium. Annon poeta noster Ethicus quoque, Pope, inscius forsàn, in hunc turpissimum errorem

errorem *Stoicorum* et *Peripateticorum*, à *Cartesio* et *Leibnitzio* malè recoctum, lapsus esse videur, ubi *Deum* hallucinans *Animam Mundi* refinxerit?

“All are but *parts* of one stupendous *whole*,
Whose *body* NATURE is, and God the *soul*.”

Quanto meliùs fuisset, si à disquisitionibus spinosissimis et *periculose* plenis *alae*, quæ poëtis (qui plerumque in hisce studiis severioribus parùm sunt versati) minime conveniunt, prorsùs abstinuisset, dogmatis illius fani à se-ipso alibi prolaturè memor!

“A little learning is a dangerous thing;
Drink deep, or taste not the *PIERIAN SPRING*.”

Vide § 207.

195. Regerent hìc forsàn Poetæ, *Popùque* fautores, “nónne ipse *Newtonus* pariùr quoque peccat, ubi de *Sensorio DEI* differit in sublimi argumento de existentia *ENTIS SUPREMI* ex operibus *Naturæ* causisque finalibus patefacti?” *Optics*, p. 345.

“Does it not appear from *phænomena* that there is a *BEING incorporeal, living, intelligent, omnipresent*, who in infinite space (as it were in *his Sensorium*) sees the *things themselves* intimately, and thoroughly perceives them, and comprehends them wholly by their immediate presence to *HIMSELF*: of which things the *images* only, carried through the organs of sense into our *little sensoriums*, are there seen and beheld by *that* which in us perceives and thinks?”

But how cautiously, how correctly does *Newton* guard against misrepresentation of his meaning, in a subsequent passage, p. 379.

“Such a wonderful *uniformity* in the *planetary system*—the *uniformity* in the bodies of *animals*—the *first contrivance* of these very artificial parts—the eyes, ears, brain, muscles, &c. and other organs of sense and motion; and the *instinct* of brutes and insects, can be the *effect* of nothing else than the *wisdom* and *skill* of a *Powerful, Everliving AGENT*; who, being in all places, is more able by *his will* to move the bodies within his boundless uniform *sensorium*, and thereby to form and reform the parts of the Universe, than we are by *our will* to move the parts of our own bodies.

“And yet, we are not to consider “*the World* as the *body* of God, or the several parts thereof as the *soul* of God:”—*HE* is an *UNIFORM BEING*, void of organs, members or parts; and *They* are his *creatures*, subordinate to *HIM*, and subservient to *his will*: and *HE* is no more *the soul* of them, than the *soul* of man is the soul of the *species* of things carried through the organs of sense into the place of it's sensorium; where *It* perceives them by means of it's immediate presence, without the intervention of any *third* thing.

[N. B.] "The *Organs of Sense* are not for enabling the soul to perceive the species of things in it's sensorium; but only for conveying them thither: and God has no need of such organs; He being every where present to the things themselves."

DE ENTE SUPREMO.

196. AUDIAMUS *Newtonum* quoque, tunc temporis grandævum, gravissimè et ornatissimè de hoc mysterio maximè incomprehensibili differentem, in *Scholio generali Principiorum*, nunquàm satis laudando.

"Hic omnia regit, non ut *Anima Mundi*, sed ut *Univerforum Dominus*. Et propter dominium suum *ἡγεμονία* (id est, *Imperator Universalis*), dici solet: Nam *Deus* est vox relativa, et ad *servos* refertur; et *Deitas* est *Dominatio Dei*, non in corpus proprium, (ut sentiunt quibus *Deus* est *anima mundi*), sed in *servos*.—Vox *Deus* passim significat *Dominum* (m). *Dominatio* ENTIS SPIRITUALIS, DEUM constituit, vera verum, summa summum, ficta fictum. Et ex dominatione

DE NOMINIBUS DEI PRIMITIVIS.

(m) *Newtonus*, "*Pocock* nostrum secutus, vocem Latinam *Dei* deducit à voce Arabicâ *Du*, (et in obliquo casu, *Di*), quæ *Dominum* significat,"—At minùs rectè, ut videtur. Et forsàn, haud importunum erit, pro munere meo *professorio*, quamvis *emerito*, de sacrosanctis DEI nominibus primitivis, cautè, verecundè et circumspèctè ex rationibus *etymologicis*, sanioribus magisque scientificis, forsàn, quam hæcenus per *Lexica* quævis *Orientalia* vel *Institutiones Hebræas* prolatis, breviter differere.

ἈΓΙΑΣΘΗΤΩ ΤΟ ΟΝΟΜΑ ΣΟΥ.

"Sanctificetur Nomen Tuum."

Innuit *Pocock*, nimio Arabismi studio illectus, Linguam Arabicam esse *primævam*.—Cæterum omnium matrem esse *Hebræam*, satis constabit ex nominibus propriis, *Virorum*, *Regionum*, *Fluviorum* et *Urbium*, tam antè diluvium quàm per quinque generationes post; quæ omnia sunt purè *Hebræa*, et plurima ex cæteris dialectis primitivis exulant. Et necessariò, quidè, si Lingua *Noachidarum* erat eadem ac *Antediluvianorum*; et si "*Totus Mundus fuit unius linguæ uniusque dialecti*" usque ad *Dispersionem Babeliicam* et *Linguarum Confusionem*. *Gen. xi. 1.*

Vox Latina *Deus* rectiùs deduci potest immediatè à Græcâ *Δεὸς*, Doricè pro *Ζεὺς*,—(*Hesych.*)—transmutatione literarum *cognatarum* per omnes gentes usitatissimâ; atque hæc iterùm à voce Phœniciâ *Ievw*, qualem pronunciabant nomen *Dei* tetragrammatum apud *Hebræos* sanctissimum, יהוה, *Iehob*; (*Scaliger*, ex *Sanchoiathone*, De *Emend. Temp. Fragm. Not. p. 37.*) simili transmutatione, et pro terminatione Hebræâ *h*, *oh*, substituendo quòque terminationem *Σ*, à Lingua *Medâ*, seu Persicâ primitivâ, mutuata; (*Herodotus*, I.) Antiquitus autè, ab Oraculo *Clarii Apollinis*, pronunciabatur *Iaw*, quasi *Iabob*, sciscitantibus "*quis Deorum habendus sit qui vocatur Iaw?*" (*Macrob. I. 18.*)—Undè Priscorum Græcorum *Zas* et *Zay*—*Zavos*, et Ionicè *Zḡ*—*Zḡvos*—(*Hesych.*) et Priscorum

dominatione verâ sequitur DEUM VERUM esse *vivum, intelligentem et potentem*: ex reliquis perfectionibus *summum* esse vel *summè perfectum*.

“Deus est UNUS ET IDEM DEUS, *semper et ubique*. Omnipresens est, non per *virtutem* solam, sed etiâ per *substantiam*: nam virtus sine substantiâ subsistere non potest. In Ipso continentur et moventur Universa, sed sine mutuâ passione: Deus nihil patitur ex *corporum motibus*; illa nullam sentiunt resistentiam ex *Omnipræsentia* DEI.

“Deum

Præcorum Itolorum, *Ianus*, Deorum antiquissimus, qui “*Saliorum* quôque antiquissimis carminibus *Deorum Deus* canitur.” *Macrob.*

Hanc vocem *Iaw* contractiùs pronunciabatur *Iw*, quam malè confundunt omnes ferè editores Græci atque Latini, cum interiectione O! ut *Euripid. Bacchæ*, 583.

Iw, Iw, Δεσποτα, Δεσποτα.

“*Io, Io, Domine, Domine.*”

—pro *Iw Βαχχæ*, “*Io-Bacche*,” *Hor. Sat. I. 3. 7.*

Itémque *Virgilius*, archaismi observantissimus, *Æn. 10. 17.*

“*Io-pater, Io—Hominum Divûmque æterna potestas.*”

—quæ vulgò *Jupiter*, pronunciatur, at in casibus obliquis, *Io-vi, Io-vi*, &c. verum suum etymon declarat. Vocem *Iw* interpretatur *Hesychius*, *ενι, uno*; et confirmatur ex *Homeri* Ionicâ, in phrasi *Ιω ἡματι*, *Iliad. 6. 422.* interpretante Scholiâ, *εν μια ἡμερα*, “*uno die.*” Optimè autem consentit *Sacra Scriptura, Deut. vi. 4. et Mark xii. 29.*

“*Audi Israel, Dominus Deus noster Dominus unus est.*”

Et notatu dignissimum est quod nunquàm ab Hebræis *יהוה*, *Iaw*, nunquàm à Græcis, *Ζεὺς* aut *Zeus*, pluralitèr usurpantur, utpote *unitatem Dei* designantes.

“*Eis Zeus, εἰς Ἀδης, εἰς Ἥλιος, εἰς Διόνυσος.*”

“*Unus Jovis, unus Pluto, unus Sol, unus Bacchus.*”

Ut refert *Orpheus*, citante *Macrobio*, qui cum *Oraculo Clario* consentit.

Et vox ipsa *יהוה* ultimò deduci videtur à radice primævâ *יה Iab*; quæ nomen Dei antiquissimum apud Hebræos fuit, (sicût apud Italos, *Ianus*); et cujus propria significatio optimè forsàn ab *Homero* asservatur in phrasi *Ια γηγρυς*, *Iliad 4. 437.* interpretante Scholio, *μια και αυτη φωνη*, “*una eadèmq̃ lingua.*”—Et optimè respondet hæc significatio, *Isa. xxvi. 4.* ubi *יה* et *יהוה* simul occurrunt:

“*Fidite Domino usque in perpetuum,*

Quia in (*Iab, Iabob*) uno eodèmq̃ Domino est *Petra Sæculorum.*”

Egregiè igitur designat radix *יה* Divinam immutabilitatem et æternitatem. Confer *Malach. iii. 6. Jam. i. 17. John xiv. 1. Heb. xiii. 8.*

2. A voce Hebræâ *די Di*, quæ propriè significat *Sufficientiam, Copiam, Opes*, et undè epitheton divinum *שדי Sadi*,—“*cujus est sufficientia,*” *Gen. xvii. 1. et 2 Cor. iii. 5.* quasi ab radice exsurgunt Arabicâ, *Sadi*; Græca *Δις, Δι-ος*, quæ Deum Summum designabat; cum voce *Zeus* synonymans, et sibi postea casus obliquos *Δι-ος, Δι-ι, Δι-α*, assumens; et Latina *Dis, Di-tis*, Deum Infernum, *Plutonem*, turpiore Theologiæ Primitivæ depravatione; et indè quoque, *Dii Manes*, seu *Dæmones*, mortuorum umbræ, quas pravissimâ superstitione colebant gentes.—Hinc *Jupiter* appellatur *Dis-piter* ab *Horatio*, distinctionis gratiâ, non *Diespiter*, ut imperitè editores.

3. A voce Hebræâ *יהוה Elōh*, quæ propriè *ὁ Δυναστης* vel *Δεσποτης*, “*Dominus,*” vel “*Imperator*” significat, *1 Tim. vi. 15, vel Jude iv.* Atque Deo Summo antiquitûs à gentibus omnibus tributa

“Deum Summum *necessariò existere*, in confesso est. Et eàdem necessitate *semper* est et ubique. Unde etià totus est *SUI-SIMILIS*: Totus oculus, totus auris, totus cerebrum, totus brachium, totus vis sentiendi, intelligendi et agendi; sed more minimè humano, *more nobis prorsùs incognito*. Ut cæcus non habet ideam colorum, sic nos ideam non habemus modorum quibus Deus *Sapientissimus* sentit et intelligit omnia.

“*Corpore omni et figurâ corporeâ* prorsùs destituitur; ideòque videri non potest, nec tangi, nec sub specie rei alicujus corporeæ coli debet. Ideas habemus *attributorum* ejus, sed quid sit alicujus rei *substantia* minimè cognoscimus. Videmus tantùm corporum figuras et colores, audimus tantùm sonos, tangimus tantùm superficies externas, olfacimus odores solos, et gustamus sapes; intimas substantias nullo sensu, nullâ actione reflexâ cognoscimus; et multò minus ideam habemus substantiæ Dei.

“Hunc cognoscimus solummodò per *proprietates* ejus et *attributa*, et per *sapientissimas* et *optimas* rerum *structuras*, et *causas finales*; et *admiramur* ob *perfec-*

tributa est, ritè deducitur variabilis *Allab*, *Ullab* vel *Alo*, quæ vel hodiè cunctas ferè regiones Orientales peragravit; atque ipsa *אלֹב* *Ælôb*, pari analogiâ, ex nomine Dei longè antiquissimo omnium, *אל* *Æl*, facilè derivatur; quæ abstractè significat ἡ δύναμις, “*Potentia*,” concretè verò, ὁ δυνατός, “*Potens*.”—Et notio Dei primaria et omnium simplicissima et usitatissima erat *Potentia*, sive beneficiæ sive maleficiæ. Vid. *Gen.* xxxi. 29. ubi minuitur *Laban* *Jacobo* genero suo profugo—“Est *potentia* manus meæ te nocere,”—*אל*. Vide quoque *1 Sam.* xvii. 26. et *2 Kings*, v. 7. et *Rom.* i. 20. ubi *Paulus* pro parallelis recenset DEI SUMMI *Δυναμις* καὶ Θεότης, “*Potentia* et *Deitas*,”—quem probè sequitur *Newtonus*—“*Deitas est Dominatio*.”

4. Glissante verò Idololatriâ, malè haberi incæpit vox *אל* *Æl*, pro *Sole*, numinibûsque fictis *Mythologiæ Veterum*; unde *Solis* epitheton, *Αελ-ιος*, *Ηλ-ιος*, vel *‘Ηλ-ιος*, in diversis *Græciæ* (in *Theologiâ mendacissimæ*) dialectis. Sic plorat *Phæthon*, fulmine ictus. *Euripid.*

Ω χρυσοφειγγες ‘Ηλι’, ὡς μ’ ἀπώλεσας;
‘Ὅθεν Σ’ Ἀπὸλλων’ εμφανως κλησῇ βροτός.

“O auræ tædâ prædite *Sol*, quomodo me perdidisti!
Undè Te *Perdentem* clarè vocant mortales.”

Atque inde pluralitèr usurpari cæperunt *אלים* et *אלֹבִים* *Ælim* et *Ælobim*, pro *Angelis*, *Divis*, *Judicibus*, *Heroibus*, &c.—Unde ad retinendam præstantiam DEI SUMMI, cum epithetis variis usi sunt *Hebræi Prophetæ*, *אלֹבִים* *Æl*, *אלֹבִים* *Ælim*, *Dan.* xi. 36. “*Deus Divorum*,”—*אלֹבִים* *Æl*, *אלֹבִים*, “*Deus Deorum*,”—*אלֹבִים* *אלֹבִים*, *Labob* *Ælobim*, “*Dominus Deorum*,”—et *אלֹבִים* *אלֹבִים* *Æl* *Ælobim* *Labob*, sicut plenissimè describitur Nomen Dei sacrosanctum, *Josb.* xxii. 22. *Ef.* i. 1. “*Dominus Deus Deorum*.”

5. Atque hinc, ellipsi frequentissimâ *אלֹבִים* *Ælobim*, licet vox pluralis, usurpatur singularitèr: ut *Gen.* i. 1. *ברא אֱלֹבִים* *Dus creavit*; subaudito vel *אלֹבִים* *Labob*, vel *אל* *Æl*, vel utroque. Atque hoc pacto *Scripturis Hebræis* grammaticè amovetur *solæcismus* iste, qui mirè torsit interpretes, atque varia absurda, imò impia, commenta, *Rabbinorum* et *Mythicorum* peperit; ut videre licet in *Lexicis* *Scriptis*que *Kimchi*, *Buxtorf*, *Parkhurst*, &c. &c. &c.

6. Ex his igitur nominibus Dei sacrosanctis, omnium vetustissimis atque maximè radicalibus, *אל* *Æl*, et *אלֹבִים* *Labob*, utpote simplicissimis, (unde deducuntur Dei nomina propria in omnibus *Linguis primitivis*, necnon in dialectis, tam *Orientalibus* quàm *Occidentalibus*;) vel ex his solis non inficiabuntur *Etymologi æqui* et *periti*, omnium matrem, ipsâque primævam linguam, fuisse *Hebræam*. Sed hæc paucis exponi non possunt.

tionis;

tiones; veneramur autem et *colimus* ob *dominium*: colimus enim ut servi, et DEUS sine *dominio*, *providentiâ* et *causis finalibus*, nihil aliud est quàm *Fatum* et *Natura*: à cæcâ *Necessitate Metaphysicâ*, quæ utique eadem est semper et ubique, nulla oritur *rerum variatio*: tota rerum conditarum *diversitas* ab *ideis* et *voluntate* ENTIS NECESSARIÒ EXISTENTIS solummodò oriri potuit.

“Dicitur autem DEUS per allegoriam, *videre, audire, loqui, ridere, amare, odio habere, cupere, dare, accipere, gaudere, irasci, pugnare, fabricare, condere, confirruere*: nam sermo omnis de Deo, à *rebus humanis* per similitudinem aliquam desumitur, non perfectam quidem, sed *aliqualem* tamen.

“Et hæc de DEO; de quo utique ex *phænominis* differere ad PHILOSOPHIAM NATURALEM pertinet.”

197. In hoc Scholio generali summè Theologico, objiciunt Metaphysici *Leibnitiani*, Newtonum denegare existentiam *spatii* et *durationis* absolutam, ubi asserit, p. 528:

“DEUS non est *æternitas* et *infinitas*, sed *æternus* et *infinitus*; non est *duratio* et *spatium*, sed *durat* et *adeſt*: *durat semper* et *adeſt ubique*; et *existendo semper et ubique, durationem et spatium, [æternitatem et infinitatem] constituit.*”

198. Cæterùm responsio facilis: Si non existeret DEUS, quo pacto *spatium* et *duratio* ab ullo *Ente intelligente* cernerentur? Nam, ut ipsi loquuntur Metaphysici, *de non apparentibus et de non existentibus eadem est ratio.*—Newtono adſtipulatur sublimis illa Theologia Patriarcharum et Evangelistarum: docuit enim *Orpheus*, *Mosem* secutus, quod “*Deus est Lux—Lucem habitans inaccessibilem,*” et confirmat *Paulus*, in celeberrimâ illâ DEI SUMMI descriptione, omni laude longè longissimèque majore, quàm potissimùm secutus atque interpretatus est *Newtonus*: 1 Tim. vi. 15.

‘Ο ΜΑΚΑΡΙΟΣ ΚΑΙ ΜΟΝΟΣ ΔΥΝΑΣΤΗΣ,

‘Ο ΒΑΣΙΛΕΥΣ ΤΩΝ ΒΑΣΙΛΕΥΟΝΤΩΝ,

ΚΑΙ ΚΥΡΙΩΣ ΤΩΝ ΚΥΡΙΟΝΤΩΝ·

‘Ο μόνος ἔχων ἀθανασίαν·

Φως οὐκ ὄντων ἀπρόσβλεπτον·

‘Οὐκ εἶδεν ἑδεις ἀνθρώπων, ἑδὲ εἶδεν δύνασθαι·

‘Ω ΤΙΜΗ ΚΑΙ ΚΡΑΤΟΣ ΑἰΩΝΙΟΝ. ΑΜΗΝ.

“FELIX ATQUE SOLUS IMPERATOR,
 REX REGENTIUM ET DOMINUS DOMINANTIUM;
Solus habens immortalitatem,
Lucem habitans inaccessibilem,
Quem vidit nemo mortalium, nec videre potest:
 CUL HONOS ET DOMINIUM ÆTERNUM. AMEN.”

199. Confirmat quodque *Johannes*:

‘Ο ΘΕΟΣ Φως ἐστὶ, καὶ Σκοτία ἐν Αὐτῷ οὐκ ἐστὶν ὁδεμία.

“DEUS *Lux* est, atque in Illo haudquaquàm est *caligo*.”

Πνεῦμα ‘Ο ΘΕΟΣ—‘Ο ΘΕΟΣ Ἀγάπη ἐστὶ.

“*Spiritus*, DEUS”—“DEUS est *Charitas*.”

At si prædicantur *Lux*, *Spiritus*, *Charitas*, de Deo *abstractè*, ut loquuntur Logici, non concretè, nōne pari analogiâ in re prorsùs incognitâ, “constituat quoque *durationem* et *spatium*?”

200. Ingenti *veræ Theologiæ* detrimento fines *Philosophiæ Naturalis* (cujus est “fidelitèr Theologiæ ancillari,” ut optimè *Bacon*), temerè transgrediuntur Metaphysici; ubi malè sani profiteri audent “*Dei existentiam et attributa demonstrari à priori*,”—ut infauistâ diligentîâ, Doctus *Clarke*, speculationibus metaphysicis ex controversiâ *Leibnitzianâ* nascentibus, nimis imbutus, et iisdem periculose indulgens. Deum enim *necessariò existere*, certissimum est: sed talis existentiae idea, utcunque per se positiva, nostro conceptui est planè *negativa*; solummodò indicans “*nullam causam*” ejus dari extrinsecùs: cæterùm *ideæ negativæ scientiam positivam minimè pariunt*, ipso testante *Locke*, *Essay*, &c. IV. 1. 5.

201. *Existencia Dei* intellectum omnium creaturarum ratione præditarum transcendit, imò *Angelorum* et *Archangelorum*, præter “FILIIUM CHARITATIS EJUS,”—JESUM CHRISTUM DOMINUM NOSTRUM:—qui “*Illi proximos occupavit honores*”—*Hor.* ut ex *mythagogo* maximo, *Johanne*, discimus, *Rev.* xix. 16.

ΒΑΣΙΛΕΥΣ ΒΑΣΙΛΕΩΝ
ΚΑΙ ΚΥΡΙΟΣ ΚΥΡΙΩΝ.

“REX REGUM ET DOMINUS DOMINORUM.” (n)

Cui

(n) Q. àm accuratè distinguunt *Mythagogi* omnium maximi, *Paulus* et *Johannes*, inter Θεον τον Πατέρα, “*Deum Patrem*,” atque Υἱον τον Μονογενη, “*Filium unice Genuinum*, seu *Dilectum*,” *John*, i. 16. *Paulus* enim PATREM designat titulo plenissimo maximèque augusto, ὁ Βασιλεὺς των Βασιλευσων, καὶ Κυριος των Κυριων, quem haudquaquàm assequitur exilitas Linguae Latinae, articulo *emphatico*, ὁ, carentis; sed qui *Anglicè* efferrì potest—“THE KING OF THE REIGNING [*Kings*], AND LORD OF THE RULING [*Lords*].”—*Johannes* verò FILIUM, titulo planè proximo: Βασιλεὺς Βασιλεων, καὶ Κυριος Κυριων—“KING OF KINGS, AND LORD OF LORDS,”—uno eodémque SPIRITU DIVINO ambo afflati.

Nec

Cui inditum est Dei gratiâ, Ονομα το ὑπερ πάντων ονομα, "*Nomen suprâ omne nomen*" (vel auctoritatem) tûm in *Cælis*, tûm in *Terris*, tûm in *Infernis*, —εις δοξαν ΘΕΟΥ ΠΑΤΡΟΣ, "*in laudem DEI PATRIS*;" nunc, sub novo fœdere, *Phil. ii. 9*; sicut olim, sub primo fœdere, *Exod. xxiii. 21*.—Nam asserit ipse 'Ο ΛΟΓΟΣ ΤΟΥ ΘΕΟΥ, "*Oraculum Dei*:" *Matt. xi. 27*.

Πάντα ΜΟΙ παραδοθή ὑπο ΤΟΥ ΠΑΤΡΟΣ,
Και ὅδεις ἐπιγινώσκει ΤΟΝ 'ΥΙΟΝ, εἰ μὴ 'Ο ΠΑΤΗΡ.
Οὐδε ΤΟΝ ΠΑΤΕΡΑ τις ἐπιγινώσκει, εἰ μὴ 'Ο 'ΥΙΟΣ,
ΚΑΙ ὃ εἰν βεβλήται 'Ο 'ΥΙΟΣ ἀποκαλύψαι.

"*Omnia MIHI commissa sunt à PATRE;*
Et nemo intimè cognoscit FILIUM, nisi PATER:
Neque PATREM quis intimè cognoscit, nisi FILIUS,
Et cuicunque voluerit FILIUS revelare."

Nec Ethnicos latuit hæc distinctio: Audi *Horatium*.

"*Quid prius dicam solitis PARENTIS*
Laudibus, qui res Hominum ac Deorum,
Qui Mare et Terras, variisque Mundum
Temperat horis?
Unde nil majus generatur IPSO:
Nec viget quicquam simile aut secundum.
Proximos ILLI tamen occupavit
PALLAS honores."

Hæc certè ex *Theologia primæ* reliquiis extraxit *Horatius*.—"PARENS" enim est ὁ Πατήρ Ἀνδρῶν ὁ Θεός, *Orphei*, *Homeri* et *Hesiodi*, licet quid hæc appellatione maximè honorificè verè significaretur, parùm callebant: PALLAS verò, gentium *Dea Bellatrix*, à Παλλῶ, "*vibro*," [*fulmen, telum, &c.*] derivatur; hæc enim vox *epitheton* tantùm designat—Παλλὰς Ἀθηνῆ, ut *Homero* passim. Ejus autem nomen Ἀθηνῆ ab *Egyptiacâ Νῆϋθ*, seu SAPIENTIA, retrò legendo derivatur; ut *Not. (p)* monuimus: quæ eadem primitus fuit cum דָּבָר דָּבָר, *Dabar Iabob* Testamenti Veteris, *Gen. xv. 1*. seu Novi—'Ο ΛΟΓΟΣ ΤΟΥ ΘΕΟΥ, "*Oraculum Dei*," *Rev. xix. 13*.—ΕΚΕΙΝΟΣ ἐξηγήσατο—"ILLE exposuit."—*John, i. 18*.

Notatu dignissimum est quòd, referente *Horatio*, PALLAS, non arripuit, non arrogavit sibi, sed DEI gratiâ, "*Proximos ILLI occupavit honores*,"—occupavit, quasi *jure hæreditario*; sicut optimè exponit *Paulus*, de *JESU CHRISTO*—Ὁς ἐν μορφῇ Θεοῦ ὑπαρχων, ἐν ἀρπαγμῶν ἡγήσατο το εἶναι Ἰσα-θεῶν ἀλλ' ἐαυτὸν ἐκένωσε, &c.—"Qui in formâ divinâ [licet] subsistens, non instar præde duxit fieri Deo-similem; sed se-ipsam exhaussit," &c.—*Phil. ii. 7*.

Fabulantur Mythologi Ethnici, DEAM SAPIENTIAM panopliâ instructam ex Διὸς κεφαλῆς, "*è Jovis capite*" emicuisse, plenè adultam, non infantem; temerè confundentes, ut videtur, vocem Hebræam ראש *Raôsh*, ambigüe significantem tûm caput tûm principium, in sublimi illâ descriptione ontûs *Sapientiæ primæ*, *Prov. viii. 22*.

"DOMINUS genuit ME, principium viæ suæ,
Ante opera sua, primitus:
Ab æternitate ordinatus sum, à capite,
A primordiis Terræ."

Ubi phrasis Hebræa lineæ penultimæ, ראש *Meraôsh*, quæ ad literam redditur, "*à capite*," seu "*ex capite*," æquipollet phrasi lineæ primæ, רֵאשִׁית *Raôsh derebû*—"principium viæ suæ;"—quam optimè interpretatur *Johannes*, *Rev. iii. 14*.—"Ἡ ἀρχὴ τῆς κτίσεως Θεοῦ," "*Primordium creationis Dei*," seu πρωτότοκος πάσης κτίσεως, "*Primogenitus omnis creationis*," *Coloss. i. 15*.—DE *JESU CHRISTO*. Et pari ambiguitate, phrasim *Orphei*, Ζεὺς κεφαλῆ, mutat *Plutarchus* in Ζεὺς ἀρχῆ. *Not. (g)*.

202. Non nisi *stultissimi* igitur autumant se à priori de Deo SUMMO philosophari *Cartesiani, Leibnitziani, Clarkiani et Unitarii*; quorum idcirco—"dubitans, circumspiciens, hesitans, multa adversa revertens, tanquam in rate in mari immenso, omnis vehitur cratio,"—(ut plorat Cicero de *Mythologis antiquis*):—"Relegens errata retrorsum littora"—*Virgil.*—"Infanientis dum Sapientiæ consulti errant;—et Retrorsum vela dare, atque iterare cursus coguntur relietos." *Hor.*

"Quantò satiùs est ire apertâ viâ et rectâ [THEOLOGIÆ REVELATÆ], quàm TIBI IPSI flexus disponere; quos cum magnâ molestiâ debeas relegere?" *Seneca.*

203. Gravissimèque perstringit *Paulus* hallucinationem inexcusabilem veterum philosophantium (o): *Rom. i. 19.*

Διοτι το γνωστον του θεου φανερον εστιν εν αυτοις· 'Ο γαρ θεος εμφανιως αυτοις (τα γαρ αορατα αυτου, απο κτισεως κοσμου, τοις ποιημασι νοημενα καθοραται· 'Ητε αιδι· αυτου ΔΥΝΑΜΙΣ και ΘΕΙΟΤΗΣ) εις το ειναι αυτου αναπολογητου.

"Quocircà, quicquid scibile de Deo, manifestum est inter illos: DEUS enim manifestabat illis (nam ab usque mundo condito, [attributa] ejus invisibilia;—nempè ÆTERNA ejus POTENTIA et DEITAS,—operibus intelligenda, cernere erat), ut illi forent inexcusabiles."

204. Paritérque *Johannes ακολαλησιαν* Mundi docet:

Και το φως εν τη σκοτια φαινει,
Και 'Η Σκοτια αυτου ε κατελαβεν.

"Et Lux in Caligine lucet;
Sed Caligo eam non comprehendebat."

Etsi gestiebant Veteres, docente *Paulo*: *Act. xvii. 27.*

Ζητειν τον κτηριον, ει αρα ψηλαφησειαν
Αυτον, και ευροιεν·

—"Quærere DOMINUM, si fortè palparent Illum,
Atque invenirent."

205. Testante itèd *Socrate* (Græcorum sapientissimo, secundùm *Oraculi Delphici* effatum, quòd, se nihil scire præter suam inscitiam de rebus divinis, professus est, in *Phædone*, undè *Paulus* proculdubio excerptit:

(o) Turpissimè hallucinatur vel ipse *Aristoteles*, referente *Cicerone*, *Nat. Deorum*, II. 16. qui, ex ordine et constantiâ motuum cœlestium, astris ipsis sensum atque intelligentiam stultissimè tribuit!

—"Ordo autem astrorum, et in omni æternitate constantia, neque naturam significat, ut enim plena rationis; neque fortunam, quæ, amica varietati, constantiam respuit: sequitur ergò, ut ipsa sua sponte, suo sensu, ac divinitate moveantur."—

"Cœlestium ergò admirabilem ordinem, incredibilemque constantiam, ex quâ conservatio et salus omnium omnis oritur, qui vagare mente putat, is ipse mentis expertus habendus est." *Nat. D.or. II. 21.*

Το γὰρ μὴ διελεσθαι ὅσον τ' εἶναι, ὅτι ἄλλο μὲν τι ἐστὶ τὸ αἰῖον τῷ οὐσί, ἄλλο δ' ἐκείνο ἀνεῦ ἐ το αἰῖον καὶ ἂν ποτ' εἴη αἰῖον· ὃ δὲ μοι φαίνονται ψηλαφῶντες οἱ πολλοί, ὥσπερ ἐν σκοτεινῷ ἀλλοτρίῳ ὁμμασί προσχρωμένοι, ὡς αἰῖον αὐτοῦ προσαγορεύειν.

“Nam distinguendi non capax esse, quod unum quidem est *ipsa rei alicujus existentis causa*, aliud verò, *illa, sine quâ non foret unquam causa*: quoad hoc sanè, mihi videntur plerique, *palpantes quasi in caligine*; alienis oculis usi, siquidem *causam ipsam*, [immediatam, ultimam falsò] appellant.” (p).

206. Optiméque et luculentissimè quidèṃ exponit *Paulus*, argumentum illud acutissimum omniumque invictissimum quo *Newtonus*, ex “*rerum variatione*,” et *tempestatum diversitate*, NUMEN SUPREMUM colligit: *Æt.* xiv. 17.

(p) Præclarum quasi *Socratis* testimonium de hallucinatione veterum philosophorum DEI SUMMI naturam investigare satagentium, affert *Lucianus*, in *Halepne*, Vol. I. p. 179.

Ὡ φίλε Χαίρεφῶν, εἰκάμεν ἡμεῖς τῶν δυναίων τε καὶ ἀδυναίων ΑΜΒΑΥΩΠΙΟΙ τινες εἶναι κριταὶ πάντεσσι. Δοκιμαζόμεν γὰρ δὴ, κατὰ δυνάμιν ἀνθρωπίνην, [ΤΗΝ ΔΥΝΑΜΙΝ] Αἰνῶστον· ἔσαν καὶ Αἰπιστόν καὶ Δοπατόν· Πολλὰ ἐν φαίνεται ἡμῖν καὶ τῶν εὐπορίων ἀπορα, καὶ τῶν ἐφικτῶν ἀνεφικτὰ· συχνὰ μὲν καὶ δι' ἀπειρίαν, συχνὰ δὲ καὶ διὰ νηπιότητά φρενῶν· τῷ οὐσί γὰρ νηπιὸς εἰσμεν εἶναι πᾶς ἀνθρώπος· καὶ ὁ πᾶν γέρον· ἐπεὶ μὲν μικρὸς πᾶν καὶ νεογνὸς ὁ τῆ βίᾳ χρόνος, πρὸς τὸν πᾶν αἰῶνα. Τί δ' αὖν, ὦ γὰθε, οἱ ἀγνοήντες τὰς τῶν θεῶν καὶ δαίμονων δυνάμεις, ἐχρῶμεν ἀν εἰπεῖν πλείον δυνάμιν ἢ ἀδυναμίαν τι τῶν τοιαύτων.

“O chare Chærophon, Nos quidem tam *possibilium* quàm *impossibilium* videmur esse *Judices* HAL-LUCINANTES prorsus. Æstimamus enim, *secundum potentiam humanam*, [POTENTIAM] NON SCIBILEM, INCREDIBILEM, INVISIBLEM: Multa igitur ex facilibus difficilia, et ex attingendis non attingenda, nobis apparent. Crebrò quidèṃ *propter imperitiam*, crebrò autèṃ *propter infantiam mentis*: Reverà enim *Infans* videtur esse *omnis homo*, etiāsi valdè senex; quoniam exiguum prorsus et pusillum tempus est vitæ, cum omni ævo comparatum. Quà ratione autem, O bone! possint, qui nesciunt DEORUM DÆMONUMQUE POTENTIAS, dicere *utrum talium aliquid sit possibile vel impossibile?*”

Atque hinc optimè exponenda Inscriptio Altaris *Athenis*, *Æt.* xvii. 23.

Αἰνῶστον θεῷ.

“NON SCIBILI DEO.”

Inscriptio forsàn derivanda ab Inscriptiōne Templi *Nēith* seu *Sapientiæ* apud *Sais* in *Ægypto* Superiori, unde primitus Colonia *Athenis* deducta fuerat. Et forsàn vox *Athenæ* (Ἀθῆναι), ab *Ægyptiā* *Nēith* vel *Nēithas*, si ordine retrogrado legatur, optimè deduci potest; sicut coniecit doctissimus *Cudworth*, *Intellect. Syst.* p. 309, ex *Platone* in *Timæo*, referens:

“Ὅτι τῆς πόλεως [Σαῖς] Θεὸς ἀρχηγός ἐστιν Αἰγυπτίσι μὲν τένομα *Nēith*, Ἑλλήνισι δὲ, ὡς ὁ ἐκεῖνον λόγος, Ἀθῆναι.

“Urbis [*Sais*] *Deus presul*, est *Nēith*, nomine *Ægyptiaco*; *Athēna* autem, *Græco*; sicut eorum sermo.”

Inscriptio *Saitica*, a *Plutarcho* servatur:

Εγὼ εἰμι ΠΑΝ, τὸ ΓΕΓΟΝΟΣ, καὶ ΟΝ, καὶ ΕΞΟΜΕΝΟΝ.

Καὶ τὸν ἐμὸν πεπλὸν εἴδεις πῶς θνήσκῃς ἀπεκαλύψεν.

“Ego sum OMNE, quod FUIT, quod EST, et quod ERIT:

Atque meum pepulum nullus adhuc Mortalis revelavit.”

Atqui nemo quidem Mortalium revelare potuit, donec Ipse demissus est e sinu PATRIS, ὁ μονογενής, “*unicè dilectus FILIUS*, qui naturam cultumque Dei cæcutientibus mortalibus exponeret.” *John* i. 18.

Καίτοιγε ἔκ ἀμαρτυροῦ Ἐαυτοῦ ἀφῆκεν (Ὁ ΘΕΟΣ) ἀγαθοποιῶν, ἐρανοθεν ἡμῶν
ὕψος διδῶς καὶ καιρὸς καρποφύρας, ἐμπιπλῶν τρυφῆς καὶ εὐφροσύνης τὰς καρδίας ἡμῶν.

“Veruntamen, se ipsum *haud intestatum* relinquebat (DEUS) utique *benefaciendo*, cœliuð *imbres* nobis ac *tempestates fructuosas præbendo*, corda nostra *cibo* atque *letitiâ* implendo.”

Haud rarò enîm omnia fiunt contraria: puniuntur immemororum atque ingratorum corda, ultione divinâ, regionum siccitate, tempestatibus infructuosis, fame et pavore palpitantium:

“*Terra enim ferax in salsuginem convertitur,
Propter iniquitatem incolentium ejus.*”

Atqui hoc argumentum ex *causis finalibus* petitum, omnium conceptibus accommodatissimùm, vix, et ne vix quidem, attigerunt Sophistæ veteres; dum *Legibus Naturæ generalibus*, secundùm quas ordinantur anni cardines tempestatumque vicissitudines revolutionibus periodisq̃ue certis et æquabilibus, *mechanicè* quasi agentibus, tantùm incumbentes, CAUSÆ PRIMARIÆ obliti, in pravissimùm *Atheismum* delapsi sunt.—Vide *Aristotelem*, astris (propter ordinem et constantiam motuum quibus in cœlis circumferri observantur,) sensum et intelligentiam asserentem.—*Nat. Deor.* II. 16.

“*Extat igitur*, ut præclarè loquitur *Cotefius*, “*eximium NEWTONI OPUS adversus ATHEORUM impetus munitissimùm præsidium*: neque enim alicunde feliciùs, [ORACULIS DIVINIS exceptis], quàm ex hâc pharetrâ, contrâ impiam catervam *vela depromiseris.*”—*Præfat.* ad secundam editionem Principiorum Newtoni, anno 1713 à Cotefio factam.

207. Coronidis loco subungere libet sententiam auream cedròque dignam, quam protulit magnus ille BACON, *Philosophiæ Experimentalis Pater*, et NEWTONI *præceptor*:

“A *superficial taste* of PHILOSOPHY may perchance incline the mind to ATHEISM; but a *full draught* thereof brings it back again to RELIGION.

“In the entrance of Philosophy, when the *second causes*, most obvious to the senses, offer to the mind, we are apt to cleave unto them, and dwell too much upon them; so as to *forget* what is superior in Nature: but when we pass further, and behold the *dependency* and *confederacy of causes*; and the *works* of PROVIDENCE; then, according to the Poets, we easily perceive, that *the highest link of NATURE’s chain must be tied to the foot of JUPITER’s chair—that PHILOSOPHY, like JACOB’s vision, discovers to us a LADDER, whose top reaches up to the footstool of the throne of God.*”

HINC OMNE PRINCIPIUM, HUC REFER EXITUM.

A LET-

A LETTER

FROM

JAMES GLENIE, Esq.

TO

FRANCIS MASERES, Esq.

CONTAINING

A DEMONSTRATION OF SIR ISAAC NEWTON'S
BINOMIAL THEOREM.

LETTER

1784

TO THE

FRANCIS MARRIAGE

CONTAINING

DEMONSTRATION OF THE NEWTON'S

UNIVERSAL THEORY

A LETTER

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A DEMONSTRATION OF SIR ISAAC NEWTON'S
BINOMIAL THEOREM.

DEAR SIR,

*River St. John, New Brunswick,
April 20th, 1799.*

YOUR Letter of the 27th of November last I have just had the satisfaction to receive and peruse.—And when the copies, you propose to send to me, of the 3d Volume of the *Scriptores Logarithmici* arrive here, I will take care to distribute them agreeably to your directions.

Before I proceed to give you a general demonstration of the Binomial Theorem, I will briefly shew you how easily it may be derived, in the case of integral powers, from a theorem which I demonstrated by means of Euclid's Elements.

In a Paper of mine, read before the Royal Society of London the 6th March 1777, and published in the Philosophical Transactions, you will find it demonstrated geometrically, that, A and B being any two homogeneous magni-

tudes, the ratio of $A + \frac{m-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot A \cdot \frac{A-B}{B}^2$

+ &c. to B is a multiple of the ratio of A to B by the number m (m being any whole, or integral, number whatsoever). Let us now suppose A and B to be numerical magnitudes, and B the common consequent, or standard of comparison to be represented by arithmetical unity or 1, and the difference between A and B by x . Then, if A be greater than B, it will be represented by $1+x$ and $A-B$ by x ; and, by substituting $1+x$, x , and 1, respectively,

tively, for A, $A - B$, and B, we, in this case, get $\overline{1+x} \times (1 + \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 + \&c.) : 1$ for the ratio, which is a multiple of the ratio $1+x : 1$ by the number (m), and consequently $\overline{1+x} \times (1 + \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 + \&c.) = \overline{1+x}^m$, which has to 1 the m th ratio of $1+x$ to 1. But $\overline{1+x} \times (1 + \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 + \&c.)$ is manifestly equal to $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c.$; which is therefore equal to $\overline{1+x}^m$.

But if A be less than B, it will be represented by $1 - x$, and $A - B$ by $-x$; and, by substituting $1 - x$, $-x$, and 1, respectively, for A, $A - B$, and B, we, in this case, get $\overline{1-x} \times (1 - \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 - + \&c.) : 1$ for the ratio, which is a multiple of the ratio $1-x : 1$ by the number (m), and consequently $\overline{1-x} \times (1 - \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 - + \&c.) = \overline{1-x}^m$, which has to 1 the m th ratio of $1-x$ to 1. But $\overline{1-x} \times (1 - \frac{m-1}{1} \cdot x + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot x^2 - + \&c.)$ is evidently equal to $1 - \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 - + \&c.$; which is therefore equal to $\overline{1-x}^m$.

This derivation of these two theorems, in the case of integral powers, appears to be beautifully simple.

You have proved the truth of them in a very satisfactory manner: and that they are true is likewise easily shewn by Algebraick Multiplication, which, in the demonstration of numerical theorems, (which the binomial and residual theorems unquestionably are, there being no such geometrical magnitudes as x^4 , x^5 , x^6 , &c.), is a perfectly legitimate and admissible operation.

It may likewise be proved by Multiplication, as well as by some of the methods you have made use of for proving other truths in the 2d Volume of your *Scriptures Logarithmici*, that $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$ is the n th power of $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$; or that the ratio of

1 +

$1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$ to 1 is a multiple of the ratio of
 $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$ to 1 by the number (n). This is also
 easily proved in the following manner. Let $1 + y = 1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$
 Then $1 + y^n$, or $1 + \frac{n}{1} \cdot y + \frac{n}{1} \cdot \frac{n-1}{2} \cdot y^2 + \frac{n}{1} \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot y^3 + \&c.$ is equal to the n th power of $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$
 Now, if the term, in which x is found in the value of y , be multiplied by $\frac{n}{1}$, the co-efficient of y , we get $\frac{np}{q} \cdot x$ for the second term
 of the expression (in terms of n, p, q , and x), which is equal to $1 + y^n$. And
 if the term, in which x^2 is found in the value of y , be multiplied by $\frac{n}{1}$, the
 co-efficient of y , and the term, in which x^2 is found in the value of y^2 , be
 multiplied by $\frac{n}{1} \cdot \frac{n-1}{2}$ the co-efficient of y^2 , the sum of these products
 will give us $\frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2$ for the third term. And by thus proceeding,
 we get $1 + y^n = 1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$

Precisely in the same way it is proved that the expression $1 - \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 - + \&c.$ is the n th power of $1 - \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 - + \&c.$

This truth, then, which is demonstrable in a variety of ways, being pre-
 mised, namely, that the ratio of $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$ to 1
 is a multiple of the ratio of $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$ to 1 by
 the number n (n being any whole number, and the ratio of p to q equal to any
 ratio whatsoever.) I proceed now to prove generally, that if $1 + x^{\frac{p}{q}}$ be ex-
 panded in the customary way, the expression $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$ thence arising has to 1 a ratio, which is to the ratio of $1 + x : 1$

or $\overline{1+x}^1 : 1$ as $p : q$ or $\frac{p}{q} : 1$; and, of course, that the expansion of $\overline{1+x}^{\frac{p}{q}}$ is universally true, whatever be the ratio of p to q .

Let np , then, be a multiple of p by any whole or integral number n , and mq a multiple of q by any whole or integral number m ; let a multiple by n be taken of the ratio $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c. : 1$ (which multiple, as is shewn above, is the ratio $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c. : 1$); and let a multiple by m be also taken of the ratio $1 + x : 1$ or $\overline{1+x}^1 : 1$ (which multiple is the ratio $\overline{1+x}^m : 1$ or $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c. : 1$, the binomial theorem, expanded in the usual way, being above shewn to be true in the case of integral powers). For the sake of perspicuity, let the magnitudes and their multiples stand in the following manner:

<i>The Magnitudes.</i>						<i>Their Multiples.</i>					
1st,	-	-	-	-	p	}	Of the 1st,	-	-	-	np
2d,	-	-	-	-	q		Of the 2d,	-	-	-	mq
3d,	$1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c. : 1$						Of the 3d,	$1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c. : 1$			
4th,	-	-	-	-	$1 + x : 1$		Of the 4th,	$1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c. : 1$			

Now it is evident that, if np be equal to mq , by substituting mq for np , the ratio $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c. : 1$ is equal to the ratio $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c. : 1$.

It is equally evident that, if np be greater than mq , the antecedent $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$ is greater than $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c.$ since

$$\frac{np}{q},$$

are then respectively greater than $\frac{np}{q}, \frac{np}{q} \cdot \frac{np-q}{2q}, \&c.$ }
 or than $\frac{mq}{q}, \frac{mq}{q} \cdot \frac{mq-q}{2q}, \&c.$ }
 or than $\frac{m}{1}, \frac{m}{1} \cdot \frac{m-1}{2}, \&c.$ }

and the ratio $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c. : 1$ is, of course, a greater ratio of greater inequality than the ratio $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c. : 1$ is.

And it is no less evident that, if np be less than mq , the antecedent $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c.$ is less than the antecedent $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c.$ since $\frac{np}{q}, \frac{np}{q} \cdot \frac{np-q}{2q}, \&c.$ are then respectively less than $\frac{mq}{q}, \frac{mq}{q} \cdot \frac{mq-q}{2q}, \&c.$ or than $\frac{m}{1}, \frac{m}{1} \cdot \frac{m-1}{2}, \&c.$; and the ratio $1 + \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 + \&c. : 1$ is then, of course, a less ratio of greater inequality than the ratio $1 + \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 + \&c. : 1$ is.

Here, then, are four magnitudes; and any equimultiples whatsoever being taken of the 1st and 3d, and also any equimultiples whatsoever being taken of the 2d and 4th, it is proved that, if the multiple of the 1st be equal to the multiple of the 2d, the multiple of the 3d is also equal to the multiple of the 4th; if greater, greater; and if less, less. Wherefore (5 Def. E. 5.) the magnitudes themselves are proportional, or the ratio $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c. : 1$ is to the ratio $1 + x : 1$ as $p : q$ or $\frac{p}{q} : 1$. But the ratio of

$\frac{p}{1+x \cdot \frac{p}{q}}$ to 1 is to the ratio of $1 + x$ to 1 or $\frac{p}{1+x \cdot \frac{p}{q}}$ to 1 as $\frac{p}{q} : 1$. Con-

sequently $\frac{p}{1+x \cdot \frac{p}{q}}$ is $= 1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$ universally, whatever be the ratio of p to q .

E c 2

Thus

Thus is demonstrated, by means of genuine algebræick principles and the doctrine of proportion, the truth of the Binomial $\sqrt[p]{1+x}$ universally, when expanded in the usual way, or when p has to q not only the ratio of one integral number to another, but the ratio of any two homogeneous magnitudes whatsoever.

In the same way is demonstrated the truth of the Residual $\sqrt[p]{1-x}$ universally, when expanded in the usual manner, care only being taken not to confound the magnitudes of the two ratios of less inequality, $1 - \frac{np}{q} \cdot x + \frac{np}{q} \cdot \frac{np-q}{2q} \cdot x^2 - + \&c. : 1$, and $1 - \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 - + \&c. : 1$, with the definition of the less ratio delivered by Euclid, when the antecedent of the first of these two ratios is less or greater than the antecedent of the second.

Thus, for instance, $1-x$ being less than 1, its square $1-2x+x^2$, or $\sqrt{1-x^2}$, must be less than $1-x$; and by (8. E. 5.) $1-x$ has a greater ratio to 1 than $\sqrt{1-x^2}$ has to 1 (which it really has in respect of quantity or the degree of magnitude), though the ratio $\sqrt{1-x^2} : 1$ is the duplicate ratio of $1-x : 1$, or is just twice as great a ratio of less inequality.

Now, since $\sqrt[p]{1+x}$ is universally equal to $1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.$ it is evident, that its reciprocal $\sqrt[p]{1+x}^{-1}$, or $\frac{1}{\sqrt[p]{1+x}}$, is equal to $\frac{1}{1 + \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 + \&c.}$, which is $= 1 - \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p+q}{2q} \cdot x^2 - \frac{p}{q} \cdot \frac{p+q}{2q} \cdot \frac{p+2q}{3q} \cdot x^3 + - \&c.$ by common algebræick division, and corresponds with the expression which $\sqrt[p]{1+x}^{-1}$ gives, when expanded in the usual way. And as $\sqrt[p]{1+x}^{-1}$, or $\frac{1}{\sqrt[p]{1+x}}$, is the algebræick or arithme-

tical

tical magnitude that arises from decomposing the ratio $\sqrt[p]{1+x} : 1$ with the ratio of equality $1 : 1$, it is a third proportional to $\sqrt[p]{1+x}$ and 1 , or is equal to $\frac{1 \times 1}{\sqrt[p]{1+x}}$ or $\frac{1}{\sqrt[p]{1+x}}$. Division is therefore the most natural way of deriving the value of $\sqrt[p]{1+x}$, when expanded, from the value of $\sqrt[p]{1+x}$ expanded.

In like manner, since $\sqrt[p]{1-x}$ is $= 1 - \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 -$
 $+ \&c$, it follows that $\sqrt[p]{1-x}$, or $\frac{1}{\sqrt[p]{1-x}}$, will be equal to
 $\frac{1}{1 - \frac{p}{q} \cdot x + \frac{p}{q} \cdot \frac{p-q}{2q} \cdot x^2 - + \&c}$, which, by division, is equal to $1 + \frac{p}{q} \cdot x$
 $+ \frac{p}{q} \cdot \frac{p+q}{2q} \cdot x^2 + \frac{p}{q} \cdot \frac{p+q}{2q} \cdot \frac{p+2q}{3q} \cdot x^3 + \&c$.

These theorems, I believe, have never been demonstrated in a general way, or when p has to q any ratio whatsoever, by any other person, by means of genuine and common algebræ operations and the doctrine of proportion.—At least, I can truly aver, that I have never either seen or heard of such demonstrations.

Mr. Landen's Demonstration of the Binomial Theorem, published at page 170 in the 2d Volume of your *Scriptores Logarithmici*, is a very limited and confined one, extending only to the cases in which p and q are any two integers. But, in the demonstration I have now given of it, p may have to q the ratio of any two homogeneous magnitudes whatsoever; as, for instance, the ratio of any two lines, of any two surfaces, of any two solids, of any two numbers, whole, fractional, or surd, or of the magnitudes of any two ratios. And the demonstration itself rests on the same basis with the 5th Book of Euclid's Elements, and other mathematical truths, grounded on his definition of proportionality.

It is natural to suppose that Sir Isaac Newton must have oftener than once attempted to find a general algebræ investigation of a theorem, of which he made such frequent use for the advancement both of science and of his own reputation; and had he succeeded, it is more than probable that such an investigation

of it would have appeared in some part or other of his works. Had that very ingenious man only thought of taking any multiples of the two ratios $1 + \frac{p}{q} \cdot x + \&c. : 1$ and $1 + x : 1$ and correspondent, or the same multiples of p and q , he could not have met with any difficulty in investigating the theorem generally by algebraïck operations alone. He must have been well aware, that he could not, with any sort of propriety, have recourse to his Method of Fluxions to investigate generally this arithmetical theorem, as he

had no where determined the fluxion of $\sqrt[p]{1+x}$ when p has to q any ratio whatsoever. In the introduction to his *Quadrature of Curves*, he takes the truth of this theorem for granted, and expanding, by means of it, the quantity $(x+0)^n$, or the n th power of the binomial quantity $x+0$, determines the ratio of the fluxions of x and x^n . In like manner, the ingenious Mr. Robins, in the 2d Volume of his *Mathematical Tracts*, takes the truth of this theorem for granted; and, expanding $\frac{x+a}{a^{n-1}}$ by means of it, proves, after the Method of Exhaustions, that the ratio of the fluxion of x^n to the fluxion of x is neither greater nor less than the ratio of $\frac{nx^{n-1}}{a^{n-1}}$ to 1, or of nx^{n-1} to a^{n-1} , or $\frac{nx^{n-1}}{a^{n-2}}$ to a . True it is, indeed, that Sir Isaac Newton, in the 2d Lemma

of the 2d Book of his *Principia*, has determined the fluxion of $\sqrt[p]{x}$, when p and q are integers, without making use of the Binomial Theorem. And his learned editor Dr. Horsley, (now Bishop of Rochester,) treading almost exactly in his steps, has determined the same in his *Geometria Fluxionum*, which is an useful and very good exposition or illustration of this Lemma. Beyond this

point, however, they do not go: they do not determine the fluxion of $\sqrt[p]{x}$ generally, or when p has to q any ratio whatsoever. They do not, for instance,

determine the fluxion of $\sqrt[1]{x}$, $\sqrt[1/2]{x}$, $\sqrt[1/3]{x}$, $\sqrt[2/3]{x}$, and so on. And Sir Isaac New-

ton, having determined the fluxion of $\sqrt[p]{1 \pm x}$ only in the case in which p and q are whole numbers, must have been fully sensible, that he could not make use of his Method of Fluxions to demonstrate the truth of the Binomial Theorem beyond this case; which is probably the reason why no demonstration whatever of it appears in any part of his works.

I can

I can easily investigate it generally by means also of my Antecedental Calculus, which does not take motion or velocity at all into consideration (being derived from the General Doctrine of Proportion and some of the Geometrical

Formulae in my Universal Comparifon), since the antecedental of $\sqrt[p]{1+x}$ is

$\frac{p}{q} \cdot \sqrt[p-q]{1+x} \cdot x^a$, let the ratio of p to q be what it will. But I have thought it most natural to deliver a general investigation of this famous arithmetical theorem by means of algebraick operations alone, without referring to any calculus. I have, indeed, long had by me geometrical demonstrations of all the general formulae in my Universal Comparifon, from several of which, when supposed to become numerical, the foregoing demonstration of the Binomial and Residual Theorems, in all cases whatsoever, is easily and immediately derivable.

Having, in the beginning of this letter, derived, in a very simple manner, the Binomial and Residual Theorems, in the case of integral powers, from a formula in a geometrical Paper of mine published in the Philosophical Transactions, before I conclude it I will briefly shew you how the same Theorems may very concisely be derived from that Paper, in different ways.

1st, If, after the same manner in which the ratios of $C : D$, $E : F$, &c. are compounded with the ratio of $A : B$, (see the geometrical figure to that Paper), the ratios $A : B$, $C : D$, $E : F$, &c. to any number p , be compounded with the ratio of equality $B : B$, we get the ratio $B + \frac{p}{1} \cdot B \cdot \frac{A-B}{B} + \frac{p}{1} \cdot \frac{p-1}{2} \cdot B \cdot \frac{A-B}{B}^2 + \&c. : B$ (when $A : B$, $C : D$, &c. are equal,) having to the ratio $A : B$ the ratio $p : 1$ (p being any integer), which, when $B = 1$, and $A - B = \pm x$, according as A is greater or less than B , becomes the ratio $1 \pm \frac{p}{1} \cdot x + \frac{p}{1} \cdot \frac{p-1}{2} \cdot x^2 \pm \&c. : 1$, or such a multiple of the ratio $1 \pm x : 1$, as is expressed by the number p .

2^{dly}, If the ratio $A : B$ be decomposed with the ratio $A + \frac{p-1}{1} \cdot A \cdot \frac{A-B}{B} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot A \cdot \frac{A-B}{B}^2 + \&c. : B$, we get the ratio $B + \frac{p-1}{1} \cdot B \cdot \frac{A-B}{B} + \frac{p-1}{1} \cdot \frac{p-2}{2} \cdot B \cdot \frac{A-B}{B}^2 + \&c. : B$ for such a multiple ratio of $A : B$ as is expressed by the number $p - 1$. Now, if $m = p - 1$, $B = 1$, and $A - B = \pm x$, according as A is greater or less than B , this ratio becomes

comes $1 \pm \frac{m}{1} \cdot x + \frac{m}{1} \cdot \frac{m-1}{2} \cdot x^2 \pm + \&c. : 1$, having to the ratio $1 \pm x : 1$ the ratio of m to 1. And so on.

I hope I have expressed myself in terms perfectly intelligible, and I am inclined to believe that the demonstration I have here furnished you with of the truth of the Binomial and Residual Theorems, in all cases whatsoever, is as concise and, at the same time, as simple as the demonstrations of such general theorems can well be rendered. And it is certainly of no small importance to science to have such an investigation of them, as proves their truth universally, or when they are extended even indefinitely beyond the limits usually prescribed to their application.

I must now subscribe myself,

Dear Sir,

With respect and esteem,

FRANCIS MASERES, *Esq.*
Cursitor Baron of the Exchequer.

Your well-wisher and humble servant,
JAMES GLENIE.

A DISCOURSE
ON
COMPOUND INTEREST,

BY

DR. EDMUND HALLEY, F.R.S.

ASTRONOMER ROYAL, AND SAVILIAN PROFESSOR OF GEOMETRY
IN THE UNIVERSITY OF OXFORD.

First published in the Year 1705 in the Introduction to Mr. Henry Sherwin's
Mathematical Tables, and afterwards in the following Editions of
those Tables. The present Edition is re-printed from
the third Edition of those Tables, published
by *Mr. William Gardiner* in
the Year 1741.

A DISCOURSE

COMPOUND INTEREST.

By EDWARD HALL, F.R.S.

LECTURER IN ROYAL AND SAVILE PROFESSOR OF GEOMETRY
IN THE UNIVERSITY OF OXFORD.

Published in the year 1753 in the Institution to Mr. Henry Bland
Mathematical Tables, and answers to the following Questions
and Tables. The present Edition is re-printed from
the third Edition of those Tables, published
by Mr. William Bland in
the Year 1753.

A DISCOURSE

ON

COMPOUND INTEREST.

By Dr. EDMUND HALLEY, F.R.S.

A PRINCIPAL Use of Logarithms, is to solve all the Cases of Compound Interest; which are not without great difficulty attainable by the Rules of common Arithmetick. But, before we proceed to the practical part, it may, perhaps, not be improper to say something of the foundation, or demonstration, of the Rules we are to give.

Therefore let p be any sum of money forborn t times; r the rate of Interest, or produce of one Pound and it's Interest in one time; that is, as 1 to r , so 1 Pound to it's amount after one year, or other space of time; and let m be the amount of the sum p forborn t times. Now, because in one year, or time, unity becomes r , by the same reason r will in another time become rr , and rr will in a third time become r^3 , &c. it appears that r^t , or r raised to the power whose Index is the number of times, will be the amount of one Pound forborn t times, and therefore:

I. The amount $m = pr^t$. Therefore, to find m , multiply the Logarithm of r by t , and to the product add the Logarithm of p ; the sum shall be the Logarithm of m .

Example. What is the amount of 15*l.* 17*s.* 6*d.* forborn 12 years at 6 *per Cent. per Annum* compound Interest?

The number 1.06 = r it's Log. 0.025,3059

Which Log. multiplied by ($t =$) 12, the number of years, produces 0.303,6708

The principal sum 15*l.* 17*s.* 6*d.* = 15.875*l.* = p , it's Log. 1.200,7137

The amount 31*l.* 18*s.* 10½*d.* = 31.94362*l.* = m , it's Log. 1.504,3845

II. $r^t = \frac{m}{p}$. Therefore, if from the Logarithm of m the Logarithm of p be subtracted, and the remainder be divided by t , the quotient is the Logarithm of r .

Example. What is the rate of compound Interest, when the sum of 15*l.* 17*s.* 6*d.* forborn 12 years amounts to 31*l.* 18*s.* 10½*d.*

The amount 31*l.* 18*s.* 10½*d.* = 31.94362*l.* = m ; it's Log. 1.504,3845

The principal sum 15*l.* 17*s.* 6*d.* = 15.875*l.* = p , it's Log. 1.200,7137

The remainder is Log. of r^t 0.303,6708

Which divided by $t = 12$ quotes Log. of (1.06 =) r 0.025,3059

Therefore the rate is 6 *per Cent. per Annum.*

III. Because $r^t = \frac{m}{p}$; divide the difference of the Logarithms of m and p by the Logarithm of r : the quotient is t , or the time wherein the sum p will amount to m at the rate r .

Example. In what time will the principal sum of 15*l.* 17*s.* 6*d.* amount to 31*l.* 18*s.* 10½*d.* at 6 *per Cent. per Annum* compound Interest?

The amount 31*l.* 18*s.* 10½*d.* = 31.94362*l.* = m , it's Log. 1.504,3845

The principal sum 15*l.* 17*s.* 6*d.* = 15.875*l.* = p , it's Log. 1.200,7137

The remainder is Log. of r^t 0.303,6708

Which, divided by 0.025,3059 = Log. of r , quotes 12 years for the time.

IV. $p = \frac{m}{r^t}$. Therefore, to find p , multiply the Logarithm of r by t , and subtract the product from the Log. of m ; the remainder shall be the Log. of p the principal sum.

Example. What is the principal sum that in 12 years, at 6 *per Cent. per Annum* compound Interest, will amount to 31*l.* 18*s.* 10½*d.*?

The amount 31*l.* 18*s.* 10½*d.* = 31.94362*l.* = m , it's Log. 1.504,3845

The number 1.06 = r it's Log. 0.025,3059

Which Log. multiplied by ($t =$) 12, the years, produces 0.303,6708

And subtracted from the Log. of the amount

The remainder is Log. of (15.875*l.* = 15*l.* 17*s.* 6*d.* =) p . = 1.200,7137

The four preceeding Rules are also readily deduced from the consideration of the rebate of money in this manner :

For

For if, in any time, r becomes 1, in the same time 1 becomes $\frac{1}{r}$, and in the second time $\frac{1}{r}$ becomes $\frac{1}{rr}$, and in the third time $\frac{1}{rr}$ becomes $\frac{1}{rrr}$, &c.; so that, putting p for the value, or present worth, of any sum m payable after t times, at the rate of r to 1, we shall have

I. The sum $m = pr^t$. Therefore, to find m , multiply the Log. of r by t , and to the product add the Log. of p ; the sum shall be Log. of m sought.

II. $r^t = \frac{m}{p}$. Therefore, to find r , from the Log. of m subtract the Log. p , and divide the remainder by t ; the quotient will be the Log. of r .

III. Since $r^t = \frac{m}{p}$; divide the difference of the Logs. of m and p by the Log. of r ; and the quotient will be t , the number of years.

IV. $p = \frac{m}{r^t}$. Therefore, to find p , multiply the Log. of r by t , and subtract the product from the Log. of m , the remainder will be the Log. of p ; which finds the value of any sum of money payable after any time assigned.

The Logarithms are also serviceable to resolve all questions concerning the amount, or present worth, of Annuities, not paid as due, or purchased to be paid for time to come.

Let a be an Annuity or yearly pension, whose successive amounts for times past are ar^t , and whose present values are $\frac{a}{r^t}$ successively, by what goes before: And the series, &c. $ar^5, ar^4, ar^3, ar^2, ar, a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4}, \frac{a}{r^5}$, &c. will be a rank of geometrical proportionals continued infinitely in the ratio of r to 1. Now the sum of all the consequents, or of the whole infinite series, will be to the said sum increased by the next greater term (or the sum of all the antecedents) as 1 to r , by *Eucl.* 5. 12. Wherefore, putting y for the said sum of the consequents, ry will be equal to $y + ar^t$, the sum of the antecedents; and $ry - y = ar^t$; and therefore $\frac{ar^t}{r-1}$ will be equal to y , the sum of all our continued proportionals, whereof ar^{t-1} is the greatest. And by the same Rule $\frac{a}{r-1}$ will be the sum of all the terms whereof $\frac{a}{r}$ is the greatest. So that, if

we subtract $\frac{a}{r-1}$ from $\frac{ar^t}{r-1}$, the difference will be the sum of all the terms, whereof ar^{t-1} is the greatest, and a the least, their number being t ; which sum we will call z . Therefore z (the amount of the annuity of a forborn t times at the rate r) = $\frac{ar^t - a}{r-1}$.

I. The annuity (a), rate of interest (r), and time (t), being given, to find (z) the amount.

From the Log. of a subtract the Log. of $r-1$, and to the remainder add the Log. of r^t ; from the number answering to this last sum, subtract the number answering to the remainder; the difference shall be z the amount sought.

Example. What will an annuity of 34.4*l.* forborn $12\frac{1}{2}$ years amount to at 6 per Cent. per Annum?

$a = 34.4$	$L. a = 1.536,5584$	$L. r = 0.025,3059$
$r - 1 = 0.06$	$L. \overline{r-1} = 8.778,1513$	$t = \dots\dots\dots 12.5$
$\frac{a}{r-1} = 573.3333$	$L. \frac{a}{r-1} = 2.758,4071$	$\begin{array}{r} 1265295 \\ 506118 \\ \hline 253059 \end{array}$
	$L. r^t = 0.316,3237$	$\begin{array}{r} 253059 \\ \hline \end{array}$
$\frac{ar^t}{r-1} = 1187.7660$	$L. \frac{ar^t}{r-1} = 3.074,7308$	$L. r^t = 0.316,32375$
$z = 614.4327\textit{l.} = \frac{ar^t}{r-1} - \frac{a}{r-1} = \text{the amount.}$		

II. The annuity (a), it's amount (z), and the rate of interest (r), being given, to find (t) the time.

By the foregoing observations $\frac{ar^t - a}{r-1}$ is = z ; therefore $z + \frac{a}{r-1} = \frac{ar^t}{r-1} \times r^t$. Wherefore, from the Log. of a subtract the Log. $\overline{r-1}$; to the number answering to the remainder add the given amount, and from the Log. of the sum subtract the afore-found remainder; and this second remainder, divided by the Log. of r , will quote the time required.

Example. In what time will an annuity of 34.4*l.* amount to 614.4327*l.* at the rate of 6 per Cent. per Annum?

$a =$

$$\begin{array}{ll}
 a = 34.4 & L. a = 1.536,5584 \\
 r - 1 = 0.06 & L. \overline{r - 1} = 8.778,1513 \\
 \frac{a}{r - 1} = 573.3333 & L. \frac{a}{r - 1} = 2.758,4071 \\
 z = 614.4327 & \\
 \frac{ar^t}{r - 1} = 1187.7660 & L. \frac{ar^t}{r - 1} = 3.074,7308 \\
 L. r = 0.025,3059 & 0.316,3237 \text{ (12.5 years = } t \text{).}
 \end{array}$$

III. *The amount (z), rate of interest (r), and time (t), being given, to find (a) the annuity.*

The former Equation, being reduced, becomes $a = \frac{z \times \overline{r - 1}}{r^t - 1}$. Wherefore, to the Log. of the amount z , add the Log. of $\overline{r - 1}$, and from the sum subtract the Log. of $r^t - 1$, the remainder is the Log. of a .

Example. An annuity forborn $12\frac{1}{2}$ years amounts, at 6 per Cent. per Annum, to the sum of 614.4327*l.* How much is that annuity?

$$\begin{array}{llll}
 z = 614.4327 & L. z = 2.788,4743 & L. r = 0.025,3059 & \\
 r - 1 = 0.06 & L. \overline{r - 1} = 8.778,1513 & t = \dots\dots\dots 12.5 & \\
 r^t = 2.071,685 & \underline{1.566,6256} & L. r^t = 0.316,3237 & \\
 r^t - 1 = 1.071,685 & L. r^t - 1 = 0.030,0671 & & \\
 L. 34.4*l.* the annuity = L. a = 1.536,5585 & & &
 \end{array}$$

IV. *The annuity (a), time (t), and amount (z), being given, to find (r) the rate.*

In order to find r , the former Equation is reduced to $\frac{z}{a} - 1 = \frac{z}{a} r - r^t$, or, in our present case, $16.8614 = 17.8614r - r^t$, which is so affected as not

* For, since a is $= z \times \frac{\overline{r - 1}}{r^t - 1}$, we shall have $\frac{a}{z} = \frac{\overline{r - 1}}{r^t - 1}$, or $1 = \frac{z}{a} \times \frac{r - 1}{r^t - 1}$, and consequently $1 \times \overline{r^t - 1} = \frac{z}{a} \times \overline{r - 1}$, or $r^t - 1 = \frac{z}{a} \times \overline{r - 1} = \frac{z}{a} \times r - \frac{z}{a}$, and $r^t - 1 + \frac{z}{a} = \frac{z}{a} \times r$, and $\frac{z}{a} - 1 = \frac{z}{a} \times r - r^t$. Q. E. D.

readily

readily to be resolved by the general method for resolution of Equations, unless we can first approach it by some other means; for which purpose take the following Rule, which will suffice where great exactness is not required.

Let $\sqrt[t]{\frac{z}{a}} = 1 + y$, and let $\frac{6}{t+1} = b$. I say, that $\sqrt{bb + 2by} - b$ is exceeding near the increase of the rate, or $r - 1$.

Wherefore take the Log. of the amount, and the complements of the Logarithms of the time and annuity, the sum (abating 2 in the place of tens in the Index) divide by $\frac{1}{2} \times t - 1$; the quotient shall be the Log. of $1 + y$: Then divide 6 by $t + 1$, and to b , the quotient, add twice y ; and to the Log. of b add the Log. of $b + 2y$; half the sum shall be the Log. of $\sqrt{bb + 2by}$; from which square root subtract b , the residue will be very near the increase, or $r - 1$; and, adding 1, r is found. If great exactness be desired, let r thus found be assumed; and $\frac{z}{a} r - r^t$, compared with $\frac{z}{a} - 1$, will always be greater than it; and, dividing the excess by $tr^{t-1} - \frac{z}{a}$, the quotient, added to r , shall verify as many more figures in the rate as were true in the assumed r .

Example. An annuity of 34.4*l.* forborn $12\frac{1}{2}$ years amounts to 614.4327*l.* It is required to find the rate of interest allowed.

$$\begin{array}{ll} z = 614.4327 & \text{L. } z = 2.788,4743 \\ t = 12.5 & \text{co. L. } t = 8.903,0900 \\ a = 34.4 & \text{co. L. } a = 8.463,4416 \end{array}$$

$$\frac{1}{2} \times t - 1 = 5.75 \quad 0.155,0059 \quad (0.026,9575 = \text{L. } (1+y) =) \quad 1.064,039$$

$$\begin{array}{ll} b = \frac{6}{t+1} = \frac{6}{13.5} = 0.444,4444 & 2y = 0.12,8078 \\ 2y \dots\dots\dots & 0.128,0780 \\ b + 2y \dots\dots\dots & 0.572,5224 \\ \sqrt{bb + 2by}^{\frac{1}{2}} \dots\dots\dots & 0.504,4353 \\ b \dots\dots\dots & 0.444,4444 \\ r - 1 \dots\dots\dots & 0.059,9909 \end{array} \quad \begin{array}{ll} \text{L. } b = 9.647,8175 \\ \text{L. } b + 2y = 9.757,7936 \\ 19.405,6111 \\ \text{L. } \sqrt{bb + 2by}^{\frac{1}{2}} = 9.702,8055 \end{array}$$

Therefore the rate of Interest sought is 6 *per Cent. per Annum.*

After the same manner the four Cases relating to the Purchase of Annuities are readily solved by Logarithms, and the Theorems discovered with the same ease;

case. For, $a, \frac{a}{r}, \frac{a}{r^2}, \frac{a}{r^3}, \frac{a}{r^4},$ &c. being a scale of continued geometrical proportionals in the *ratio* of r to 1 , put y for the sum of all the consequents infinitely continued, whereof $\frac{a}{r}$ is the first, and that sum will be to the sum of all the antecedents whereof a is the first, as 1 to r , that is, $1 : r :: y : ry$, so that $ry = y + a$, and $\frac{a}{r-1}$ will be equal to y , the value of the fee, or the sum of all the continued proportionals less than a . And, by the same Rule, $\frac{a}{r^t \times r - 1}$ will be the sum of all the continued proportionals less than $\frac{a}{r^t}$, or the value of the reversion: and (subtracting the one sum from the other) $\frac{a}{r-1} - \frac{a}{r^t \times r - 1}$ will be equal to z the sum of all the means, whereof $\frac{a}{r}$ is the greatest, and $\frac{a}{r^t}$ the least.

I. *The annuity (a), time (t), and rate of interest (r), being given, to find (z) the present value.*

The present value $z = \frac{a}{r-1} - \frac{a}{r^t \times r - 1}$; therefore from Log. of the annuity subtract the Log. of $r-1$, and from the residue subtract also the Log. of r^t , the difference of the numbers answering to the two remainders is the present value sought.

Example. What is an annuity of 70*l.* per Annum which is to continue 59 years, worth in present money, at the rate of 5 per Cent. per Annum?

$a = 70.$	L. $a = 1.845,0980$	L. $r = 0.021,1893$
$r - 1 = 0.05$	L. $r-1 = 8.698,9700$	$t = \dots\dots\dots 59$
	L. fee = 3.146,1280	1907037
Fee = 1400 <i>l.</i>	L. $r^t = 1.250,1687$	1059465
Reversion 78.6972 <i>l.</i>	L. rever. = 1.895,9593	L. $r^t = 1.250,1687$
$z = 1321.3028\%$ the present value sought.		

II. *The annuity (a), present value (z), and rate of interest (r), being given, to find (t) the time.*

Now r^t will be equal to $\frac{a}{r-1}$ (or the fee) divided by the value of the reversion, that is by $\frac{a}{r-1} - z$. Wherefore from the Log. of the annuity, subtract the Log. of $\frac{a}{r-1}$; the number answering to the remainder will be the value of the fee; from the fee subtract the present worth, the residue is the value of the reversion. Take the Log. of the reversion from the Log. of the fee, and divide the residue by the Log. of r ; and the quotient will be t the number of years sought.

Example. In what time will an annuity of 70*l.* per Annum pay off a debt of 1321.3028*l.* allowing the Creditor 5 per Cent. per Annum?

$a = 70$	$L. a = 1.845,0980$
$r - 1 = 0.05$	$L. \frac{a}{r-1} = 8.698,9700$
Fee = 1400	$L. \text{fee} = 3.146,1280$
$z = 1321.3028$	
Reversion 78.6972	$L. \text{rev.} = 1.895,9593$
	$L. r = 0.021,1893$

$$1.250,1687 \text{ (59 years = } t \text{.)}$$

III. *The present value (z), rate of interest (r), and time (t), being given, to find (a) the annuity.*

The former Equation may be reduced to this proportion, as $1 - \frac{1}{r^t}$ to z , so is $r - 1$ to a the annuity sought.

Wherefore to the complement of the Log. of $1 - \frac{1}{r^t}$ add the Logarithms of z and of $r - 1$; and the sum will be Log. of a .

Example. What annuity to continue 59 years can be purchased for 1321.3028*l.* at the rate of 5 per Cent. per Annum?

$\frac{1}{r^t} = 0.056,2123$	$L. r = 0.021,1893$
	$t = \dots\dots\dots 59$
$1 - \frac{1}{r^t} = 0.943,7877$	$L. 1 - \frac{1}{r^t} = 0.025,1257$
$z = 1321.3028$	$L. z = 3.121,0023$
$r - 1 = 0.05$	$L. \frac{a}{r-1} = 8.698,9700$
	$L. \frac{1}{r^t} = 8.749,8313$

$$L. (70\text{l.}) a \text{ the annuity sought} = 1.845,0980$$

IV. *The annuity (a), present value (z), and time (t), being given, to find (r) the rate of interest.*

This Problem being more difficult than appears at first sight, and requiring the solution of this equation $\frac{a}{z} = \frac{z+a}{z} r^t - r^{t+1}$, to which it is reduced; there

there must be applied some method of approaching the root r , which is by no means evident: And that approximation, as the number of years and rate are greater or less, cannot properly be obtained by one general rule; but rather by two, according as the value of the reversion is greater or less.

If the number of years be great (as suppose 40 or upwards) and especially if the rate of interest be high, $1 + \frac{a}{z}$ will be nearly the rate, or more accurately $\frac{z+a}{z} - \frac{z}{z+a} \times \frac{a}{z}$; call it r ; and $\frac{a}{r^t \times r - 1}$ will be exceeding near

the value of the reversion, which let be x ; then $1 + \frac{a}{z+x}$ shall approach the true rate sufficiently. But if greater exactness be desired, by repeating this process it will be obtained. Hence this rule: From the Log. of a , and also from the Log. of $z + a$, take the Log. of z ; this latter remainder shall be nearly the Log. of the rate. Multiply that Log. by t , and the complement of the product add to the first remainder; the decimal fraction answering to the sum taken from the former rate shall give a more correct rate. With this rate seek x the reversion after the time given, and add it to z . Then to the complement of the Log. of $z + x$, add the Log. of a ; and the sum will be the Log. of the increase, or of $r - 1$, sufficiently near.

Example. If 1321.3028 l. is paid for an annuity of 70 l. per Annum for 59 years to come, what is the rate of interest allowed the Purchaser?

$a = 70$	L. $a = 1.845,098,0$	L. $\frac{z+a}{z} = 3.143,421,7$
$z = 1321.3028$	L. $z = 3.121,002,3$	L. $z \dots\dots\dots 3.121,002,3$
First r is $= \frac{z+a}{z} = 1.052,978$	8.724,095,7	L. $\frac{z+a}{z} = 0.022,419,4$
	co. t L. $\frac{z+a}{z} = 8.677,255,4$	$t \dots\dots\dots 59$
Log. of 0.002,52	7.301,351,1	201 774,6
Second $r = 1.050,458$	L. $r = 0.021,378,7$	1120970
	$t = \dots\dots\dots 59$	
$z = 1321.302,8$	1924083	t L. $\frac{z+a}{z} = 1.322,744,6$
$x = 76.002$	1068935	co. L. $r^t = 8.738,656,7$
$z+x = 1397.3048$ the Fee	L. $r^t = 1.261,343,3$	co. L. $r-1 = 1.297,070,0$
	co. L. $\frac{z+x}{z} = 6.854,708,8$	L. $a = 1.845,098,0$
	L. $a \dots\dots\dots 1.845,098,0$	L. $x = 1.880,824,7$
$r-1 = 0.050,096.$	L. $r-1 = 8.699,806,8$	
Third r is $= 1.050,096.$		

So that the rate of interest sought is 5 per Cent. per Annum.

If the number of years be small, the aforesaid Rule will avail little. In this case it will be requisite to approach the rate thus. Let $\frac{t+1}{2}$ be the Index of a root* of $\frac{at}{z}$; from which root subtract 1, and the remainder call y ; and let $\frac{6}{t-1}$ be called b . I say, that $b - \sqrt{bb-2by}^{\frac{1}{2}}$ is sufficiently near to $r-1$, and will be still nearer the truth, as the number of years is smaller; and that the error will be always in excess. Hence the Rule: Divide the Log. of $\frac{at}{z}$ by $\frac{t+1}{2}$, and from the number answering to the quotient subtract 1; double the remainder, and subtract it from b ; that is, from the quotient of 6 divided by $t-1$; to the Logarithm of this remainder add the Log. of b ; then the number answering to half the sum of those Logarithms, taken from b , will leave $r-1$ the increase of the rate sought.

Example. An annuity of 20*l.* per Annum, to continue 21 years, is sold for 220*l.* It is required to find the rate of interest allowed the purchaser.

$$\begin{array}{ll} a = 20 & \text{L. } a = 1.301,0300 \\ t = 21 & \text{L. } t = 1.322,2193 \\ z = 220 & \text{co. L. } z = 7.657,5773 \end{array}$$

$$\frac{1}{2} \times \overline{t+1} = 11) 0.280,8266 \quad (0.025,5297 = \text{L. } (\overline{1+y}) =) \quad \underline{1.060,546}$$

$$2y = \underline{0.121,092}$$

$$b = \frac{6}{t-1} = \frac{6}{20} = 0.$$

$$\begin{array}{l} \text{L. } b = 9.477,1213 \\ \text{L. } \overline{b-2y} = 9.252,6298 \end{array}$$

$$2y \dots\dots\dots 0.121,092$$

$$b - 2y \dots\dots\dots 0.178,908$$

$$\overline{bb-2by}^{\frac{1}{2}} \dots\dots\dots 0.231,673$$

$$\underline{18.729,7511}$$

$$\text{L. } \overline{bb-2by}^{\frac{1}{2}} = 9.364,8755$$

$$\text{Now } b - \overline{bb-2by}^{\frac{1}{2}} = 0.068,327 = r - 1$$

Therefore $r = 1.068,327$ the rate sought.

* This $\frac{t+1}{2}$ th root of $\frac{at}{z}$ is $(= \sqrt[t+1]{\frac{at}{z}})^{\frac{1}{2}} = \sqrt[t+1]{\frac{at}{z} \times \frac{2}{2}} = \sqrt[t+1]{\frac{at}{z} \times \frac{2}{t+1}}$, or the $\frac{2}{t+1}$ power of $\frac{at}{z}$.

The rate r thus found is always some small matter too big, the true rate being 1.06814; but as the number of years are fewer, the error becomes insensible. If greater exactness be required, it will be easy by the general method for the resolution of Equations, having so near an approximation, to prosecute this enquiry as far as you please. But this seems abundantly sufficient for use, which is our principal design in this place.

Lastly, By way of Corollary to the former. Let it be required to find the rate of interest allowed the Purchaser when he pays a sum $= z$, for an annuity $= a$, wherein he has already a term $= t$, to have it prolonged for a certain time $= x$.

Example. An annuity of 20*l.* *per Annum* that is already granted for a term of years of which 21 years are still to come, may, for the sum of 40 pounds paid down, be prolonged for 10 years more, or to 31 years; what is the rate of interest required?

Put $T = 2t + x + 1$, and $\frac{1}{2}T$ shall be the Index of a root of $\frac{ax}{z}$. Let $\sqrt[\frac{1}{2}T]{\frac{ax}{z}}$ be equal to $1 + y$, and $\frac{6T+6}{xx}$ be $= b$. I say, $r - 1$ is very near to $b - \overline{bb - 2by}^{\frac{1}{2}}$.

$$\begin{array}{ll} a = 20 & \text{L. } a = 1.301,0300 \\ x = 10 & \text{L. } x = 1.000,0000 \\ z = 40 & \text{co.L. } z = 8.397,9400 \end{array}$$

$$\frac{1}{2}T = 26.5 \quad 0.698,9700 \quad (0.026,3762 = \text{L. } (\overline{1+y}) =) \quad 1.062,616$$

$$b = \frac{6T+6}{xx} = 3.24$$

$$2y \dots\dots\dots 0.125,232$$

$$2y = 0.125,232$$

$$\text{L. } b = 0.510,5450$$

$$\text{L. } \overline{b-2y} = 0.493,4257$$

$$\underline{1.003,9707}$$

$$\overline{bb - 2by}^{\frac{1}{2}} = 3.176,767$$

$$\text{L. } \overline{bb - 2by}^{\frac{1}{2}} = 0.501,9853$$

$$b - \overline{bb - 2by}^{\frac{1}{2}} = 0.063,233 = r - 1$$

Therefore $r = 1.063,233$ the rate sought.

As will be readily proved by seeking the value of the reversion of an annuity of 20*l.* *per Annum* for 10 years after 21; at the rate of 1.063,233 *per Cent. per Annum*. See the Work.

$Q =$

$a = 20$	$L. a = 1.301,0300$	$L. r = 0.026,6284$
$r - 1 = 0.063,233$	$co. L. \frac{r}{r-1} = 1.199,0562$	21
	$co. L. r^{\frac{1}{2}} = 9.440,8036$	<u>266284</u>
Rever. 87.275/.	$L. rev. = 1.940,8898$	<u>532568</u>
	$co. L. r^x = 9.733,7160$	$L. r^{\frac{1}{2}} = 0.559,1964$
Rever. 47.2722/.	$L. rev. = 1.674,6058$	
<u>40.0028/.</u> the value sought.		

Thus it appears that 40/ and about three farthings is the true value of the difference of the reversion at the rate of interest before found; by which it may be judged how near an approximation the foregoing Rule affords towards finding the rate of interest, when the value of an annuity for a term of years to commence after such a distant time is proposed.

N O T E S
ON
SOME DIFFICULT PASSAGES
OF THE
FOREGOING DISCOURSE OF DR. HALLEY
ON
COMPOUND INTEREST.

By FRANCIS MASERES, Esq. F.R.S.
CURSITOR BARON OF THE COURT OF EXCHEQUER.

NOTE I.

Article 1. **T**HE first passage in this Discourse which seems to stand in need of explanation, is the following Problem and it's Solution, which occur in page 17 of the Introduction to the 3d Edition of Sherwin's Mathematical Tables, published in the year 1741, and in pages 223 and 224 of the present Volume, and are as follows.

PROBLEM IV.

"The annuity a , time t , and amount z , being given, to find r , the rate.

SOLUTION.

"In order to find r , the former equation is reduced to $\frac{z}{a} - 1 = \frac{z}{a} \times r - r^t$,

"or, in our present case, to the equation $16.8614 = 17.8614r - r^t$, or to

" $17.8614r - r^t = 16.8614$; which equation is so affected as not readily to be
"resolved.

“ resolved by the general method for the resolution of equations, unless we can
 “ first approach it by some other means : for which purpose take the following
 “ Rule, which will suffice where great exactness is not required.

R U L E.

“ Let $\frac{z}{at} \sqrt[t-1]{2}$ be $= 1 + y$; and let $\frac{6}{t+1}$ be $= b$. I say, that
 “ $\sqrt[bb + 2by]{\frac{1}{2}} - b$ is exceeding near the increase of the rate, or $r - 1$.”

Art. 2. To this expression of the value of $r - 1$ Dr. Halley afterwards adds a Correction, which will be useful when great exactness is required. This Correction is expressed in these words.

A Correction of the foregoing Expression.

“ If great exactness be desired, let r , thus found, be assumed. And
 “ $\frac{z}{a} \times r - r^t$, compared with $\frac{z}{a} - 1$, will always be greater than it : and,
 “ dividing the excess by $tr^{t-1} - \frac{z}{a}$, the quotient, added to r , shall verify
 “ as many figures in the rate as were true in the assumed r .”

The foregoing Correction expressed more clearly.

Art. 3. This last passage concerning the correction of the expression $\sqrt[bb + 2by]{\frac{1}{2}} - b$, given above for the first near value of $r - 1$, may be expressed more clearly in the following manner.

If greater exactness be required, let the value of r resulting from the foregoing near value of $r - 1$, to wit, $\sqrt[bb + 2by]{\frac{1}{2}} - b$, (which value of r will evidently be $1 + \sqrt[bb + 2by]{\frac{1}{2}} - b$), be denoted by the letter c ; and let the value of the binomial quantity $\frac{z}{a} \times c - c^t$ be computed. This value will always be greater than the binomial quantity $\frac{z}{a} \times r - r^t$, and consequently than, it's equal, the known quantity $\frac{z}{a}$; and therefore the said known quantity

tity $\frac{z}{a} - 1$ may be subtracted from the said binomial quantity $\frac{z}{a} \times c - c^t$. Let it be so subtracted; and let the difference, or remainder, be called d . Then will the fraction $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$, or the quotient of the division of the said difference d by the excess of $t \times c^{t-1}$ above $\frac{z}{a}$, be the quantity that ought to be added to the former near value of r , to wit, $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, in order to obtain a much nearer value of r , of which twice as many figures will be exact as there were in the former near value of it, or $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$.

An Investigation of the foregoing Correction by Mr. Raphson's Method of resolving Affected Equations by Approximation.

Art. 4. This second near value of r , or correction of it's first near value, $1 + \sqrt{bb + 2by} - b$, or c , may be derived from the said first near value of it by Mr. Raphson's method of approximation in the following manner.

Since $1 + \sqrt{bb + 2by} - b$, or c , is nearly equal to, but somewhat less than, the true value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, let e be put for the excess of the said true value above c , so that r shall be equal to $c + e$; and let $c + e$ be substituted, instead of r , in the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Then, since r is $= c + e$, we shall have $\frac{z}{a} \times r (= \frac{z}{a} \times \overline{c + e}) = \frac{z}{a} \times c + \frac{z}{a} \times e$, and $r^t = \overline{c + e}^t =$ (by Sir Isaac Newton's binomial theorem) $c^t + t \times c^{t-1} \times e + \&c$, continued to $t + 1$ terms. Therefore the binomial quantity $\frac{z}{a} \times r - r^t$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} \frac{z}{a} \times c + \frac{z}{a} \times e \\ - c^t - t \times c^{t-1} \times e - \&c \end{array} \right\}$$

$$= \frac{z}{a} \times c - c^t + \frac{z}{a} \times e - t \times c^{t-1} \times e - \&c.$$

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But the binomial quantity $\frac{z}{a} \times r - r^t$ is $= \frac{z}{a} - 1$.

Therefore the compound quantity $\frac{z}{a} \times c - c^t + \frac{z}{a} \times e - t \times c^{t-1} \times e - \&c$ will also be $= \frac{z}{a} - 1$; and consequently (adding $t \times c^{t-1} \times e + \&c$ to both sides,) we shall have $\frac{z}{a} \times c - c^t + \frac{z}{a} \times e = \frac{z}{a} - 1 + t \times c^{t-1} \times e + \&c$, and (subtracting $\frac{z}{a} - 1$ from both sides) $\frac{z}{a} \times c - c^t - \left[\frac{z}{a} - 1 \right] + \frac{z}{a} \times e = t \times c^{t-1} \times e + \&c$, or (putting $d = \frac{z}{a} \times c - c^t - \left[\frac{z}{a} - 1 \right]$) we shall have $d + \frac{z}{a} \times e = t \times c^{t-1} \times e + \&c$, and (subtracting $\frac{z}{a} \times e$ from both sides,) $d = t \times c^{t-1} \times e + \&c - \frac{z}{a} \times e$, or $t \times c^{t-1} \times e - \frac{z}{a} \times e + \&c = d$, and (neglecting the terms expressed by the $\&c$, on account of their smallness, as they will involve e^2, e^3, e^4 , and the other higher powers of e , which, on account of the smallness of e , will be much less than e ,) we shall have $t \times c^{t-1} \times e - \frac{z}{a} \times e =$, nearly, d , or $e \times \left[t \times c^{t-1} - \frac{z}{a} \right]$ nearly $= d$. Therefore e will be nearly $= \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, and consequently r , or $c + e$, will be nearly $= c + \frac{d}{t \times c^{t-1} - \frac{z}{a}}$. Q. E. I.

I return now to Dr. Halley's first expression of the value of r , to wit, $1 + \sqrt{bb + 2by} - b$, or $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$; which may be investigated in the following manner.

The Investigation of the First Expression, $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, given above by

Dr. Halley for a near Value of r in the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Art. 5. Let r be put for 1 pound sterling, together with it's interest for a year, or for half a year, or for a quarter of a year, or for any other given portion

portion of time, during which a sum of money is put out at interest. And let an annuity of 1 pound *per annum*, or a pension of 1 pound for every half-year, or every quarter of a year, or any other given portion of time, be supposed to be granted to any person, for the space of t years, or t half-years, or t quarters of a year, or t times any other given portion of time, t being any whole number whatsoever; so that the first payment of the said sum of 1 pound shall become due at the end of one year, or of one half-year, or of one quarter of a year, or of any other given portion of time; and the second payment of 1 pound shall become due at the end of the second year, or half-year, or quarter of a year, or other given portion of time; and the third payment of 1 pound shall become due at the end of the third year, or half-year, or quarter of a year, or other given portion of time; and a like payment of 1 pound shall become due at the end of every following year, or half-year, or quarter of a year, or other given portion of time; of the t years, or t half-years, or t quarters of a year, or t other given portions of time, during which the annuity, or pension, is to continue.

Further, let it be supposed that the person who has granted the annuity, or pension, finding it inconvenient to pay it at the end of every year, or of every half year, or quarter of a year, or other given portion of time, at which it will become due, as he ought to do according to his agreement, and hoping that, before the expiration of the t years, or half-years, or quarters of a year, or other given portions of time, for which the pension is granted, he shall come into the possession of a good estate, that will enable him to pay all the said yearly, or half-yearly, or other, payments of 1 pound each, together with the interest that will then be due upon them to the grantee of the annuity, or pension, (in consequence of his having forborn to receive the said annual, or other, payments at the ends of the several years, or half-years, or quarters of a year, or other given portions of time, at which they became due,) in one payment at the end of the said t years, or half-years, or quarters of a year, or other given portions of time;—I say, let it be supposed that the grantor of the said annuity, or pension, prevails upon the grantee of it to consent to such a delay of payment. Then will the sum that will be due to the grantee of the said annuity, or pension, from the grantor of it at the end of the whole t years, or half-years, or quarters of a year, or other given portions of time, during which the said annuity, or pension, was to continue, be what Dr. Halley calls *the amount* of the said annuity, or pension, of 1 pound *per annum*, or 1 pound for every half-year, or quarter of a year, or other given portion of time, at the end of the said t years, or half-years, or quarters of a year, or other equal portions of time, during which the said annuity, or pension, was to continue.

To avoid the repetition of the phrase expressing the different portions of time at the ends of which the payments of the said sums of 1 pound each may, by the agreement of the parties, be made to become due, I will now suppose them to become due only at the end of every year, or the pension to be an annual pension, or *an annuity* in the strict sense of the word.

Art. 6. Now, if t is $= 1$, or the annuity continues for only one year, the payment to be made to the grantee at the end of the said year will be only 1 pound. But, if t is $= 2$, or the annuity continues for two years, the payment to be made to the grantee at the end of the two years will be more than two pounds, and will be $= 1 + r$ pounds; because the pound that was due at the end of the first year, and was not paid, will carry interest during the second year, and thereby increase from 1*l.* to r ; as, for example, if the interest of money is 5 per cent, it will increase from 1*l.* to 1*l.* + $\frac{5}{100}$ *l.*, or to 1.05*l.* And, for the same reason, the payment to be made to the grantee at the end of three years will be $1 + r + r^2$; because the pound that became due at the end of the first year, will now be grown-up in the space of two years from 1*l.* to r^2 , and the pound that was due at the end of the second year will be grown-up from 1*l.* to r , and consequently the whole sum due at the end of the third year will be $r^2 + r + 1$, or $1 + r + r^2$. And, in like manner, the payment to be made to the grantee at the end of four years will be $r^3 + r^2 + r + 1$, or $1 + r + r^2 + r^3$; and at the end of five years it will be $r^4 + r^3 + r^2 + r + 1$, or $1 + r + r^2 + r^3 + r^4$, and at the end of t years it will be $r^{t-1} + r^{t-2} + r^{t-3} + r^{t-4} + r^{t-5} + \&c + r^{t-t}$, or $r^{t-1} + r^{t-2} + r^{t-3} + r^{t-4} + r^{t-5} + \&c + r^0$, or $r^{t-1} + r^{t-2} + r^{t-3} + r^{t-4} + r^{t-5} + \&c + 1$, or $1 + r + r^2 + r^3 + r^4 + r^5 + \&c + r^{t-1}$, which is a series of quantities in continued geometrical proportion, consisting of t terms, of which the first term 1*l.* is the least, and the last term r^{t-1} is the greatest.

Art. 7. Now the sum of all the terms of a geometrical progression, such as $1 + r + r^2 + r^3 + r^4 + r^5 + \&c + r^{t-1}$, consisting of t terms, is equal to a fraction, of which the numerator is the excess of the square of the greatest term above the product of the multiplication of the greatest term but one into the least term, and the denominator is the excess of the greatest term above the greatest term but one. This may be easily demonstrated by means of the 12th Proposition of the 5th Book of Euclid's Elements, in the following manner:

Of the Sum of the Terms of a Geometrical Progression.

A THEOREM.

The sum of the terms of every geometrical progression is equal to the quotient that arises by dividing the excess of the square of it's greatest term
above

above the product of the multiplication of it's greatest term but one into it's least term, by the excess of it's greatest term above it's greatest term but one.

DEMONSTRATION.

Let E, D, C, B, A be a series of quantities of any kind in continued geometrical proportion, of which E is the least, and A is the greatest.

Then we shall have $A : B :: B : C$
 and $A : B :: C : D$
 and $A : B :: D : E$.

Therefore, (by El. 5, 12) the sum of all the antecedents A, B, C, D will be to the sum of all the consequents B, C, D, E, in the same proportion as the first antecedent A is to the first consequent B.

But the sum of all the antecedents A, B, C, D is the sum of all the terms of the series, except the least term E; and the sum of all the consequents B, C, D, E is the sum of all the terms of the series, except the greatest term A. Therefore, if the sum of all the terms of the series be called S, we shall have $S - E : S - A :: A : B$. Therefore $B \times S - E$ will be $= A \times S - A$, or $BS - BE$ will be $= AS - AA$. Therefore (adding AA to both sides,) we shall have $AA + BS - BE = AS$; and (subtracting BS from both sides; which is evidently possible, because B is less than A, and consequently BS is less than AS, and therefore than the other side of the equation;) we shall have $AA - BE = AS - BS$. But $AS - BS$ is $= S \times \overline{A - B}$. Therefore $AA - BE$ will be $= S \times \overline{A - B}$; and consequently S will be $= \frac{AA - BE}{A - B}$; that is, the sum of all the terms A, B, C, D, E, or E, D, C, B, A, will be equal to the quotient that arises by dividing $AA - BE$, or the excess of AA, or the square of the greatest term A, above BE, or the product of the multiplication of B, the greatest term but one, into E, the least term, by $A - B$, or the excess of A, the greatest term, above B, the greatest term but one.

Q. E. D.

And it is easy to see that the same reasonings will take place, and consequently that the same conclusion will follow from them, if the series E, D, C, B, A, instead of containing only five terms, had contained any other number of terms whatsoever.

Art. 8. From this Proposition it follows that the sum of all the terms of the geometrical progression $1/. + r + r^2 + r^3 + r^4 + \&c + r^{t-1}$, consisting of t terms,

t terms, in which $1l.$ is the least term, and r^{t-1} is the greatest term, and r^{t-2} is the greatest term but one, will be equal to the quotient of the division of $r^{t-1} \times r^{t-1} - r^{t-2} \times 1l.$ by $r^{t-1} - r^{t-2}$, or to the fraction

$$\frac{r^{t-1} \times r^{t-1} - r^{t-2} \times 1l.}{r^{t-1} - r^{t-2}}, \text{ which is } (= \frac{r^{2t-2} - r^{t-2} \times 1l.}{r^{t-1} - r^{t-2}} =$$

$$\frac{r^2 \times r^{2t-2} - r^2 \times r^{t-2} \times 1l.}{r^2 \times r^{t-1} - r^2 \times r^{t-2}} = \frac{r^{2t} - r^t \times 1l.}{r^{t+1} - r^t} = \frac{r^t \times r^t - 1l.}{r^t \times r - 1l.}) = \frac{r^t - 1l.}{r - 1}.$$

Therefore the sum of all the terms of the geometrical series $1l. + r + r^2 + r^3 + r^4 + r^5 + \&c + r^{t-1}$, consisting of t terms, will be $=$ the fraction $\frac{r^t - 1l.}{r - 1}$, or $\frac{r^t - 1}{r - 1}$. Therefore, if the annuity of 1 pound *per annum* be forborn for t years, the sum due to the annuitant at the end of the said t years, or *the amount* of the said annuity at that time, will be $=$ the fraction $\frac{r^t - 1}{r - 1}$.

Art. 9. And therefore, if the annuity is not 1 pound *per annum*, but a pounds, or a parts of a pound, *per annum*, a being any whole number, mixt number, or fraction, whatsoever, the sum due to the annuitant at the end of the said t years, or *the amount* of the said annuity a at that time, will be $= a \times \frac{r^t - 1}{r - 1}$, or $\frac{ar^t - a}{r - 1}$. This amount Dr. Halley denotes by the letter z .

These preliminaries being settled, the investigation of Dr. Halley's first expression of the value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, to wit, the expression $1 + \sqrt[bb+2by]{b} - b$, or $1 + \sqrt[bb+2by]{b}^{\frac{1}{2}} - b$, may be made in the following manner.

Art. 10. Since z is $= \frac{ar^t - a}{r - 1}$, it follows that $\frac{z}{a}$ will be $= \frac{r^t - 1}{r - 1}$. Therefore (multiplying both sides into $r - 1$, we shall have $\frac{z}{a} \times r - \frac{z}{a} = r^t - 1$; and (adding $\frac{z}{a}$ to both sides,) we shall have $\frac{z}{a} \times r = r^t - 1 + \frac{z}{a}$; and (subtracting r^t from both sides,) we shall have $\frac{z}{a} \times r - r^t = -1$

$+\frac{z}{a}$, or $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; which is the equation which is now to be resolved.

Art. 11. Let v be put for the unknown interest of 1 pound for a year, or for the excess of r (which is equal to 1*l.* together with it's interest for a year,) above 1*l.*, or for $r - 1$.

Then will r be $= v + 1$, or $1 + v$, and consequently r^t will be $= \overline{1+v}^t$ (by Sir Isaac Newton's binomial theorem in the case of integral powers,) to the series $1 + tv + t \times \frac{t-1}{2} \times v^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c$, continued to $t + 1$ terms. Therefore $r^t - 1$ will be $=$ the series $tv + t \times \frac{t-1}{2} \times v^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c$, continued to t terms; and consequently $\frac{r^t - 1}{r - 1}$, or $\frac{r^t - 1}{v}$, will be $=$ the series $t + t \times \frac{t-1}{2} \times v + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms. Therefore $\frac{z}{a}$ (which is $= \frac{r^t - 1}{r - 1}$), will also be $=$ the series $t + t \times \frac{t-1}{2} \times v + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms; and consequently (dividing both sides by t), the fraction $\frac{z}{at}$ will be $=$ the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times v^2 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms. Therefore any power whatsoever, integral or fractional, of the fraction $\frac{z}{at}$ will be equal

equal to the same power of the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times v^2$
 $+ \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 +$
 &c, continued to t terms.

Art. 12. Now let the fraction $\frac{z}{at}$ (which is a known quantity, because z ,
 a ; and t are all supposed to be given,) be raised to the power of which the
 fraction $\frac{2}{t-1}$, (or the reciprocal of the fraction $\frac{t-1}{2}$, which is the co-effi-
 cient of v in the last series,) is the index; which may be easily done by means
 of a Table of Logarithms: and let the last series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5}$
 $\times v^4 + \&c$ (which is equal to the fraction $\frac{z}{at}$;) be likewise raised to the same
 power, of which the fraction $\frac{2}{t-1}$ is the index, by means of Mr. De Moivre's
 multinomial theorem, or theorem for raising a multinomial quantity to any given
 power; the said theorem being supposed to be true when the power to which
 the multinomial quantity is to be raised is a fractional power, as well as when
 it is an integral power, though Mr. De Moivre's demonstration of it seems to

relate only to the case of integral powers. Then will $\left[\frac{z}{at}\right]^{\frac{2}{t-1}}$, or the said
 $\frac{2}{t-1}$ th power of the fraction $\frac{z}{at}$ be equal to the said $\frac{2}{t-1}$ th power of the said
 series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3$
 $+ \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms. This
 celebrated Theorem of Mr. De Moivre, as it is given in the Second Volume
 of the *Miscellanea Curiosa*, page 183, is as follows.

Mr.

Mr. De Moivre's Theorem for raising any Power of an Infinite Series.

Art. 13. The m th power of the infinite series $az + bz^2 + cz^3 + dz^4 + ez^5 + \&c$ is equal to the infinite series

$$\begin{aligned}
 & a^m z^m + \frac{m}{1} \times a^{m-1} b z^{m+1} + \frac{m}{1} \times \frac{m-1}{2} \times a^{m-2} b^2 z^{m+2} \\
 & \quad + \frac{m}{1} \times a^{m-1} c z^{m+2} \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3} b^3 z^{m+3} \\
 & \quad + \frac{m}{1} \times \frac{m-1}{1} \times a^{m-2} b c z^{m+3} \\
 & \quad + \frac{m}{1} \times a^{m-1} d z^{m+3} \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times a^{m-4} b^4 z^{m+4} \\
 & \quad + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} \times a^{m-3} b^2 c z^{m+4} \\
 & \quad + \frac{m}{1} \times \frac{m-1}{1} \times a^{m-2} b d z^{m+4} \\
 & \quad + \frac{m}{1} \times \frac{m-1}{2} \times a^{m-2} c^2 z^{m+4} \\
 & \quad + \frac{m}{1} \times a^{m-1} c z^{m+4} \\
 & + \&c.
 \end{aligned}$$

Coroll. 1. Therefore, if a is $= 1$, or the series that is to be raised to the m th power is $z + bz^2 + cz^3 + dz^4 + ez^5 + \&c$, the m th power of the said series will be the infinite series

$$\begin{aligned}
 & z^m + \frac{m}{1} \times b z^{m+1} + \frac{m}{1} \times \frac{m-1}{2} \times b^2 z^{m+2} \\
 & \quad + \frac{m}{1} \times c z^{m+2}
 \end{aligned}$$

$$\begin{aligned}
& + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3 \times z^{m+3} \\
& \quad + \frac{m}{1} \times \frac{m-1}{1} \times bc \times z^{m+3} \\
& \quad + \frac{m}{1} \times d \times z^{m+3} \\
& + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times b^4 \times z^{m+4} \\
& \quad + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} \times b^2c \times z^{m+4} \\
& \quad + \frac{m}{1} \times \frac{m-1}{1} \times bd \times z^{m+4} \\
& \quad + \frac{m}{1} \times \frac{m-1}{2} \times c^2 \times z^{m+4} \\
& \quad + \frac{m}{1} \times e \times z^{m+4}
\end{aligned}$$

+ &c.

Coroll. 2. The series $z + bz^2 + cz^3 + dz^4 + ez^5 + \&c$ is $= z \times$ the series $1 + bz + cz^2 + dz^3 + ez^4 + \&c$. Therefore the m th power of the series $z + bz^2 + cz^3 + dz^4 + ez^5 + \&c$ will be $= z^m \times$ the m th power of the series $1 + bz + cz^2 + dz^3 + ez^4 + \&c$; and consequently the m th power of the series $1 + bz + cz^2 + dz^3 + ez^4 + \&c$ will be equal to the quotient that arises by dividing the m th power of the series $z + bz^2 + cz^3 + dz^4 + ez^5 + \&c$, or the series which is set down in the foregoing corollary, by z^m ; that is, the m th power of the series $1 + bz + cz^2 + dz^3 + ez^4 + \&c$ will be the infinite series

$$\begin{aligned}
1 + \frac{m}{1} bz^1 + \frac{m}{1} \times \frac{m-1}{2} \times b^2z^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3z^3 \\
+ \frac{m}{1} \times cz^2 + \frac{m}{1} \times \frac{m-1}{1} \times bc \times z^3 \\
+ \frac{m}{1} \times d \times z^3
\end{aligned}$$

+

$$\begin{aligned}
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times b^4 \times z^4 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} \times b^3 c \times z^4 \\
 & + \frac{m}{1} \times \frac{m-1}{1} \times b d \times z^4 \\
 & + \frac{m}{1} \times \frac{m-1}{2} \times c^2 \times z^4 \\
 & + \frac{m}{1} \times e \times z^4 \\
 & + \&c.
 \end{aligned}$$

Coroll. 3. Therefore, if v be substituted instead of z in these last serieses, the m th power of the series $1 + bv + cv^2 + dv^3 + ev^4 + \&c$ will be equal to the infinite series

$$\begin{aligned}
 1 + \frac{m}{1}bv + \frac{m}{1} \times \frac{m-1}{2} \times b^2v^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3v^3 \\
 + \frac{m}{1} \times cv^2 + \frac{m}{1} \times \frac{m-1}{1} \times bcv^3 \\
 + \frac{m}{1} \times dv^3 \\
 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times b^4 \times v^4 \\
 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{1} \times b^3c \times v^4 \\
 + \frac{m}{1} \times \frac{m-1}{1} \times bd \times v^4 \\
 + \frac{m}{1} \times \frac{m-1}{2} \times c^2 \times v^4 \\
 + \frac{m}{1} \times e \times v^4 \\
 + \&c.
 \end{aligned}$$

Coroll. 4. Now let the index m be equal to the fraction $\frac{2}{t-1}$.

Then will $m-1$ be $(= \frac{2}{t-1} - 1 = \frac{2-t+1}{t-1}) = \frac{3-t}{t-1}$, and $m-2$ will be $= \frac{2}{t-1} - 2 (= \frac{2}{t-1} - \frac{2t+2}{t-1}) = \frac{4-2t}{t-1}$, and $m-3$ will be $= \frac{2}{t-1} - 3 (= \frac{2}{t-1} - \frac{3t+3}{t-1}) = \frac{5-3t}{t-1}$; and consequently $\frac{m}{1} \times \frac{m-1}{2}$ will be

$$\begin{aligned} \text{be } (= \frac{m}{2} \times \frac{m-1}{1} = \frac{2}{2 \times t-1} \times \frac{3-t}{t-1} = \frac{1}{t-1} \times \frac{3-t}{t-1}) = \frac{3-t}{(t-1)^2}, \text{ and} \\ \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \text{ will be } (= \frac{3-t}{(t-1)^2} \times \frac{m-2}{3} = \frac{3-t}{(t-1)^2} \times \frac{4-2t}{3 \times t-1} = \\ \frac{3-t \times 4-2t}{3 \times (t-1)^3}) = \frac{2t^2 - 10t + 12}{3 \times (t-1)^3}, \text{ and } \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \text{ will} \\ \text{be } = \frac{2t^2 - 10t + 12}{3 \times (t-1)^3} \times \frac{5-3t}{4 \times t-1} = \frac{60 - 86t + 40t^2 - 6t^3}{12 \times (t-1)^4}. \end{aligned}$$

Therefore the $\frac{2}{t-1}$ th power of the series $1 + bv + cv^2 + dv^3 + ev^4 + \&c$ will be equal to the infinite series

$$\begin{aligned} 1 + \frac{2}{t-1} \times bv + \frac{3-t}{(t-1)^2} \times b^2v^2 + \frac{2t^2 - 10t + 12}{3 \times (t-1)^3} \times b^3v^3 \\ + \frac{2}{t-1} \times cv^2 + \frac{2}{t-1} \times \frac{3-t}{t-1} \times bcv^3 \\ + \frac{2}{t-1} \times dv^3 \\ + \frac{60 - 86t + 40t^2 - 6t^3}{12 \times (t-1)^4} \times b^4 \times v^4 \\ + \frac{3-t}{(t-1)^2} \times \frac{4-2t}{t-1} \times b^3c \times v^4 \\ + \frac{2}{t-1} \times \frac{3-t}{t-1} \times bd \times v^4 \\ + \frac{3-t}{(t-1)^2} \times c^2 \times v^4 \\ + \frac{2}{t-1} \times e \times v^4 \end{aligned}$$

+ &c = the infinite series

$$\begin{aligned} 1 + \frac{2}{t-1} \times bv + \frac{3-t}{(t-1)^2} \times b^2 \times v^2 + \frac{2t^2 - 10t + 12}{3 \times (t-1)^3} \times b^3 \times v^3 \\ + \frac{2}{t-1} \times c \times v^2 + \frac{6-2t}{(t-1)^2} \times bc \times v^3 \\ + \frac{2}{t-1} \times dv^3 \end{aligned}$$

+

$$\begin{aligned}
 & + \frac{60 - 86t + 40t^2 - 6t^3}{12 \times (t-1)^4} \times b^4 \times v^4 \\
 & + \frac{2t^2 - 10t + 12}{(t-1)^3} \times b^2 c \times v^4 \\
 & + \frac{6 - 2t}{(t-1)^2} \times bd \times v^4 \\
 & + \frac{3 - t}{(t-1)^2} \times c^2 \times v^4 \\
 & + \frac{2}{t-1} \times e \times v^4
 \end{aligned}$$

+ &c.

Coroll. 5. Now let the terms of the series $1 + bv + cv^2 + dv^3 + ev^4 + \&c$ be respectively equal to the corresponding terms of the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times v^2 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$.

Then will b be $= \frac{t-1}{2}$, and c will be $= \frac{t-1}{2} \times \frac{t-2}{3} = b \times \frac{t-2}{3}$, and d will be $= \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} = c \times \frac{t-3}{4}$, and e will be $= \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} = d \times \frac{t-4}{5}$.

Let $\frac{t-1}{2}$ be substituted instead of b in the last series of Coroll. 4. And it will become = the infinite series

$$\begin{aligned}
 & 1 + \frac{2}{t-1} \times \frac{t-1}{2} \times v + \frac{3-t}{(t-1)^2} \times \frac{(t-1)^2}{4} \times v^2 \\
 & + \frac{2}{t-1} \times c \times v^2 \\
 & + \frac{2t^2 - 10t + 12}{3 \times (t-1)^3} \times \frac{(t-1)^3}{8} \times v^3 \\
 & + \frac{6-2t}{(t-1)^2} \times \frac{t-1}{2} \times c \times v^3 \\
 & + \frac{2}{t-1} \times d \times v^3
 \end{aligned}$$

+

$$\begin{aligned}
& + \frac{60 - 86t + 40t^2 - 6t^3}{12 \times (t-1)^4} \times \frac{(t-1)^4}{16} \times v^4 \\
& + \frac{2t^2 - 10t + 12}{(t-1)^3} \times \frac{(t-1)^2}{4} \times c \times v^4 \\
& + \frac{6 - 2t}{(t-1)^2} \times \frac{t-1}{2} \times d \times v^4 \\
& + \frac{3-t}{(t-1)} \times c^2 \times v^4 \\
& + \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{3-t}{4} \times v^2 & + \frac{2t^2 - 10t + 12}{24} \times v^3 \\
+ \frac{2}{t-1} \times c \times v^3 & + \frac{3-t}{t-1} \times c \times v^3 \\
& + \frac{2}{t-1} \times d \times v^3 \\
& + \frac{60 - 86t + 40t^2 - 6t^3}{192} \times v^4 \\
& + \frac{2t^2 - 10t + 12}{4 \times (t-1)} \times c \times v^4 \\
& + \frac{3-t}{t-1} \times d \times v^4 \\
& + \frac{3-t}{(t-1)^2} \times c^2 \times v^4 \\
& + \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c, which (by substituting in it's terms, instead of c , it's value $\frac{t-1}{2} \times \frac{t-2}{3}$), becomes = the infinite series

$$\begin{aligned}
1 + v + \frac{3-t}{4} \times v^2 & + \frac{2t^2 - 10t + 12}{24} \times v^3 \\
+ \frac{2}{t-1} \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 & + \frac{3-t}{t-1} \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 \\
& + \frac{2}{t-1} \times d \times v^3
\end{aligned}$$

+

$$\begin{aligned}
 &+ \frac{60 - 86t + 40t^2 - 6t^3}{192} \times v^4 \\
 &+ \frac{2t^2 - 10t + 12}{4 \times t - 1} \times \frac{t - 1}{2} \times \frac{t - 2}{3} \times v^4 \\
 &+ \frac{3 - t}{t - 1} \times d \times v^4 \\
 &+ \frac{3 - t}{t - 1}^2 \times \frac{t - 1}{4} \times \frac{t - 2}{9} \times v^4 \\
 &+ \frac{2}{t - 1} \times e \times v^4
 \end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
 1 + v + \frac{3 - t}{4} \times v^2 &+ \frac{2t^2 - 10t + 12}{24} \times v^3 \\
 + \frac{t - 2}{3} \times v^2 &+ \frac{3 - t}{2} \times \frac{t - 2}{3} \times v^3 \\
 &+ \frac{2}{t - 1} \times d \times v^3 \\
 &+ \frac{60 - 86t + 40t^2 - 6t^3}{192} \times v^4 \\
 &+ \frac{2t^2 - 10t + 12}{4} \times \frac{t - 2}{6} \times v^4 \\
 &+ \frac{3 - t}{t - 1} \times d \times v^4 \\
 &+ \frac{3 - t}{4} \times \frac{t^2 - 4t + 4}{9} \times v^4 \\
 &+ \frac{2}{t - 1} \times e \times v^4
 \end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
 1 + v + \frac{t + 1}{12} \times v^2 &+ \frac{2t^2 - 10t + 12}{24} \times v^3 \\
 &+ \frac{5t - t^2 - 6}{6} \times v^3 \\
 &+ \frac{2}{t - 1} \times d \times v^3
 \end{aligned}$$

+

$$\begin{aligned}
& + \frac{60 - 86t + 40t^2 - 6t^3}{192} \times v^4 \\
& + \frac{2t^3 - 14t^2 + 32t - 24}{24} \times v^4 \\
& + \frac{3-t}{t-1} \times d \times v^4 \\
& + \frac{12 - 16t + 7t^2 - t^3}{36} \times v^4 \\
& + \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{t+1}{12} \times v^2 & + \frac{2t^2 - 10t + 12}{24} \times v^3 \\
& + \frac{20t - 4t^2 - 24}{24} \times v^3 \\
& + \frac{2}{t-1} \times d \times v^3 \\
& + \frac{60 - 86t + 40t^2 - 6t^3}{192} \times v^4 \\
& + \frac{16t^3 - 112t^2 + 256t - 192}{192} \times v^4 \\
& + \frac{3-t}{t-1} \times d \times v^4 \\
& + \frac{192 - 256t + 112t^2 - 16t^3}{576} \times v^4 \\
& + \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{t+1}{12} \times v^2 & + \frac{10t - 2t^2 - 12}{24} \times v^3 \\
& + \frac{2}{t-1} \times d \times v^3 \\
& + \frac{10t^3 - 72t^2 + 170t - 132}{192} \times v^4 \\
& + \frac{3-t}{t-1} \times d \times v^4 \\
& + \frac{192 - 256t + 112t^2 - 16t^3}{576} \times v^4 \\
& + \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

1 +

$$\begin{aligned}
 1 + v + \frac{t+1}{12} \times vv + \frac{10t - 2t^2 - 12}{24} \times v^3 \\
 + \frac{2}{t-1} \times d \times v^3 \\
 + \frac{30t^3 - 216t^2 + 510t - 396}{576} \times v^4 \\
 + \frac{3-t}{t-1} \times d \times v^4 \\
 + \frac{192 - 256t + 112t^2 - 16t^3}{576} \times v^4 \\
 + \frac{2}{t-1} \times e \times v^4
 \end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
 1 + v + \frac{t+1}{12} \times vv + \frac{10t - 2t^2 - 12}{24} \times v^3 \\
 + \frac{2}{t-1} \times d \times v^3 \\
 + \frac{14t^3 - 104t^2 + 254t - 204}{576} \times v^4 \\
 + \frac{3-t}{t-1} \times d \times v^4 \\
 + \frac{2}{t-1} \times e \times v^4
 \end{aligned}$$

+ &c, which (if we substitute in it's terms, instead of d , the value of d , to wit, $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4}$,) becomes = the infinite series

$$\begin{aligned}
 1 + v + \frac{t+1}{12} \times vv + \frac{10t - 2t^2 - 12}{24} \times v^3 \\
 + \frac{2}{t-1} \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 \\
 + \frac{14t^3 - 104t^2 + 254t - 204}{576} \times v^4 \\
 + \frac{3-t}{t-1} \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 \\
 + \frac{2}{t-1} \times e \times v^4
 \end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{t+1}{12} \times vv &+ \frac{10t - 2t^2 - 12}{24} \times v^3 \\
&+ \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 \\
&+ \frac{14t^3 - 104t^2 + 254t - 204}{576} \times v^4 \\
&+ \frac{3-t}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 \\
&+ \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{t+1}{12} \times vv &+ \frac{10t - 2t^2 - 12}{24} \times v^3 \\
&+ \frac{t^2 - 5t + 6}{12} \times v^3 \\
&+ \frac{14t^3 - 104t^2 + 254t - 204}{576} \times v^4 \\
&\quad - \frac{t^3 + 8t^2 - 21t + 18}{24} \times v^4 \\
&+ \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$\begin{aligned}
1 + v + \frac{t+1}{12} \times vv &+ \frac{10t - 2t^2 - 12}{24} \times v^3 \\
&+ \frac{2t^2 - 10t + 12}{24} \times v^3 \\
&+ \frac{14t^3 - 104t^2 + 254t - 204}{576} \times v^4 \\
&\quad - \frac{24t^3 + 192t^2 - 504t + 432}{576} \times v^4 \\
&+ \frac{2}{t-1} \times e \times v^4
\end{aligned}$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + 0 \times v^3 \\ - \frac{10t^3 + 88t^2 - 250t + 228}{576} \times v^4 \\ + \frac{2}{t-1} \times e \times v^4$$

+ &c, (which, if we substitute in it, instead of e , the value of e , to wit, $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5}$), will be = to the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{10t^3 + 88t^2 - 250t + 228}{576} \times v^4 \\ + \frac{2}{t-1} \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{10t^3 + 88t^2 - 250t + 228}{576} \times v^4 \\ + \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{10t^3 + 88t^2 - 250t + 228}{576} \times v^4 \\ + \frac{t^2 - 9t^2 + 26t - 24}{60} \times v^4$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{10t^3 + 88t^2 - 250t + 228}{576} \times v^4 \\ + \frac{96t^3 - 864t^2 + 2496t - 2304}{5760} \times v^4$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{100t^3 + 880t^2 - 2500t + 2280}{5760} \times v^4 \\ + \frac{96t^3 - 864t^2 + 2496t - 2304}{5760} \times v^5$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{4t^3 + 16t^2 - 4t - 24}{5760} \times v^4$$

+ &c = the infinite series

$$1 + v + \frac{t+1}{12} \times vv + * \\ - \frac{t^3 + 4t^2 - t - 6}{1440} \times v + \&c.$$

Art. 14. Thus, after many tedious and troublesome substitutions, we have at last discovered that the $\frac{2}{t-1}$ th power of the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ will be equal to the infinite series $1 + v + \frac{t+1}{12} \times vv +$

$0 \times v^3 - \frac{t^3 + 4t^2 - t - 6}{1440} \times v^4 \&c.$ Therefore $\left[\frac{z}{at} \right]^{\frac{2}{t-1}}$ will be = to the said series $1 + v + \frac{t+1}{12} \times vv + 0 \times v^3 - \frac{t^3 + 4t^2 - t - 6}{1440} \times v^4 \&c.$ and

consequently (subtracting 1 from both sides,) $\left[\frac{z}{at} \right]^{\frac{2}{t-1}} - 1$ will be = the series $v + \frac{t+1}{12} \times vv + 0 \times v^3 - \frac{t^3 + 4t^2 - t - 6}{1440} \times v^4 \&c.$

Art. 15. Now let the term $-\frac{t^3 + 4t^2 - t - 6}{1440} \times v^4$ and all the following terms of this series, included under the mark of &c, be omitted, on account of their smallness in comparison of the two first terms $v + \frac{t+1}{12} \times vv$; than which they must be much less, because, as v is but a small quantity in comparison

rison of 1*l.*, (being only $\frac{5}{100}$, or $\frac{6}{100}$, or $\frac{7}{100}$, or some other such small part of it, according as the interest of money is 5, or 6, or 7, &c per cent,) it is evident that v^4 , and v^5 , v^6 , v^7 , &c, will be much less than v and vv . And we

shall then have $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1 =$, nearly, $v + \frac{t+1}{12} \times vv$, or $v + \frac{t+1}{12} \times vv$,

nearly, $= \left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$.

Put y for the known quantity $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$. And we shall then have $v + \frac{t+1}{12} \times vv = y$; and consequently $\frac{12v}{t+1} + vv = \frac{12}{t+1} \times y$, or $vv + \frac{12}{t+1} \times v = \frac{12}{t+1} \times y$.

Now let b be put $= \frac{6}{t+1}$. And we shall then have $vv + \frac{12}{t+1} \times v = vv + 2b \times v$, and $\frac{12}{t+1} \times y = 2b \times y = 2by$, and consequently $vv + 2b \times v = 2by$. Therefore (adding bb to both sides,) we shall have $vv + 2bv + bb = bb + 2by$, and (extracting the square-roots of both sides,) $v + b = \sqrt{bb + 2by}$, or $\sqrt{bb + 2by}^{\frac{1}{2}}$, and (subtracting b from both sides,) we shall have $v = \sqrt{bb + 2by} - b$, or $\sqrt{bb + 2by}^{\frac{1}{2}} - b$; that is, $r - 1$ will be nearly $= \sqrt{bb + 2by}^{\frac{1}{2}} - b$, and consequently r , or the root of the proposed equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, will be nearly $= 1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$; which is the expression given us for the value of the said root by Dr. Halley. Q. E. I.

The substance of this investigation of the value of r was communicated to me by my learned friend Mr. William Morgan, the Actuary of the Society for Equitable Assurances on Lives in Chatham Square, near Black Friars' Bridge, London.

Art. 16. If t is either equal to, or greater than, 4, the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, obtained in the foregoing articles, will always be less than the true value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; as may be shewn in the manner following.

When

When t is $= 4$, t^3 is $= 4t^2$, and consequently $t^3 + t + 6$ will be greater than $4t^2$; and, when t is greater than 4, t^3 will be greater than $4t^2$, and therefore $t^3 + t + 6$ will, *a fortiori*, be greater than $4t^2$. Therefore, when t is either equal to, or greater than, 4, the four first terms of the series $1 + v + \frac{t+1}{12} \times vv - \frac{t^3 + 4t^2 - t - 6}{1440} \times v^4 + \&c$ will be less than the three first terms of it; the subtraction of $\frac{t^3 + t + 6}{1440} \times v^4$ (or the part of the fourth term $-\frac{t^3 + 4t^2 - t - 6}{1440} \times v^4$ which is marked with the sign $-$, or is to be subtracted from the three former terms $1 + v + \frac{t+1}{12} \times vv$) diminishing the said three terms more than the addition of $\frac{4t^2}{1440} \times v^4$ (or the part of the fourth term $-\frac{t^3 + 4t^2 - t - 6}{1440} \times v^4$ which is marked with the sign $+$, or is to be added to the said three former terms $1 + v + \frac{t+1}{12} \times vv$) will increase them. Therefore in these cases the three first terms $1 + v + \frac{t+1}{12} \times vv$ will be greater than the four first terms $1 + v + \frac{t+1}{12} \times vv - \frac{t^3 + 4t^2 - t - 6}{1440} \times v^4$, and therefore, we may suppose, than the whole series, and consequently than the quantity $\left[\frac{z}{at} \right]^{\frac{2}{t-1}}$, to which the whole series is equal. Therefore, when we suppose the said three terms $1 + v + \frac{t+1}{12} \times vv$ to be equal to $\left[\frac{z}{at} \right]^{\frac{2}{t-1}}$, (as we do in the foregoing investigation,) we suppose them to be less than they really are; and consequently the value of v resulting from that supposition will be less than it's true value; that is, $\overline{bb + 2by}^{\frac{1}{2}} - b$ will be less than the true value of v . And consequently $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$ will be less than the true value of $1 + v$, or than the true value of r . Q. E. D.

Art. 17. The example given by Dr. Halley of the application of this expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$ to the resolution of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or to the finding of the value of r in the said equation, is as follows.

An

An Example of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the foregoing Expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$.

An annuity of 34.4*l.*, being forborn 12 $\frac{1}{2}$ years, amounts to 614.4327*l.* It is required to find the rate of interest allowed.

RESOLUTION.

Here z is = 614.4327, and t is = 12.5, and a is = 34.4.

Therefore $\frac{z}{a}$ is = $\frac{614.4327}{34.4} = 17.861,415,7$, and consequently $\frac{z}{a} - 1$ is = 16.861,415,7. Therefore the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ becomes in this case $17.861,415,7 \times r - r^t = 16.861,415,7$, or (because t is = 12.5) it becomes $17.861,415,7 \times r - r^{12.5} = 16.861,415,7$.

Now, since $\frac{z}{a}$ is = 17.861,415,7, we shall have $\frac{z}{at} (= \frac{z}{a} \times \frac{1}{t} = \frac{z}{a} \times \frac{1}{12.5} = \frac{17.861,415,7}{12.5}) = 1.428,913,2$, the logarithm of which is = 0.155,

005,8. Therefore the logarithm of $\left(\frac{z}{at}\right)^{\frac{2}{t-1}}$ will be $(= 0.155,005,8 \times \frac{2}{t-1}) = \frac{2 \times 0.155,005,8}{t-1} = \frac{0.310,011,6}{t-1} = \frac{0.310,011,6}{12.5-1} = \frac{0.310,011,6}{11.5} = 0.026,957,5$,

which is the logarithm of 1.064,038,9. Therefore $\left(\frac{z}{at}\right)^{\frac{2}{t-1}}$, or $\left(\frac{z}{at}\right)^{\frac{2}{11.5}}$, will

be = 1.064,038,9. Therefore y , or $\left(\frac{z}{at}\right)^{\frac{2}{t-1}} - 1$, or $\left(\frac{z}{at}\right)^{\frac{2}{11.5}} - 1$, will be $(= 1.064,038,9 - 1) = 0.064,038,9$.

Further, b , or $\frac{6}{t+1}$, will be $(= \frac{6}{12.5+1} = \frac{6}{13.5} = 0.444,444,4$. Therefore bb will be $= 0.444,444,4^2 = 0.197,530,860,246,913,6$, and by will be $(= 0.444,444,4 \times 0.064,038,9) = 0.028,461,733,048,716$, and consequently $2by$ will be $(= 2 \times 0.028,461,733,048,716) = 0.056,923,466,097,432$, and $bb + 2by$ will be $(= 0.197,530,860,246,913,6 + 0.056,923,466,097,432) = 0.254,454,326,344,345,6$. Therefore $\sqrt{bb + 2by}$ will be $(= \sqrt{0.254,454,326,344,345,6}) = 0.504,434,6$, and $\sqrt{bb + 2by} - b$ will be $(= 0.504,434,6$

$= 0.444,444,4) = 0.059,990,2$. Therefore v , or $r - 1$, will be very nearly $= 0.059,990,2$; and consequently $1 + v$, or r , will be very nearly $= 1.059,990,2$, or 1.06 , or the interest of money allowed in the case supposed will be 6 per cent. Q. E. I.

This value of v , or $\sqrt{bb + 2by} - b$, or $\overline{bb + 2by}^{\frac{1}{2}} - b$, agrees with the number found for it by Dr. Halley in all but the last figure, Dr. Halley's number being $0.059,990,9$.

Art. 18. This expression $\overline{bb + 2by}^{\frac{1}{2}} - b$, given by Dr. Halley for the value of $r - 1$, comes, in the foregoing example, very near to the true value of $r - 1$, which is 0.06 ; and it seems probable that in most cases it will be pretty near the truth, but less so when t , or the number of years during which the annuity is forborn, is a great number, (as 50 or 60 years,) than when it is a small number; because, when t is a great number, the co-efficients of v^4 , v^5 ,

v^6 , &c in the terms of the infinite series which is equal to $\frac{z}{at} \left| \frac{2}{t-1} \right|$ (into which co-efficients the quantities $\frac{t-1}{2}$, $\frac{t-2}{3}$, $\frac{t-3}{4}$, $\frac{t-4}{5}$, &c. enter,) may, perhaps, be so great as to counter-act the diminution of the said terms arising from the smallness of v^4 , v^5 , v^6 , &c, and make those terms too considerable to be neglected. But in that case a second near value of r , derived from the former near value of it, to wit, $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, by means of one process of Mr. Raphson's method of approximation, in the manner described above in art. 4, will come much nearer to the truth than the said first value did, and, probably, as near to it as need to be desired for any useful purpose.

Art. 19. Though the expression of the near value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, given us by Dr. Halley and investigated in the foregoing articles, is a very useful one, on account of it's near approach to the truth in many cases, yet the difficulty of investigating it by the application of Mr. De Moivre's Multinomial Theorem, and the remoteness and subtlety of the principles on which that investigation is grounded, have induced me to endeavour to find some other expression for a first near value of r that should be more easily attainable, and by the help of simpler and more natural principles. And the result of my inquiries has been the discovery of another expression, which is much more easily obtained than the foregoing one of Dr. Halley, and is near enough to the truth to be highly useful in most cases, and to be in all cases an excellent ground-work, or basis, for a further approach to the true value of r by a process of Mr. Raphson's method of approximation. This expression may be found in the following manner.

The

The Investigation of another Expression for a near Value of r in the Equation

$$\frac{z}{a} \times r - r^t = \frac{z}{a} - 1.$$

Art. 20. It has been shewn above in art. 11, that the fraction $\frac{z}{at}$ is equal to the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms. Therefore (subtracting 1 from both sides) we shall have $\frac{z}{at} - 1 =$ the series $\frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to $t-1$ terms, or $\frac{z-at}{at} =$ the same series $\frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to $t-1$ terms. Therefore, if we multiply both sides of this equation into the fraction $\frac{2}{t-1}$, we shall have $\frac{2z-2at}{at^2-at} =$ the series $v + \frac{t-2}{3} \times vv + \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to $t-1$ terms; and (by multiplying both sides of this last equation into the fraction $\frac{3}{t-2}$, we shall have $(\frac{6z-6at}{at^3-at^2-2at^2+2at}, \text{ or } \frac{6z-6at}{at^3-3at^2+2at}, =$ the series $\frac{3}{t-2} \times v + vv + \frac{t-3}{4} \times v^3 + \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to $t-1$ terms.

Art. 21. Now, because v is $= r - 1$, or the interest of 1 pound for a year, and therefore is much less than 1*l*. (as, for example, $\frac{4}{100}$ or $\frac{5}{100}$, or $\frac{6}{100}$, or some other such small part of it,) it is evident that vv will be much less than v , and that v^3 will be much less than vv , and, *a fortiori*, than the sum of vv and $\frac{3}{t-2} \times v$. Therefore, if we omit the third term $\frac{t-3}{4} \times v^3$ and the fourth

term $\frac{t-3}{4} \times \frac{t-4}{5} \times v^4$, and all the following terms of the foregoing series $\frac{3}{t-2} \times v + vv + \frac{t-3}{4} \times v^3 + \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, on account of their smallness in comparison of the two first terms, $\frac{3}{t-2} \times v$ and vv , of the said series, the said two terms $\frac{3}{t-2} \times v + vv$ will be nearly equal to, though somewhat less than, the known quantity $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$, to which the said series is equal. Let this known quantity, (or rather quantity composed of known quantities combined together by multiplication, addition, and subtraction,) be computed, and it's value be called g . And we shall then have $vv + \frac{3}{t-2} \times v$, nearly, $= g$; and, putting $2e = \frac{3}{t-2}$, or $e = \frac{3}{2t-4}$, we shall have $vv + 2ev = g$. Therefore (adding e^2 to both sides,) we shall have $vv + 2ev + ee = g + ee$, and (extracting the square-roots of both sides,) $v + e = \sqrt{g + ee}$, and (subtracting e from both sides) $v = \sqrt{g + ee} - e$. Therefore $1 + v$, or r , will be, nearly, $= 1 + \sqrt{g + ee} - e$. Q. E. I.

Art. 22. This value of v is always a little greater than the truth, because it results from the supposition that the two first terms $\frac{3}{t-2} \times v + vv$ of the series $\frac{3}{t-2} \times v + vv + \frac{t-3}{4} \times v^3 + \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to $t - 1$ terms, are equal to the whole of the said series, than which they are evidently less. But, if t , or the number of years during which the annuity is forborn, is not a great number, as, for example, if it is only 10, or 12, years, the difference will not be considerable; as will appear by applying this expression to the foregoing example given by Dr. Halley of the resolution of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by his expression of the value of r above investigated, to wit, $1 + \sqrt[bb + 2by]{\frac{1}{2}} - b$. This application of the expression $1 + \sqrt{g + ee} - e$ to the said example may be made in the following manner.

Art. 23. *An Example of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the foregoing Expression of the Value of r that has been just now investigated, to wit, the Expression $1 + \sqrt{g + ee} - e$, in which e is $= \frac{3}{2t - 4}$, and g is $=$ the Fraction $\frac{6z - 6at}{a^3 - 3at^2 - 2at}$.*

A PROBLEM.

An annuity of 34.4*l.*, being forborn for $12\frac{1}{2}$ years, amounts to 614.4327*l.* It is required to find the rate of interest allowed.

SOLUTION.

Here z is $= 614.4327$ l., and a is $= 34.4$ l., and t is $= 12\frac{1}{2}$, or 12.5, years. Therefore $6z$ is $(= 6 \times 614.4327) = 3686.5962$, and at is $(= 34.4 \times 12.5) = 430$, and consequently $6at$ is $(= 6 \times 430) = 2580$, and $6z - 6at$ is $(= 3686.5962 - 2580) = 1106.5962$; and t^2 is $(= 12.5^2) = 156.25$, and t^3 is $(= 12.5^3) = 1953.125$, and at^3 is $(= 34.4 \times 1953.125) = 67,187.5000$; and $3a$ is $(= 3 \times 34.4) = 103.2$, and $3at^2$ is $(= 103.2 \times 156.25) = 16,125$, and $2at$ is $(= 2 \times 430) = 860$, and consequently $at^3 - 3at^2 + 2at$ is $(= 67,187.5000 - 16,125 + 860) = 51,922.5000$. Therefore the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$ will be $= \frac{1106.5962}{51,922.5000} = 0.021,312,4$; that is, g will be $= 0.021,312,4$. And e is $(= \frac{3}{2t - 4} = \frac{3}{2 \times 12.5 - 4} = \frac{3}{2 \times 10.5} = \frac{3}{21} = \frac{1}{7}) = 0.142,857,1$, and ee is $(= \frac{1}{49}) = 0.020,408,0$, and consequently $g + ee$ is $(= 0.021,312,4 + 0.020,408,0) = 0.041,720,4$. Therefore $\sqrt{g + ee}$, or $\sqrt{0.041,720,4}$, will be $(= 0.204,255,7)$, and $\sqrt{g + ee} - e$ will be $(= 0.204,255,7 - 0.142,857,1) = 0.061,398,6$; that is, v , or $r - 1$, will be nearly equal to, but somewhat less than, 0.061,398,6. Therefore r , or $1 + v$, will be nearly equal to, but somewhat less than, 1.061,398,6, or 1.061,398,6. Q. E. I.

This sum 1.061,398,6*l.* is less than 1.0614*l.*, which exceeds 1.06*l.*, or the true value of r , by only 0.0014. And therefore the interest upon 100 pounds for one

year would, according to this value of r obtained by means of the expression $1 + \sqrt{g+ee}^{\frac{1}{2}} - e$, be somewhat less than $(100 \times 0.0614l., \text{ or } 6.14l., \text{ or } 6 \text{ pounds, } 2 \text{ shillings, and ten pence, instead of being exactly } 6 \text{ pounds. This difference of } 2s. 10d. \text{ upon } 6 \text{ pounds seems to be but trifling; and therefore this expression } 1 + \sqrt{g+ee} - e, \text{ or } 1 + \sqrt{g+ee}^{\frac{1}{2}} - e, \text{ (which was obtained so much more easily than the expression } 1 + \sqrt{bb+2by}^{\frac{1}{2}} - b \text{ given by Dr. Halley,) seems to be a pretty good approximation to the true value of } r \text{ in this example.}$

A further Approach to the true Value of r in the foregoing Equation $17.8614r - r^{12.5} = 16.8614$ by a Process of Mr. Raphson's Method of Approximation.

Art. 24. But, if 1.061,398,6 should not be thought to be near enough to the true value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or $17.8614r - r^t = 16.8614$, or $17.8614r - r^{12.5} = 16.8614$, it will, at least, serve for an excellent basis, or ground-work, for a further approach towards it's true value by Mr. Raphson's method of approximation, which may be performed in the following manner.

It has been shewn above, in art. 22, that this value of r , to wit, 1.061,398,6, is somewhat greater than it's true value. And consequently 1.0614 (which is a little greater than 1.061,398,6) will, *a fortiori*, be greater than the true value of r . Let their difference, or the excess of 1.0614 above r , be denoted by the letter w .

Then we shall have $r = 1.0614 - w$, and consequently $17.8614 \times r (= 17.8614 \times 1.0614 - w = 17.8614 \times 1.0614 - 17.8614 \times w) = 18.958,089,96 - 17.8614 \times w$. And $r^{12.5}$ will be $= \overline{1.0614 - w}^{12.5} = (\text{by the binomial theorem}) \text{ to the series } \overline{1.0614}^{12.5} - 12.5 \times \overline{1.0614}^{11.5} \times w + \&c$, in which the value of $\overline{1.0614}^{12.5}$ may be found by means of a Table of Logarithms in the manner following.

The logarithm of 1.0614 is $= 0.025,879,1$. Therefore the logarithm of $\overline{1.0614}^{12.5}$ will be $(= 12.5 \times 0.025,879,1) = 0.323,488,75$, which is the logarithm of 2.106,147,3. Therefore $\overline{1.0614}^{12.5}$ is $= 2.106,147,3$. And consequently $\overline{1.0614}^{11.5}$ will be $(= \frac{\overline{1.0614}^{12.5}}{1.0614} = \frac{2.106,147,3}{1.0614}) = 1.984,310,6$. Therefore $12.5 \times \overline{1.0614}^{11.5}$ will be $(= 12.5 \times 1.984,310,6) = 24.803,882,5$.

Therefore

Therefore $r^{12.5}$ will be $(= \overline{1.0614})^{12.5} - 12.5 \times \overline{1.0614}^{11.5} \times w + \&c)$
 $= 2.106,147,3 - 24.803,882,5 \times w + \&c$, and consequently $17.8614r$
 $- r^t$, or $17.8614r - r^{12.5}$, will be = the compound quantity

$$\left\{ \begin{array}{l} 18.958,089,96 - 17.8614 \times w \\ - 2.106,147,3 + 24.803,882,5 \times w - \&c \\ 16.851,942,66 + 6.942,482,5 \times w - \&c. \end{array} \right\} =$$

But $17.8614r - r^{12.5}$ is = 16.8614.

Therefore $16.851,942,66 + 6.942,482,5 \times w - \&c$ will also be = 16.8614. And consequently $6.942,482,5 \times w - \&c$ will be $(= 16.8614 - 16.851,942,66) = 0.009,457,34$, and w will be = $\frac{0.009,457,34}{6.942,482,5} = 0.001,362,2$.

Therefore r , or $1.0614 - w$, will be $(= 1.0614 - 0.001,362,2) = 1.060,037,8$; that is, the value of r , or the rate of interest sought, will be, nearly; = 1.060,037,8. Q. E. I.

This value of r , or the rate of interest sought, is exceedingly near to it's true value, which is 1.06.

The Investigation of a General Alebräick Expression for the second near Value of r in the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, obtained in Art. 24 by a Process of Mr. Raphson's Method of Approximation, after having obtained a first near Value of it by means of the Expression $1 + \overline{g + ee}^{\frac{1}{t}} - e$.

Art. 25. This second near value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ may be expressed by a general Alebräick expression, which may be found in the manner following.

Let c be put for the first near value of r , which was obtained by means of the expression $1 + \overline{g + ee}^{\frac{1}{t}} - b$, and which will always be greater than it's true value; and let the excess of c above r be denoted by the letter w .

Then, since w is = $c - r$, we shall have $w + r = c$, and $r = c - w$. Therefore $\frac{z}{a} \times r$ will be $(= \frac{z}{a} \times \overline{c - w}) = \frac{z}{a} \times c - \frac{z}{a} \times w$; and r^t will be $= \overline{c - w}^t =$ (by the binomial theorem) to the series $c^t - t \times c^{t-1} \times w + \&c$.

Therefore $\frac{z}{a} \times r - r^t$ will be $(= \frac{z}{a} \times c - \frac{z}{a} \times w - \text{the series } c^t - t \times c^{t-1} \times w + \&c = \frac{z}{a} \times c - \frac{z}{a} \times w - c^t + t \times c^{t-1} \times w - \&c)$
 $= \frac{z}{a} \times c - c^t + t \times c^{t-1} \times w - \frac{z}{a} \times w - \&c.$

But $\frac{z}{a} \times r - r^t$ is $= \frac{z}{a} - 1.$

Therefore $\frac{z}{a} \times c - c^t + t \times c^{t-1} \times w - \frac{z}{a} \times w - \&c$ will also be
 $= \frac{z}{a} - 1.$

But $\frac{z}{a} \times c - c^t$ will be less than $\frac{z}{a} - 1$, as may be shewn in the following manner.

Art. 26. If x be a variable quantity that increases from 0 *ad infinitum*, and n , P , and Q , be certain known quantities, the binomial quantity $Px - x^n$ will at first increase from 0 to a certain finite magnitude, and then decrease from that finite magnitude and become a second time equal to 0, when x^n is become $= Px$, or when $\frac{x^n}{x}$, or x^{n-1} , is become equal to the given quantity P , or x is become $= \sqrt[n-1]{P}$, or $P^{\frac{1}{n-1}}$.

Therefore, if the quantity Q be less than the greatest magnitude of the binomial quantity $Px - x^n$, the said binomial quantity will, at two different instants of time during the increase of x from 0 to $P^{\frac{1}{n-1}}$, become equal to the given quantity Q , namely, once before it attains it's said greatest magnitude, and a second time after it has attained it's said greatest magnitude and is decreasing from it towards 0; and consequently the equation $Px - x^n = Q$ would have two roots, of which the lesser would be less than the value of x at the time that the binomial quantity is of the greatest magnitude, and the greater would be greater than the value of x at that time. And, while x increases from 0 to that intermediate magnitude which it has when the binomial quantity is of the greatest magnitude, and which we will call M , the binomial quantity $Px - x^n$ will increase at the same time that x increases; but, when x is greater than that intermediate quantity called M , the binomial quantity $Px - x^n$ will decrease while x increases.

Art. 27. Now in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which we are now considering, the lesser of it's two roots is 1. For, if r is $= 1$, we shall have $r^t = 1^t$, which is also $= 1$; and consequently $\frac{z}{a} \times r - r^t$ will be $= \frac{z}{a} \times 1 - 1,$

-1 , which is $= \frac{z}{a} - 1$, or the absolute term of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. Therefore 1 is one of the roots of this equation. But this root of this equation is not the root which we are now seeking, or which will answer the conditions of the problem from which the equation was derived; because r , being the sum of 1 pound together with it's interest for a year, must of necessity be greater than 1 £, or 1 . Therefore r , in the present Problem, must stand for the greater of the two roots of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. Therefore, since c is greater than r , it follows, from what has been shewn in the last article, that the binomial quantity $\frac{z}{a} \times c - c^t$ must be less than the binomial quantity $\frac{z}{a} \times r - r^t$, and consequently than, it's equal, the absolute term $\frac{z}{a} - 1$. Q. E. D.

Art. 28. Therefore $\frac{z}{a} \times c - c^t$ may be subtracted from $\frac{z}{a} - 1$, and consequently from the other, or first, side of the equation $\frac{z}{a} \times c - c^t + t \times c^{t-1} \times w - \frac{z}{a} \times w - \&c = \frac{z}{a} - 1$. Let it be so subtracted. And we shall then have $t \times c^{t-1} \times w - \frac{z}{a} \times w - \&c = \frac{z}{a} - 1 - \left[\frac{z}{a} \times c - c^t \right]$.

Now let d be put for the quantity $\frac{z}{a} - 1 - \left[\frac{z}{a} \times c - c^t \right]$, or the excess of $\frac{z}{a} - 1$ above $\frac{z}{a} \times c - c^t$. And we shall then have $t \times c^{t-1} \times w - \frac{z}{a} \times w - \&c = d$, or $t \times c^{t-1} - \frac{z}{a} \times w - \&c = d$. Therefore w will be $= \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, and r , or $c - w$, will be $= c - \frac{d}{t \times c^{t-1} - \frac{z}{a}}$. Q. E. I.

A Computation of the second near Value of r in the numeral Equation $17.8614r^{12.5} = 16.8614$ by means of the foregoing Algebraick Expression

$$c = \frac{d}{t \times c^{t-1} - \frac{z}{a}}$$

Art. 29. We will now compute the value of r , or $c - w$, in the foregoing equation $17.8614r^{12.5} = 16.8614$, by means of this general expression.

$$c = \frac{d}{t \times c^{t-1} - \frac{z}{a}}$$

Now

Now in this equation z is = 614.4327 $\frac{1}{2}$., and a is = 34.4 $\frac{1}{2}$., and t is = 12.5 years, and $\frac{z}{a}$ is = 17.8614, and $\frac{z}{a} - 1$ is = 16.8614, and c is = 1.0614. And by means of these values of z , a , t , and c we are now to compute the value of $c - \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, or of c — the quotient that arises by dividing the excess of $\frac{z}{a} - 1$, or 16.8614, above $\frac{z}{a} \times c - c^t$, or 17.8614 $\times$ 1.0614 = $\overline{1.0614}^{12.5}$, by $t \times c^{t-1} - \frac{z}{a}$, or 12.5 $\times$ $\overline{1.0614}^{11.5}$ — 17.8614.

In the first place $\frac{z}{a} \times c$, or 17.8614 $\times$ 1.0614, is = 18.958,089,96, and the logarithm of 1.0614 is = 0.025,879,1. Therefore the logarithm of $\overline{1.0614}^{12.5}$ will be (= 12.5 $\times$ 0.025,879,1) = 0.323,488,75, which is the logarithm of 2.106,147,3. Therefore c^t , or $\overline{1.0614}^{12.5}$, will be = 2.106,147,3. Therefore $\frac{z}{a} \times c - c^t$, or 17.8614 $\times$ 1.0614 — $\overline{1.0614}^{12.5}$, will be (= 18.958,089,96 — 2.106,147,3) = 16.851,942,66, and $\frac{z}{a} - 1$ — $\left[\frac{z}{a} \times c - c^t \right]$ will be (= 16.8614 — 16.851,942,66) = 0.009,457,34; that is, d is = 0.009,457,34.

And c^{t-1} is (= $\frac{c^t}{c} = \frac{2.106,147,3}{1.0614}$) = 1.984,310,6; and $t \times c^{t-1}$ is = 12.5 $\times$ 1.984,310,6 = 24.803,882,5; and $t \times c^{t-1} - \frac{z}{a}$ is (= 24.803,882,5 — 17.8614) = 6.942,482,5. Therefore $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$ will be = $\frac{0.009,457,34}{6.942,482,5}$ = 0.001,362,2; that is, w will be = 0.001,362,2. Therefore r , or $c - w$, or $c - \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, will be (= 1.0614 — 0.001,362,2) = 1.060,037,8; that is, the second near value of r , obtained by means of the general Algebraïck expression $c - \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, will be = 1.060,037,8, which

which was the number obtained for it before in art. 24 by resolving the equation $17.8614r - r^{12.5} = 16.8614$ by Mr. Raphson's method of approximation.

Q. E. I.

Another Example of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the two foregoing Expressions $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$ and $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$, when t , or the Number of Years during which the Annuity is forborn, is 70 Years.

Art. 30. I will now try the two foregoing expressions of the value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ in a case in which t is a very great number.

Let us therefore suppose the annuity to be, as before, an annuity of 34.4*l.*, or 34*l.* 8*s.* per annum, but the number of years during which it is forborn to be 70 instead of 12 $\frac{1}{2}$, or 12.5; and let the amount z of this annuity at the end of the said 70 years be first computed, upon a supposition that the interest of money is 6 per cent per annum.

The Computation of the Amount z of the proposed Annuity at the End of the 70 Years.

On these suppositions we shall have $a = 34.4$, and $t = 70$, and $r = 1.06$. Therefore (by art. 9) z , or the amount of the said annuity at the end of the said 70 years will be $= \frac{ar^t - a}{r - 1}$, or $a \times \frac{r^t - 1}{r - 1}$, or $34.4 \times \frac{1.06^{70} - 1}{1.06 - 1}$, or $34.4 \times \frac{1.06^{70} - 1}{0.06}$; which expression we must therefore now compute. This may be done in the following manner.

The logarithm of 1.06 is $= 0.025,3059$. Therefore the logarithm of 1.06^{70} will be $(= 70 \times 0.025,3059) = 1.771,4130$, which is the logarithm of

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the number 59.076,260,3. Therefore $\overline{1.06}^{70}$ is = 59.076,260,3. Therefore $\overline{1.06}^{70} - 1$ will be (= 59.076,260,3 - 1) = 58.076,260,3, and $\frac{\overline{1.06}^{70} - 1}{0.06}$ will be (= $\frac{58.076,260,3}{0.06} = \frac{100 \times 58.076,260,3}{100 \times 0.06} = \frac{5807.626,03}{6}$) = 967.937,676,6, and $34.4l. \times \frac{\overline{1.06}^{70} - 1}{0.06}$ will be (= $34.4l. \times 967.937,676,6$) = 33,297.055,903,04l.; that is, z , or the amount fought, will be 33,297.055,903,04l. Q. E. I.

The Computation of r by means of Dr. Halley's Expression, $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$.

Art. 31. Having thus found that, if the interest of money is 6 per cent, the amount of the said annuity of 34.4l. *per annum* forborn during 70 years will be = 33,297.055,903,04, we will now reverse the problem, and suppose that z , or the amount, is known to be = 33,297.055,903,04, and that r , or the rate of interest, is required to be found from it.

Now Dr. Halley's expression for the value of r (which, by the computation of z just now performed, is known to be equal to 1.06) is $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, in which y is = $\left[\frac{z}{at} \right]^{\frac{2}{t-1}} - 1$, and b is = $\frac{6}{t+1}$.

Now z is = 33,297.055,903,04l., and a is = 34.4l., and t is = 70. Therefore at will be (= 34.4×70) = 2408, and consequently $\frac{z}{at}$ will be (= $\frac{33,297.055,903,04l.}{2408}$) = 13.827,681,02l., the logarithm of which is =

1.140,750,1. Therefore the logarithm of $\left[\frac{z}{at} \right]^{\frac{2}{t-1}}$ will be = $\frac{2}{t-1} \times 1.140,750,1$ = 750,1 (= $\frac{2}{70-1} \times 1.140,750,1 = \frac{2}{69} \times 1.140,750,1 = \frac{2.281,500,2}{69}$) = 0.033,065,2, which is the logarithm of the number 1.079,108,7. Therefore

$\left[\frac{z}{at} \right]^{\frac{2}{t-1}}$, or $\left[\frac{z}{at} \right]^{\frac{2}{69}}$, will be = 1.079,108,7; and consequently y , or $\left[\frac{z}{at} \right]^{\frac{2}{t-1}} - 1$, will be = 0.079,108,7.

And

And b will be $= \frac{6}{t+1}$ ($= \frac{6}{70+1} = \frac{6}{71}$) $= 0.084,507,0$, and $2b$ will be ($= 2 \times 0.084,507,0$) $= 0.169,014,0$, and $2by$ will be ($= 0.169,014,0 \times 0.079,108,7$) $= 0.013,370,477,821,80$, and bb will be ($= 0.084,507,0^2$) $0.007,141,433,049,00$. Therefore $bb + 2by$ will be ($= 0.007,141,433,049,00 + 0.013,370,477,821,80$) $= 0.020,511,910,870,80$, and consequently $\sqrt{bb + 2by}$ will be ($= \sqrt{0.020,511,910,870,80}$) $= 0.143,219,7$, and $\sqrt{bb + 2by} - b$ will be ($= 0.143,219,7 - 0.084,507,0$) $= 0.058,712,7$. Therefore $1 + \sqrt{bb + 2by} - b$, or $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, will be $= 1.058,712,7$; that is, r will be $= 1.058,712,7$. Q. E. I.

Art. 32. This number $1.058,712,7$, obtained by Dr. Halley's expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, is tolerably near the true value of r even in this case, in which the number of years during which the annuity is forborn is so great a number as 70. For the true value of r is 1.06 ; so that, upon the sum of 100 pounds, the interest for one year, as given by this expression, would be ($= 100 \times 0.0587$) $= 5.87\%$, or $5\text{ l. } 17\text{ s. } 5\text{ d.}$, instead of 6 l. , of which it falls short only by the small sum of $2\text{ s. } 7\text{ d.}$ We may conclude therefore that this expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, given us by Dr. Halley for the value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ will, in all magnitudes of t that are not greater than 70, be a very useful approximation to the true value of r , though more so when t , (or the number of years during which the annuity is forborn,) is small, than when it is great.

A further Approximation to the true Value of r by means of the Correction of the former Expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, which is given by Dr. Halley, and is set forth above in Art. 2 and 3.

Art. 33. But, if to this first near value of r , to wit, $1.058,712,7$, (obtained by means of the expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, we add the correction given above by Dr. Halley in art. 2 and more clearly expressed in art. 3, to wit, the fraction $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$, (in which c stands for the first expression

$1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, and d for the excess of $\frac{z}{a} \times c - c^t$ above $\frac{z}{a} - 1$) the

second near value of r that will be thereby obtained, to wit, $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$
 $+ \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, or $c + \frac{d}{t \times c^{t-1} - \frac{z}{a}}$, or $1.058,712,7 + \frac{d}{t \times c^{t-1} - \frac{z}{a}}$,

will, even in this last case, (in which t is $= 70$,) be as near to the true value of r , or to 1.06 , as need to be desired; as will appear by computing the said correction in the following manner.

Since c is $= 1.058,712,7$, and t is $= 70$, c^t will be $= \sqrt[70]{1.058,712,7}^{70}$. Now the logarithm of $1.058,712,7$ is $= 0.024,778,1$; and consequently the logarithm of $\sqrt[70]{1.058,712,7}$ is $(= 70 \times 0.024,778,1) = 1.734,467,0$, which is the logarithm of the number $54.258,400,0$. Therefore $\sqrt[70]{1.058,712,7}$ will be $= 54.258,400,0$; that is, c^t will be $= 54.258,400,0$.

And, since c^t is $= 54.258,400,0$, we shall have $c^{t-1} (= \frac{c^t}{c} = \frac{54.258,400,0}{1.058,712,7}) = 51.249,408,8$, and $t \times c^{t-1} (= 70 \times 51.249,408,8) = 3587.458,616,0$.

And, since z is $= 33,297.055,903,04$, and a is $= 34.4$, we shall have $\frac{z}{a} (= \frac{33,297.055,903,04}{34.4}) = 967.937,671,6$, and consequently $\frac{z}{a} - 1 (= 967.937,671,6 - 1) = 966.937,671,6$.

Therefore $t \times c^{t-1} - \frac{z}{a}$ will be $(= 3587.458,616,0 - 967.937,671,6) = 2619.520,944,4$; and $\frac{z}{a} \times c$ will be $(= 967.937,671,6 \times 1.058,712,7) = 1024.767,905,731,349,32$, and $\frac{z}{a} \times c - c^t$ will be $(= 1024.767,905,731,349,32 - 54.258,400,0) = 970.509,505,731,349,32$. Therefore $\frac{z}{a} \times c - c^t - \left[\frac{z}{a} - 1 \right]$ will be $(= 970.509,505,731,349,32 - 966.937,671,6) = 3.571,834,131,349,32$; that is, d will be $= 3.571,834,131,349,32$.

Therefore $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$ will be $= \frac{3.571,834,131,349,32}{2619.520,944,4} = 0.001,363,5$;

and consequently $c + \frac{d}{t \times c^{t-1} - \frac{z}{a}}$ will be $(= 1.058,712,7 + 0.001,363,5)$

$= 1.060,076,2$; that is, the second near value of r , which is obtained by means

means of Dr. Halley's former expression $1 + \sqrt[2]{bb + 2by} - b$ together with this correction of it, to wit, $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$, will be $= 1.060,076,2$, which

exceeds the true value of it, to wit, 1.06, by only the small difference 0.000,076,2, which, upon the interest of 100 pounds for one year, or 6 pounds, would amount to only $(100 \times 0.000,076,2)$, or 0.007,62 of a pound, or something less than two pence. This seems to be sufficiently near the truth for all useful purposes; and therefore I conclude that this expression of Dr. Halley, together with it's correction, will always give us the value of r in the equation $\frac{z}{a} \times r - c^t = \frac{z}{a} - 1$ to a sufficient degree of exactness when t , (or the number of years during which the annuity is forborn,) is a great number (such as 70, or 72, or 75, or, perhaps, any number not greater than 80,) as well as when it is a small number. And, if t is a small number, not exceeding 20, the said expression alone, without the correction, will be sufficiently exact.

The Computation of r by means of the Expression $1 + \sqrt[2]{g + ee} - e$.

Art. 34. I will now investigate the value of r in this last example (in which t is $= 70$ years,) by means of the other expression given above in art. 20 and 21, to wit, the expression $1 + \sqrt[2]{g + ee} - e$, or $1 + \sqrt[2]{g + ee} - e$, in which g stands for the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$, and e stands for $\frac{3}{2t - 4}$, in order to discover how near the value of r , obtained by means of that expression, will, in this difficult case, approach to it's true value, which we know to be 1.06.

Now, since a is $= 34.4$, and t is $= 70$, and z is $= 33,297.055,903,04$, we shall have at ($= 34.4 \times 70$) $= 2408$, and $6at$ ($= 6 \times 2408$) $= 14,448$, and $6z$ ($= 6 \times 33,297.055,903,04$) $= 199,782.335,418,24$, and $6z - 6at$ ($= 199,782.335,418,24 - 14,448$) $= 185,334.335,418,24$; that is, the numerator of the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$ will be $= 185,334.335,418,24$.

And at^3 will be ($= at \times t^2 = 2408 \times 70^2 = 2408 \times 4900$) $= 11,799,200$, and at^2 will be ($= at \times t = 2408 \times 70$) $= 168,560$, and $3at^2$ will be ($= 3 \times 168,560$) $= 505,680$, and $2at$ will be ($= 2 \times 2408$) $= 4816$; and consequently $at^3 - 3at^2 + 2at$ will be ($= 11,799,200 - 505,680 + 4816$) $= 11,298,336$; that is, the denominator of the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$ is $= 11,298,336$.

Therefore

Therefore the said fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$ will be $= \frac{185,334,335,418,24}{11,298,336} = 0.016,403,68$; that is, g will be $= 0.016,403,68$.

Further, e is $= \frac{3}{2t - 4}$ ($= \frac{3}{2 \times 70 - 4} = \frac{3}{140 - 4} = \frac{3}{136}$) $= 0.022,058,8$.
Therefore ee will be $(= 0.022,058,8)^2 = 0.000,486,590,657,44$.

Therefore $g + ee$ will be $(= 0.016,403,68 + 0.000,486,590,657,44) = 0.016,890,270,657,44$, and $\sqrt{g + ee}$ will be $(= \sqrt{0.016,890,270,657,44}) = 0.129,962,5$, and $\sqrt{g + ee} - e$ is $= 0.129,962,5 - 0.022,058,8 = 0.107,903,7$. Therefore $1 + \sqrt{g + ee} - e$, or the value of r , as obtained by this expression, will be $1 + 0.107,903,7$, or $1.107,903,7$. Q. E. I.

Art. 35. This value of r is so much greater than it's true value, 1.06 , that we may justly conclude from it that the expression $1 + \sqrt{g + ee} - e$, will be of little, or no, use in resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, when t , or the number of years during which the annuity is forborn, is so great a number as 70 . But, when t is nearly equal to 12 , we have seen above in art. 23 that this expression gives the value of r sufficiently exact for most purposes: and I therefore conjecture that it may safely be employed when t is not greater than 20 . And, if we correct it in the manner described in art. 24, 25, &c - - 29, by putting $c = 1 + \sqrt{g + ee} - e$, and subtracting $\frac{z}{a} \times c - c^t$ from $\frac{z}{a} - 1$, and putting d for the remainder thence arising, and, lastly, by subtracting from c , or $1 + \sqrt{g + ee} - e$, (which is always greater than the true value of r ,) the fraction $\frac{d}{t \times c^{t-1} - \frac{z}{a}}$, it will, probably,

give us the value of r , or the rate of interest sought, with sufficient exactness, even when t is greater than 20 , and of any magnitude not greater than 30 . But this can only be ascertained by trying it in other examples, in which t is taken of different magnitudes, such as $20, 30, 40, 50$, &c between $12\frac{1}{2}$ and 70 , in which two magnitudes of t it has been tried in the foregoing articles. But the trials of Dr. Halley's expression $1 + \overline{bb + 2by}^{\frac{1}{2}} - b$, that have been made above, prove that that expression is in all cases more exact than the expression $1 + \sqrt{g + ee} - e$, or $1 + \overline{g + ee}^{\frac{1}{2}} - e$, and likewise that it is sufficiently exact for most purposes even when t is so large a number as 70 , and that, with the correction of it given by Dr. Halley in art. 2, it becomes much more exact than before, and as exact, even in those difficult cases, as any calculator need desire.

Art. 36.

Art. 36. Before I quit the consideration of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, I will give one more example of the resolution of it by the two foregoing expressions, $1 + \sqrt[bb + 2by]{\frac{1}{2}} - b$ and $1 + \sqrt[g + ee]{\frac{1}{2}} - e$, when the interest of money is much lower than 6 per cent, and t , or the number of years during which the annuity is forborn, is but a small number.

*Another Example of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by the two foregoing Expressions $1 + \sqrt[bb + 2by]{\frac{1}{2}} - b$ and $1 + \sqrt[g + ee]{\frac{1}{2}} - e$, when the Interest of Money is only 3*l.* 6*s.* 8*d.*, or 3.333,333, &c, per Cent.*

Let the interest of money be 3*l.* + $\frac{1}{3}$ *l.*, or 3*l.* 6*s.* 8*d.* per cent *per annum*, and let an annuity of 10 pounds a year be forborn for 10 years. And let it, first, be required to find the amount of the said annuity at the end of the said 10 years.

Here 100*l.* increases in the course of one year to 100*l.* + 3*l.* + $\frac{1}{3}$ *l.*, or to 103*l.* $\frac{1}{3}$ *l.* Therefore 1*l.* will increase in the same time to $\frac{100 \frac{1}{3}}{100} \text{ l.} + \frac{1}{300} \text{ l.}$, or to 1.03 + 0.003,333,333, &c, or to 1.033,333,333, &c; that is, r will be = 1.033,333,333, &c. And a is 10*l.*; and t is = 10 years. Therefore z , or the amount of the said annuity at the end of the said 10 years, (which is universally equal to $a \times \frac{r^t - 1}{r - 1}$), will be ($= 10 \text{ l.} \times \frac{1.033,333,3 \text{ \&c}^{10} - 1}{1.033,333,3 \text{ \&c} - 1} = 10 \text{ l.} \times \frac{1.033,333,3 \text{ \&c}^{10} - 1}{0.033,333,3 \text{ \&c}} = 10 \text{ l.} \times \frac{1.033,333, \text{ \&c}^{10} - 1}{\frac{1}{30}} = 10 \text{ l.} \times$

$$1.033,333,3 \text{ \&c}^{10} - 1 \times \frac{30}{1}) = 300 \text{ l.} \times \sqrt[1.033,333,3 \text{ \&c}^{10}]{1.033,333,3 \text{ \&c}^{10} - 1}.$$

Now the logarithm of 1.033,333,3 is = 0.014,240,4. Therefore the logarithm of $1.033,333,3^{10}$ will be ($= 10 \times 0.014,240,4$) = 0.142,404,0, which is the logarithm of the number 1.388,046,3. Therefore $1.033,333,3^{10}$ is = 1.388,048,3, and consequently $1.033,333,3^{10} - 1$ is ($= 1.388,046,3 - 1$) = 0.388,046,3. Therefore $300 \text{ l.} \times 1.033,333,3^{10} - 1$ will be = $300 \text{ l.} \times 0.388,$

0.388,046,3 = 116.413,890,0l.; that is, the amount, z , of the annuity of 10 pounds *per annum* at the end of the 10 years during which it will have been forborn, will be 116.413,890,0l., or 116l. 8s. 3d. Q. E. I.

Art. 37. Having thus found that, if the interest of money is $3\frac{1}{3}\%$ per cent *per annum*, the amount of the said annuity of 10l. *per annum*, forborn during 10 years, will be 116.413,890,0l., or 116l. 8s. 3d., we will now reverse the problem, and suppose that z , or the amount of the said annuity, is known to be = 116.413,890,0l., and that r , or the rate of interest, is required to be found from it.

Now Dr. Halley's expression for the value of r is $1 + \sqrt[2]{bb + 2by} - b$, in which y is $= \left(\frac{z}{at} \right)^{\frac{2}{t-1}} - 1$, and b is $= \frac{6}{t+1}$.

Now z is, in this case, = 116.413,890,0l., and a is = 10l., and t is = 10 years. Therefore at will be ($= 10 \times 10$) = 100, and $\frac{z}{at}$ will be ($= \frac{116.413,890,0}{100}$) = 1.164,138,900; the logarithm of which is = 0.066,004,8.

Therefore the logarithm of $\left(\frac{z}{at} \right)^{\frac{2}{t-1}}$ will be $\frac{2}{t-1} \times 0.066,004,8$ ($= \frac{2}{10-1} \times 0.066,004,8 = \frac{2}{9} \times 0.066,004,8 = \frac{0.132,009,6}{9}$) = 0.014,667,7; which is

the logarithm of the number 1.034,349. Therefore $\left(\frac{z}{at} \right)^{\frac{2}{t-1}}$ is = 1.034,349,

and consequently $\left(\frac{z}{at} \right)^{\frac{2}{t-1}} - 1$ will be ($= 1.034,349 - 1$) = 0.034,349; that is, y will be = 0.034,349.

Further, b is $= \frac{6}{t+1}$ ($= \frac{6}{10+1} = \frac{6}{11}$) = 0.545,454,5 &c, and bb is ($= \frac{36}{121}$) = 0.297,520,6, and $2by$ is ($= 2 \times \frac{6}{11} \times 0.034,349 = \frac{12 \times 0.034,349}{11} = \frac{0.412,188}{11}$) = 0.037,471,6. Therefore $bb + 2by$ will be ($= 0.297,520,6 + 0.037,471,6$) = 0.334,992,2, and $\sqrt[2]{bb + 2by}$, or $\sqrt{bb + 2by}$, will be ($= \sqrt{0.334,992,2}$) = 0.578,785,1, and $\sqrt[2]{bb + 2by} - b$ will be ($= 0.578,785,1 - 0.545,454,5$) = 0.033,330,6, and $1 + \sqrt[2]{bb + 2by} - b$ will be ($= 1 + 0.033,330,6$) = 1.033,330,6; that is, r will be = 1.033,330,6.

Q. E. I.

This

This value of r , obtained by the expression $1 + \sqrt[2]{bb + 2by} - b$, given us by Dr. Halley, is wonderfully near the true value of r , which is 1.033,333,3 &c. For, according to this value of r , the interest of 100 pounds for one year would be $(= 100 \times 0.033,330,6l.) = 3,333,06l.$, or 3*l.* 6*s.* 7*d.* and $\frac{2}{1000}$ of a penny, or more than 3*l.* 6*s.* 7 $\frac{1}{2}$ *d.*, which falls short of the true value of that interest (which is 3*l.* 6*s.* 8*d.*) by less than one farthing. This expression therefore in this case may be considered as perfectly exact for every useful purpose.

Art. 38. I will now find the value of r in this example by means of the other expression $1 + \sqrt{g + ee} - e$, investigated above in art. 20; in which expression g is = the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$, and e is = $\frac{3}{2t - 4}$.

Now, since a is = 10*l.*, and t is = 10 years, and z is = 116.413,890,0*l.*, we shall have at ($= 10 \times 10$) = 100, and $6at$ = 600, and $6z - 6at$ ($= 6 \times 116.413,890,0 - 600 = 698.483,340,0 - 600$) = 98.483,340,0; that is, the numerator of the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$, (which is equal to g), will be = 98.483,340,0. And t^2 will be ($= 10^2$) = 100, and t^3 will be ($= 10^3$) = 1000, and at^3 will be ($= 10 \times 1000$) = 10,000, and $3at^2$ will be ($= 3 \times a \times 100 = 3 \times 10 \times 100 = 3 \times 1000$) = 3000, and $2at$ will be ($= 2 \times 100$) = 200; and consequently $at^3 - 3at^2 + 2at$ will be ($= 10,000 - 3000 + 200 = 7000 + 200$) = 7200; that is, the denominator of the fraction $\frac{6z - 6at}{at^3 - 3at^2 + 2at}$ will be = 7,200. Therefore the said fraction will be = $\frac{98.483,340,0}{7200} = 0.013,678,2$; that is, g will be = 0.013,678,2.

And e is = $\frac{3}{2t - 4}$ ($= \frac{3}{2 \times 10 - 4} = \frac{3}{20 - 4} = \frac{3}{16}$) = 0.1875, and consequently ee is ($= 0.1875^2$) = 0.035,156,25. Therefore $g + ee$ will be ($= 0.013,678,2 + 0.035,156,25$) = 0.048,834,45, and $\sqrt{g + ee}$ will be ($= \sqrt{0.048,834,45}$) = 0.220,985,1, and $\sqrt{g + ee} - e$ will be ($= 0.220,985,1 - 0.1875$) = 0.033,485,1. Therefore $1 + \sqrt{g + ee} - e$, or $1 + \sqrt{g + ee} - e$, will be ($= 1 + 0.033,485,1$) = 1.033,485,1; that is, the value of r , obtained by means of the expression $1 + \sqrt{g + ee} - e$, will be = 1.033,485,1.

Q. E. I.

This value of r is a little greater than it's true value, which is 1.033,333,3 &c. But the difference is so small as not to be worth attending to. For the interest of an hundred pounds for one year would, according to this expression, be = $(100 \times 0.033,485,1l.)$, or 3.348,51*l.*, or 3*l.* 6*s.* 11 $\frac{1}{2}$ *d.*, which is only three pence, half-penny, more than 3*l.* 6*s.* 8*d.*, or the interest that would have been due upon it if the value of r had been obtained with perfect exactness.

Therefore in this case this expression seems to be almost as exact as Dr. Halley's expression. And it certainly has the advantage of having been investigated with much greater ease and perspicuity.

Remarks on the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the Expression $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$, together with some Processes of Mr. Raphson's Method of Approximation, when t , or the Number of Years during which the Annuity is forborn, is some great Number, such as 70.

Art. 39. Further, if we make use of the expression $1 + \sqrt{g + ee} - e$, or $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$, as a first near value of r , that is greater than it's true value, and derive from it one or more nearer values of r by means of one or more processes of Mr. Raphson's method of approximation, we may always obtain at last as near a value of r as we shall desire. But the misfortune of this way of proceeding is, that it is in some cases, when t , (or the number of years during which the annuity is forborn,) is very great, as, for example, 70 or 80 years, exceedingly tedious and laborious, from the number of processes which will be necessary to be performed in order to arrive at the wished-for degree of exactness. Thus, for example, if, after having found by the computation in art. 34, that $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$, or the first near value of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or $967.937,671,6 \times r - r^{70} = 966.937,671,6$, is $= 1.107,903,7$, (which is much greater than it's true value, 1.06,) we were to endeavour to find a nearer value of r by means of Mr. Raphson's method of approximation, by putting w for the excess of 1.107,903,7, or 1.108, above r , and substituting $1.108 - w$ instead of r in the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, and resolving the transformed equation thereby obtained, as if it were a mere simple equation, we should find w to be $= 0.0147$, and consequently $1.108 - w$, or the second near value of r , to be $= (1.108 - 0.0147)$, or 1.0933 , which is still much greater than 1.06, or the true value of r ; and a second process of Mr. Raphson's method of approximation would give us $(1.0933 - 0.0132)$, or 1.0801 for a third near value of r ; and a third process of the same method would give us $(1.0801 - 0.0106)$, or 1.0695 for a fourth near value of r ; and a fourth process of the same method would give us $(1.0695 - 0.0067)$, or 1.0628 for a fifth near value of r ; and a fifth process of the same method would give us $(1.0628 - 0.0024)$, or 1.0604 for a sixth near value of r ; and a sixth process of the same method would give us $(1.0604 - 0.00039)$, or 1.06001 for a seventh near value of r , which is so very little greater than it's true value 1.06, that it may be considered as quite exact. So that six processes of Mr. Raphson's method

method of approximation would be necessary to be gone through, besides the computation of the original expression $1 + g + ee^{\frac{1}{2}} - e$, to obtain the value of r , or the root of the proposed equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$ to the wished-for degree of exactness. These several processes will be as follows.

The First Process of Mr. Raphson's Method of Approximation for finding the Value of r in the Equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, after it has been found that $1.107,903,7$, or 1.108 , is a first near Value of it greater than the truth.

Art. 40. Let w be put for the excess of 1.108 above r .

And we shall then have $r = 1.108 - w$, and consequently $967.937,671,6 \times r$
 $(= 967.937,671,6 \times 1.108 - w = 967.937,671,6 \times 1.108 - 967.937,671,6 \times w) = 1072.474,940,132,8 - 967.937,671,6 \times w$.

And r^{70} will be $(= 1.108 - w)^{70} =$, by the binomial theorem, $1.108^{70} - 70 \times 1.108^{69} \times w + \&c = 1.108^{70} - 70 \times \frac{1.108^{70}}{1.108} \times w + \&c = 1311.553 - 70 \times \frac{1311.553}{1.108} \times w + \&c = 1311.553 - 70 \times 1183.712 \times w + \&c = 1311.553 - 82,859.840 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be = the compound quantity

$$\left\{ \begin{array}{l} 1072.474,940,132,8 - 967.937,671,6 \times w \\ - 1311.553 + 82,859.840 \times w - \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 1072.474,940,132,8 \\ - 1311.553 + 81,891.902,328,4 \times w - \&c. \end{array} \right.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is = $966.937,671,6$.

Therefore the compound quantity $1072.474,940,132,8 - 1311.553 + 81,891.902,328,4 \times w - \&c$ will also be = $966.937,671,6$.

Therefore (adding 1311.553 to both sides) we shall have $1072.474,940,132,8 + 81,891.902,328,4 \times w - \&c = 2278.490,671,6$, and (subtracting

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1072.474,940,132,8 from both sides) $81,891.902,328,4 \times w - \&c = 1206.015,731,467,2$. Therefore $w - \&c$ will be $= \frac{1206.015,731,467,2}{81,891.902,328,4} = 0.0147$, and w will be $= 0.0147 + \&c$. Therefore r , or $1.108 - w$ will be $(= 1.1080 - 0.0147 - \&c) = 1.0933 - \&c$, or will be less than 1.0933 ; that is, the second near value of r , or of the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.0933 - \&c$, or will be less than 1.0933 . Q. E. I.

The Second Process of Mr. Raphson's Method of Approximation.

Art. 41. Let w be put for the excess of 1.0933 above r .

Then will r be $= 1.0933 - w$; and consequently $967.937,671,6 \times r$ will be $(= 967.937,671,6 \times 1.0933 - w = 967.937,671,6 \times 1.0933 - 967.937,671,6 \times w) = 1058.246,256,360,28 - 967.937,671,6 \times w$.

And r^{70} will be $(= \overline{1.0933 - w})^{70} =$, by the binomial theorem, $\overline{1.0933}^{70} - 70 \times \overline{1.0933}^{69} \times w + \&c = \overline{1.0933}^{70} - 70 \times \frac{\overline{1.0933}^{70}}{1.0933} \times w + \&c = 514.9333 - 70 \times \frac{514.9333}{1.0933} \times w + \&c = 514.9333 - 70 \times 470.9899 \times w + \&c) = 514.9333 - 32,969.2930 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1058.246,256,360,28 - 967.937,671,6 \times w \\ - 514.9333 + 32,969.2930 \times w - \&c \end{array} \right\} = 543.312,956,360,28 + 32,001.355,328,4 \times w - \&c.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is $= 966.937,671,6$.

Therefore the compound quantity $543.312,956,360,28 + 32,001.355,328,4 \times w - \&c$ will also be $= 966.937,671,6$.

Therefore (subtracting $543.312,956,360,28$ from both sides) we shall have $32,001.355,328,4 \times w - \&c = 423.624,715,239,72$, and consequently $w - \&c (= \frac{423.624,715,239,72}{32,001.355,328,4}) = 0.0132$, and $w = 0.0132 + \&c$. Therefore r , or $1.0933 - w$, will be $(= 1.0933 - 0.0132 - \&c) = 1.0801 - \&c$,
or

or will be less than 1.0801; that is, the third near value of r , or the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.0801 - \&c$, or will be less than 1.0801. Q. E. I.

The Third Process of Mr. Raphson's Method of Approximation.

Art. 42. Let w be put for the excess of 1.0801 above r :

Then will r be $= 1.0801 - w$; and consequently $967.937,671,6 \times r$ will be $(= 967.937,671,6 \times 1.0801 - w = 967.937,671,6 \times 1.0801 - 967.937,671,6 \times w) = 1045.469,479,095,16 - 967.937,671,6 \times w$.

And r^{70} will be $(= 1.0801 - w)^{70} =$, by the binomial theorem, $1.0801^{70} - 70 \times 1.0801^{69} \times w + \&c = 1.0801^{70} - 70 \times \frac{1.0801^{70}}{1.0801} \times w + \&c = 220.0290 - 70 \times \frac{220.0290}{1.0801} \times w + \&c = 220.0290 - 70 \times 203.7116 \times w + \&c = 220.0290 - 14,259.8120 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1045.469,479,095,16 - 967.937,671,6 \times w \\ - 220.029,0 + 14,259.8120 \times w - \&c \end{array} \right\} \\ = 825.440,479,095,16 + 13,291.874,328,4 \times w - \&c.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is $= 966.937,671,6$.

Therefore the compound quantity $825.440,479,095,16 + 13,291.874,328,4 \times w - \&c$ will also be $= 966.937,671,6$.

Therefore (subtracting $825.440,479,095,16$ from both sides) we shall have $13,291.874,328,4 \times w - \&c = 141.497,192,504,84$, and consequently $w - \&c (= \frac{141.497,192,504,84}{13,291.874,328,4}) = 0.0106$, and $w = 0.0106 + \&c$. Therefore r , or $1.0801 - w$, will be $(= 1.0801 - 0.0106 - \&c) = 1.0695 - \&c$, or will be less than 1.0695; that is, the fourth near value of r , or the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.0695 - \&c$, or will be less than 1.0695. Q. E. I.

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The Fourth Process of Mr. Raphson's Method of Approximation.

Art. 43. Let w be put for the excess of 1.0695 above r .

Then will r be $= 1.0695 - w$; and consequently $967.937,671,6 \times r$ will be $(= 967.937,671,6 \times 1.0695 - w = 967.937,671,6 \times 1.0695 - 967.937,671,6 \times w) = 1035.209,339,776,20 - 967.937,671,6 \times w$.

And r^{70} will be $(= \overline{1.0695 - w}^{70} =, \text{ by the binomial theorem, } \overline{1.0695}^{70} - 70 \times \overline{1.0695}^{69} \times w + \&c = \overline{1.0695}^{70} - 70 \times \frac{\overline{1.0695}^{70}}{1.0695} \times w + \&c$
 $= 110.3204 - 70 \times \frac{110.3204}{1.0695} \times w + \&c = 110.3204 - 70 \times 103.1513 \times w + \&c) = 110.3204 - 7220.5910 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{rcl} 1035.209,339,776,20 & - & 967.937,671,6 \times w \\ - 110.3204 & + & 7220.5910 \times w - \&c \end{array} \right\}$$

$$= 924.888,939,776,20 + 6252.653,328,4 \times w - \&c.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is $= 966.937,671,6$.

Therefore the compound quantity $924.888,939,776,20 + 6252.653,328,4 \times w - \&c$ will also be $= 966.937,671,6$.

Therefore $6252.653,328,4 \times w - \&c$ will be $(= 966.937,671,6 - 924.888,939,776,20) = 42.048,731,823,80$, and consequently $w - \&c$ will be $(= \frac{42.048,731,823,80}{6252.653,328,4}) = 0.0067$, and w will be $= 0.0067 + \&c$.

Therefore r , or $1.0695 - w$, will be $(= 1.0695 - 0.0067 - \&c) = 1.0628 - \&c$, or will be less than 1.0628; that is, the fifth near value of r , or the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.0628 - \&c$, or will be less than 1.0628.

Q. E. I.

The Fifth Process of Mr. Raphson's Method of Approximation.

Art. 44. Let w be put for the excess of 1.0628 above r .

Then will r be $= 1.0628 - w$; and consequently $967.937,671,6 \times r$ will be $(= 967.937,671,6 \times \overline{1.0628 - w} = 967.937,671,6 \times 1.0628 - 967.937,671,6 \times w) = 1028.724,157,376,48 - 967.937,671,6 \times w$.

And r^{70} will be $(= \overline{1.0628 - w}^{70} =, \text{ by the binomial theorem, } \overline{1.0628}^{70} - 70 \times \overline{1.0628}^{69} \times w + \&c = \overline{1.0628}^{70} - 70 \times \frac{\overline{1.0628}^{70}}{1.0628} \times w + \&c$
 $= 71.056,69 - 70 \times \frac{71.056,69}{1.0628} \times w + \&c = 71.056,69 - 70 \times 66.858,007 \times w + \&c) = 71.056,69 - 4680.060,490 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{ll} 1028.724,157,376,48 & - 967.937,671,6 \times w \\ - 71.056,69 & + 4680.060,490 \times w - \&c \end{array} \right\}$$

$$= 957.667,467,376,48 + 3712.122,818,4 \times w - \&c.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is $= 966.937,671,6$.

Therefore the compound quantity $957.667,467,376,48 + 3712.122,818,4 \times w - \&c$ will also be $= 966.937,671,6$.

Therefore $3712.122,818,4 \times w - \&c$ will be $(= 966.937,671,6 - 957.667,467,376,48) = 9.270,204,223,52$, and consequently $w - \&c$ will be $(= \frac{9.270,204,223,52}{3712.122,818,4}) = 0.0024$, and w will be $= 0.0024 + \&c$.

Therefore r , or $1.0628 - w$, will be $(= 1.0628 - 0.0024 - \&c) = 1.0604 - \&c$, or will be less than 1.0604; that is, the sixth near value of r , or the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.0604 - \&c$, or will be less than 1.0604.

Q. E. I.

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The Sixth Process of Mr. Raphson's Method of Approximation.

Art. 45. Let w be put for the excess of 1.0604 above r .

Then will r be $= 1.0604 - w$; and consequently $967.937,671,6 \times r$ will be $(= 967.937,671,6 \times 1.0604 - w = 967.937,671,6 \times 1.0604 - 967.937,671,6 \times w) = 1026.401,106,964,64 - 967.937,671,6 \times w$.

And r^{70} will be $(= \overline{1.0604 - w})^{70} =$, by the binomial theorem, $\overline{1.0604}^{70} - 70 \times \overline{1.0604}^{69} \times w + \&c = \overline{1.0604}^{70} - 70 \times \frac{\overline{1.0604}^{70}}{1.0604} \times w + \&c = 60.656,73 - 70 \times \frac{60.656,73}{1.0604} \times w + \&c = 60.656,73 - 70 \times 57.201,74 \times w + \&c = 60.656,73 - 4004.121,80 \times w + \&c$.

Therefore the binomial quantity $967.937,671,6 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1026.401,106,964,64 - 967.937,671,6 \times w \\ - 60.656,73 + 4004.121,80 \times w - \&c \end{array} \right\} \\ = 965.744,376,964,64 + 3036.184,128,4 \times w - \&c.$$

But the binomial quantity $967.937,671,6 \times r - r^{70}$ is $= 966.937,671,6$.

Therefore the compound quantity $965.744,376,964,64 + 3036.184,128,4 \times w - \&c$ will also be $= 966.937,671,6$.

Therefore $3036.184,128,4 \times w - \&c$ will be $(= 966.937,671,6 - 965.744,376,964,64) = 1.193,294,635,36$, and consequently $w - \&c$ will be $(= \frac{1.193,294,635,36}{3036.184,128,4}) = 0.000,39$, and w will be $= 0.000,39 + \&c$.

Therefore r , or $1.0604 - w$, will be $(= 1.06040 - 0.000,39 - \&c) = 1.06001 - \&c$, or will be less than 1.060,01; that is, the seventh near value of r , or of the root of the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$, will be $= 1.060,01 - \&c$, or will be less than 1.060,01. Q. E. I.

This number, 1.060,01, exceeds the true value of r (which is 1.06,) by only 0.000,01; which is too small a quantity to be worth attending to. For, according to this value of r , the interest of an hundred pounds for one year would be $(= 100 \times 0.060,01\% = 6.001\%) = 6\% \text{ os. od.}$ and less than one farthing, instead of being exactly equal to 6%.

Art. 46.

Art. 46. The foregoing number of processes of Mr. Raphson's method of approximation which are necessary to be performed in order to derive from 1.107,903,7, or 1.108, or the first near value of r in the equation $967.937,671,6 \times r - r^{70} = 966.937,671,6$ (which had been obtained by computing the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$,) it's almost exact value 1.060,01, is so great, and is attended with so much labour, as to make it by no means expedient to make use of the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ for obtaining a first near value of r in an equation of that kind, when t , or the number of years during which the annuity is forborn, is so great a number as 70. But yet, when we have computed that expression in one of these equations, and have likewise discovered in what proportion the value of r , obtained by means of such computation of it, is greater than the true value of r , (as we have done in the foregoing equation, in which t is = 70 years,) the said expression may, by being multiplied by the fraction which will reduce it to an equality with the true value of r , be made the means of obtaining with ease a very convenient first near value of r in another equation of the same general form $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, in which t shall be somewhat greater than it was in the first equation, or a first near value of r that shall be so very little greater than it's true value, as to need only one process of Mr. Raphson's method of approximation to obtain a second near value of r that shall be sufficiently exact for most useful purposes. Thus, for example, if t was = 73 years, and the equation to be resolved was $\frac{z}{a} \times r - r^{73} = \frac{z}{a} - 1$, instead of $\frac{z}{a} \times r - r^{70} = \frac{z}{a} - 1$, we might make use of the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ with good success, by reasoning in the following manner.

When t was = 70, we found that the expression $1 + \overline{g + ee}^{\frac{1}{2}} - e$ was equal to 1.107,903,7, instead of being equal to 1.06, which was the true value of r in that equation $\frac{z}{a} \times r - r^{70} = \frac{z}{a} - 1$, or $967.937,671,6 \times r - r^{70} = 966.937,671,6$. Now in these two numbers 1.107,903,7 and 1.06, the excess of the former above 1 is 0.107,903,7, and the excess of the latter above 1 is 0.06; and $\frac{100}{179} \times 0.107,903,7$ is $(= \frac{10.790,37}{179}) = 0.0602$, or is a little greater than, but very nearly equal to, 0.06. Therefore, if in the first equation $\frac{z}{a} \times r - r^{70} = \frac{z}{a} - 1$, or $967.937,671,6 \times r - r^{70} = 966.937,671,6$, we multiply 0.107,903,7, or $\overline{g + ee}^{\frac{1}{2}} - e$, into the fraction $\frac{100}{179}$, the quantity

$1 + \frac{100}{179} \times \sqrt{g + ee}^{\frac{1}{2}} - e$ will be very nearly equal to, but a little greater than, 1.06, or the true value of r . But in the second equation $\frac{z}{a} \times r - r^{73} = \frac{z}{a} - 1$, in which t is equal to 73 instead of 70, the proportion of $\sqrt{g + ee}^{\frac{1}{2}} - e$ to the true value of v , or $r - 1$, will be greater than when t is equal only to 70. Therefore $\frac{100}{179} \times \sqrt{g + ee}^{\frac{1}{2}} - e$ will be somewhat greater than the true value of v , or $r - 1$, in the said second equation, and consequently $1 + \frac{100}{179} \times \sqrt{g + ee}^{\frac{1}{2}} - e$ will be somewhat greater than the true value of $1 + v$, or of $(1 + r - 1)$, or r . But, as 73 is not much greater than 70, this difference cannot be great; and therefore, if we put c for $1 + \frac{100}{179} \times \sqrt{g + ee}^{\frac{1}{2}} - e$, and w for the excess of c above r , and substitute $c - w$ instead of r in the proposed equation $\frac{z}{a} \times r - r^{73} = \frac{z}{a} - 1$, and resolve the transformed equation resulting from such substitution, as if it were a mere simple equation, according to the directions of Mr. Raphson's method of approximation, the value of r , or $c - w$, or $1 + \frac{100}{179} \times \sqrt{g + ee}^{\frac{1}{2}} - e - w$, that will be thereby obtained, will be very nearly equal to the true value of r in the said second equation $\frac{z}{a} \times r - r^{73} = \frac{z}{a} - 1$. Q. E. I.

Art. 47. And I believe that the same thing will be true, if t , or the number of years during which the annuity is forborn, is not only equal to 73, but to 75 or 76, or any number not exceeding 80. And, if it is true in those cases, it would, perhaps, be expedient to make such calculations as have been made above of the degree of inaccuracy of the value of r , obtained by means of the expression $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$, when t is of the several different values of 20 years, 30 years, 40 years, 50 years, 60 years, 80 years, and 90 years, as well as when it is 70 years, which is the case that has been above considered. For then, by means of these different known degrees of inaccuracy, corresponding to these different magnitudes of t , the said expression $1 + \sqrt{g + ee}^{\frac{1}{2}} - e$ might always be so corrected, in any of the intermediate magnitudes of r , as to become a very convenient first near value of r , from which a second near value of it, derived by a single process of Mr. Raphson's method of approximation, would be sufficiently near the truth for all useful purposes.

NOTE

NOTE II.

THE next passage in the foregoing Tract of Dr. Halley on Compound Interest that seems to want explanation, is in pages 226, 227, of the present Volume, and in page 20 of the Introduction to Sherwin's Mathematical Tables, (3d Edition published in the year 1741,) and is as follows.

PROBLEM IV.

The annuity (a), present value (z), and time (t), being given, to find (r) the rate of interest.

SOLUTION.

This problem being more difficult than it appears at first sight, and requiring the resolution of this equation $\frac{a}{z} = \frac{z+a}{z} \times r^t - r^{t+1}$, or $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, to which it is reduced, there must be applied some method of approaching the root r ; which is by no means evident. And that approximation, as the number of years and the rate of interest are greater or less, cannot properly be obtained by one general rule, but rather by two, according as the value of the reversion is greater or less.

If the number of years be great (as, suppose, 40, or upwards,) and especially if the rate of interest be high, $1 + \frac{a}{z}$ will be nearly the rate; or,

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more accurately, $\frac{z+a}{z} - \frac{z}{z+a} \Big)^t \times \frac{a}{z}$ *. Call this r ; and $\frac{a}{r^t \times r-1}$ will be exceeding near the value of the reversion, which let be x . Then $1 + \frac{a}{z+x}$ shall approach the true rate sufficiently. But, if greater exactness be desired, by repeating this process it will be obtained. Hence this rule: From the logarithm of a , and also from the logarithm of $z+a$, take the logarithm of z , &c, as above in page 227.

Dr. Halley has not given us the investigation of the foregoing expression $1 + \frac{a}{z+x}$, nor even those of the two former expressions of the value of the rate r contained in the foregoing passage, to wit, $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, and $\frac{z+a}{z} - \frac{z}{z+a} \Big)^t \times \frac{a}{z}$. This omission I shall therefore now endeavour to supply by exhibiting investigations of all the three expressions in their order.

An Investigation of the first Expression $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, given us by Dr. Halley in the foregoing Passage as a near Value of r , or the Rate of Interest sought in the foregoing Problem, or of the Root r of the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Article 1. In the first place we must shew that the foregoing Problem may be reduced to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, as Dr. Halley affirms.

Now it is shewn in the two preceeding pages of Dr. Halley's Discourse on Compound Interest, to wit, pages 224 and 225 of the present Volume, that z

* Note. This second expression $\frac{z+a}{z} - \frac{z}{z+a} \Big)^t \times \frac{a}{z}$ of the value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, given in this passage by Dr. Halley, is mis-printed in page 20 of the Introduction to Sherwin's Mathematical Tables in the Edition above-mentioned of the year 1741, being set down as follows, to wit, $\frac{z+a}{z} - \frac{z^t}{z+a} \times \frac{a}{z}$.

is $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$. Therefore (multiplying both sides into $r-1$) we shall have $rz - z = a - \frac{a}{r^t}$, and (multiplying both sides into r^t) we shall have $r^{t+1} \times z - r^t \times z = r^t \times a - a = \overline{r^t - 1} \times a$, and (dividing both sides by $r^t - 1$) we shall have $\frac{r^{t+1} \times z - r^t \times z}{r^t - 1} = a$, and (dividing both sides by z) we shall have $\frac{r^{t+1} - r^t}{r^t - 1} = \frac{a}{z}$. Therefore (multiplying both sides into $r^t - 1$) we shall have $r^{t+1} - r^t = \frac{ar^t}{z} - \frac{a}{z}$, and (adding $\frac{a}{z}$ to both sides,) we shall have $r^{t+1} - r^t + \frac{a}{z} = \frac{ar^t}{z}$, and (adding r^t to both sides,) $r^{t+1} + \frac{a}{z} = \frac{ar^t}{z} + r^t = \frac{ar^t}{z} + \frac{zr^t}{z} = \frac{z+a}{z} \times r^t$, and, lastly, (subtracting r^{t+1} from both sides,) $\frac{a}{z} = \frac{z+a}{z} \times r^t - r^{t+1}$, or $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. Q. E. D.

Art. 2. Having thus found that the foregoing problem may be reduced to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, we must shew in the next place, that $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, will be a near value of r , or the root of the said equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, but will be somewhat greater than it's true value. Now this may be shewn in the following manner.

The binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$ is $= \frac{z}{z} \times r^t + \frac{a}{z} \times r^t - r^{t+1}$
 $= 1 \times r^t + \frac{a}{z} \times r^t - r \times r^t = r^t \times 1 + \frac{a}{z} - r$.

But the said binomial quantity is $= \frac{a}{z}$.

Therefore $r^t \times 1 + \frac{a}{z} - r$ will also be $= \frac{a}{z}$.

Therefore

Therefore (dividing both sides by r^t) we shall have $1 + \frac{a}{z} - r = \frac{a}{zr^t}$, and (adding r to both sides) $1 + \frac{a}{z} = \frac{a}{zr^t} + r$, and (subtracting $\frac{a}{zr^t}$ from both sides,) $r = 1 + \frac{a}{z} - \frac{a}{zr^t}$; of which trinomial quantity the two first terms 1 and $\frac{a}{z}$ are known quantities, and the third term $\frac{a}{zr^t}$ is unknown, because it involves the unknown quantity r^t in it's denominator. But, because r is always necessarily greater than 1 , (being equal to $1.02, 1.03, 1.04, 1.05, 1.06, 1.07, 1.08, 1.09, 1.10$, &c, according as the interest of money is 2 per cent, 3 per cent, 4 per cent, 5 per cent, 6 per cent, 7 per cent, 8 per cent, 9 per cent, or 10 per cent, &c,) it is evident that $\frac{a}{zr}$ must always be less than $\frac{a}{z}$. But $\frac{a}{zr^t}$ is less than $\frac{a}{zr}$ in the proportion of r to r^t , which, when t is a great number, (as 30 , or 40 , or more,) will be a very great ratio of minority. Therefore $\frac{a}{zr^t}$ (which is much less than $\frac{a}{zr}$;) will, *à fortiori*, be much less than $\frac{a}{z}$; that is, the third term of the trinomial quantity $1 + \frac{a}{z} - \frac{a}{zr^t}$ (which is equal to r ;) will be much less than the second term, $\frac{a}{z}$ of the said quantity, and, still more, than $1 + \frac{a}{z}$, or the sum of the first and second terms, 1 and $\frac{a}{z}$, of the said trinomial quantity. Therefore $1 + \frac{a}{z}$ will be nearly equal to, or but little greater than, the whole trinomial quantity $1 + \frac{a}{z} - \frac{a}{zr^t}$, or, it's equal, r ; or $1 + \frac{a}{z}$ will be a pretty near value of r , or the rate of interest sought, but will be somewhat greater than the truth.

Q. E. D.

Art. 3. Having thus obtained the binomial quantity $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, for a first near value of r , or the rate of interest sought, we must next proceed to make use of it in investigating the second near value of r given us by Dr. Halley, which is the expression $\frac{z+a}{z} - \left[\frac{z}{z+a} \right]^t \times \frac{a}{z}$. Now this second

near

near value of r may be derived from the first near value of it, $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, by Mr. Raphson's method of approximation, in the manner following.

Let c be put for the first near value of r , to wit, $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, which is known to be somewhat greater than it's true value, or the root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. And let w be put for the excess of $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, or c , above the true value of r .

Then, since c is $= 1 + \frac{a}{z}$, we shall have $\frac{a}{z} = c - 1$, and consequently the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will be converted into the equation $c \times r^t - r^{t+1} = c - 1$.

Further, since w is $= c - r$, we shall have $w + r = c$, and $r = c - w$.

Therefore r^t will be $(= \overline{c-w})^t =$, by the binomial theorem, to the series $c^t - t \times c^{t-1} \times w + t \times \frac{t-1}{2} \times c^{t-2} \times w^2 - \&c$, continued to $t+1$ terms; and $\frac{z+a}{z} \times r^t$, or $c \times r^t$, will be $= c \times$ the said series, or to the series $c^{t+1} - t \times c^t \times w + t \times \frac{t-1}{2} \times c^{t-1} \times w^2 - \&c$, continued to $t+1$ terms; and r^{t+1} will be $= \overline{c-w}^{t+1} =$, by the binomial theorem, to the series $c^{t+1} - (t+1) \times c^t \times w + (t+1) \times \frac{t}{2} \times c^{t-1} \times w^2 - \&c$, continued to $t+2$ terms.

Therefore the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, or $c \times r^t - r^{t+1}$, will be equal to the compound quantity

$$\left\{ \begin{array}{l} c^{t+1} - t \times c^t \times w \quad + t \times \frac{t-1}{2} \times c^{t-1} \times w^2 - \&c \\ - c^{t+1} + (t+1) \times c^t \times w \quad - t \times \frac{t+1}{2} \times c^{t-1} \times w^2 + \&c \end{array} \right\}$$

$$= \quad * \quad + 1 \times c^t \times w \quad - t \times c^{t-1} \times w^2 \&c,$$

or $c^t \times w - t \times c^{t-1} \times w^2 \&c$, which is less than $c^t \times w$.

But

But the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, or $c \times r^t - r^{t+1}$, is $= \frac{a}{z}$, or $c - 1$.

Therefore the quantity $c^t \times w - t \times c^{t-1} \times w^2$ &c, or (neglecting the term $t \times c^{t-1} \times w^2$) $c^t \times w -$ &c will also be $= \frac{a}{z}$, or $c - 1$.

Therefore $c^t \times w$ will be $= c - 1 +$ &c, or will be somewhat greater than $c - 1$, and consequently w will be $= \frac{c-1}{c^t} +$ &c, or will be somewhat greater than $\frac{c-1}{c^t}$. Therefore r , or $c - w$, will be $= c - \sqrt[t]{\frac{c-1}{c^t}} -$ &c, or will be somewhat less than $c - \sqrt[t]{\frac{c-1}{c^t}}$, or than $\frac{z+a}{z} - \sqrt[t]{c-1} \times \frac{1}{c^t}$, or than $\frac{z+a}{z} - \frac{a}{z} \times \frac{1}{c^t}$, or than $\frac{z+a}{z} - \frac{a}{z} \times \frac{1}{\left(\frac{z+a}{z}\right)^t}$, or than $\frac{z+a}{z} - \frac{a}{z} \times \sqrt[t]{\frac{z}{z+a}}$;

that is, the second near value of r will be $\frac{z+a}{z} - \frac{a}{z} \times \sqrt[t]{\frac{z}{z+a}}$, which is Dr. Halley's second expression for the said quantity. Q. E. I.

This second near value of r will (as has been shewn in the foregoing investigation of it) be somewhat greater than it's true value.

Art. 4. It remains that we investigate the third expression given us by Dr. Halley for a near value of r , or the rate of interest sought, to wit, the expression $1 + \frac{a}{z+x}$. Now this expression may be found in the manner following.

Let the foregoing expression $\frac{z+a}{z} - \frac{a}{z} \times \sqrt[t]{\frac{z}{z+a}}$ be denoted by the letter d ; and let d be substituted instead of r in the fraction $\frac{a}{r^t \times r-1}$; which will thereby be converted into the fraction $\frac{a}{d^t \times d-1}$. Call this fraction x .

Then will $\frac{a}{r-1} - x$ be nearly $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$.

But it has been shewn that x is $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$.

Therefore

Therefore z will be nearly $= \frac{a}{r-1} - x$, and consequently nearly $= \frac{a - rx + x}{r-1}$.

Therefore (multiplying both sides into $r-1$) we shall have $rz - z$ nearly $= a - rx + x$, and consequently (adding rx to both sides,) $rz - z + rx = a + x$, and (adding z to both sides,) $rz + rx = a + x + z$, or $r \times \frac{z+x}{z+x} = a + x + z$, and (dividing both sides by $z+x$;) we shall have $r = \frac{a}{z+x} + \frac{x+z}{z+x} = \frac{a}{z+x} + 1$, or $1 + \frac{a}{z+x}$; that is, $1 + \frac{a}{z+x}$ will be a third near value of r , or the rate of interest sought. Q. E. I.

Art. 5. We will now apply these three expressions of the value of r , or the rate of interest sought, to the Example given us by Dr. Halley, which is as follows.

Dr. Halley's Example to the foregoing Problem.

If 1321.3028*l.* is paid for an annuity of 70*l.* *per annum* for 59 years, what is the rate of interest allowed the purchaser?

Here a is $= 70$., z is $= 1321.3028$., and t is $= 59$.

Therefore $1 + \frac{a}{z}$, or the first of the three foregoing approximations to the value of r , will be $= 1 + \frac{70}{1321.3028} = 1 + 0.052,978,0 = 1.052,978,0$; that is, the rate of interest allowed the purchaser is, according to this first expression, something more than 1.0529, but less than 1.053. Q. E. I.

According to this value of r , the interest of a hundred pounds for one year would be nearly ($= 100 \times 0.053$., or) 5.3*l.*, or 5*l.* 6*s.*

The second approximation is $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, or $1 + \frac{a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, or $1.052,978,0 - 0.052,978,0 \times \left[\frac{z}{z+a} \right]^t$, or $1.052,978,0 - 0.052,978,0 \times \left[\frac{z}{z+a} \right]^{59}$. We must therefore now compute the value of $\left[\frac{z}{z+a} \right]^{59}$.

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Now

Now we have seen that $\frac{z+a}{z}$ is = 1.052,978,0, of which the logarithm in Briggs's system is 0.022,419,3. Therefore the logarithm of $\left(\frac{z+a}{z}\right)^{59}$ will be = $59 \times 0.022,419,3 = 1.322,738,7$; which is the logarithm of the number 21.025,130,4. Therefore $\left(\frac{z+a}{z}\right)^{59}$ is = 21.025,130,4; and consequently $\left(\frac{z}{z+a}\right)^{59}$ (which is = $\frac{1}{\left(\frac{z+a}{z}\right)^{59}}$), will be = $\frac{1}{21.025,130,4} = 0.047,562,1$.

Therefore $\frac{a}{z} \times \left(\frac{z}{z+a}\right)^{59}$ will be = $\frac{a}{z} \times 0.047,562,1 = 0.052,978,0 \times 0.047,562,1 = 0.002,519,7$; and $\frac{z+a}{z} - \frac{a}{z} \times \left(\frac{z}{z+a}\right)^{59}$ will be = $1.052,978,0 - 0.002,519,7 = 1.050,458,3$; that is, the near value of r , or the rate of interest, resulting from the second approximation $\frac{z+a}{z} - \frac{a}{z} \times \left(\frac{z}{z+a}\right)^t$ will be = 1.050,458,3. Q. E. I.

According to this value of r the interest of a hundred pounds for one year would be (= $100 \times 0.050,458,3$ l. = 5.045,83l., or) 5l. 0s. 10d.

The third approximation to the value of r is $1 + \frac{a}{z+x}$, which may be computed in the following manner.

The last approximation $\frac{z+a}{z} - \frac{a}{z} \times \left(\frac{z}{z+a}\right)^t$ is now to be called d , and x is to be taken equal to the fraction $\frac{a}{d^t \times d-1}$.

Now, since $\frac{z+a}{z} - \frac{a}{z} \times \left(\frac{z}{z+a}\right)^t$, or d , has been found to be = 1.050,458,3, we shall have d^t , or d^{59} , = $\overline{1.050,458,3}^{59}$, and $d-1$ (= $1.050,458,3 - 1$) = 0.050,458,3, and $\frac{a}{d^t \times d-1}$ (= $\frac{a}{d^{59} \times d-1} = \frac{a}{\overline{1.050,458,3}^{59} \times 0.050,458,3} = \frac{70}{\overline{1.050,458,3}^{59} \times 0.050,458,3} = \frac{70}{0.050,458,3} \times \frac{1}{\overline{1.050,458,3}^{59}} = \frac{7000}{5.045,83} \times \frac{1}{\overline{1.050,458,3}^{59}} = 1387.284,153,4 \times \frac{1}{\overline{1.050,458,3}^{59}}$). We must therefore now find the value of $\overline{1.050,458,3}^{59}$; which may be done by means of a table of logarithms in the manner following.

The

The logarithm of 1.050,458,3 is = 0.021,378,8. Therefore the logarithm of $\overline{1.050,458,3}^{59}$ will be $(= 59 \times 0.021,378,8) = 1.261,349,2$; which is the logarithm of the number 18.253,628,1. Therefore $\overline{1.050,458,3}^{59}$ is = 18.253,628,1; and consequently $\frac{1387.284,153,4}{1.050,458,3}^{59}$ will be = $\frac{1387.284,153,4}{18.253,628,1} = 76.000,461,1$; that is, $\frac{a}{d^{59} \times d-1}$, or x , will be = 76.000,461,1.

Therefore $z + x$ will be = $1321.3028 + 76.000,461,1 = 1397.303,261,1$; and $\frac{a}{z+x}$ will be = $\frac{70}{1397.303,261,1} = 0.050,096,4$, and consequently $1 + \frac{a}{z+x}$ will be = $1 + 0.050,096,4$, or 1.050,096,4; that is, the third approximation to the value of r , or the rate of interest sought, will be = 1.050,096,4.

Q. E. I.

This number is very little greater than the true value of r , which is exactly 1.05. For, according to this value of r , the interest of a hundred pounds for a year would be $(= 100 \times 0.050,096,4\text{,}$ or $5.009,64\text{,}$ or 5, or $2\frac{1}{4}$, which exceeds the real interest of it (which is exactly 5,) by only two pence, farthing, which is too small a quantity to be worth attending to. Therefore this last approximation $1 + \frac{a}{z+x}$ is, in the present example, as near the truth as need to be desired.

Observations on the Three foregoing successive near Values of r in the Equation

$$\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}, \text{ given us by Dr. Halley.}$$

Art. 6. The first of these near values of r , to wit, $1 + \frac{a}{z}$, is obtained with very little trouble, and is a very good first near value of r , or a very good foundation for a further approach to it's true value by Mr. Raphson's method of approximation, or by Dr. Halley's second expression above-mentioned, to wit,

$$\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t, \text{ which is derived from the said first expression } 1 + \frac{a}{z}$$

by the said method of approximation. But the second expression, $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, and the third expression, $1 + \frac{a}{z+x}$, require a good deal of labour

in computing them, though not more than is to be expected from the application of two successive processes of Mr. Raphson's method of approximation. The labour, however, which is necessary to the computation of the second expression $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, is not to be complained of, because the value of r obtained by means of that second expression, to wit, the number 1.050, 458, 3, approaches much nearer than the first near value of r , to wit, $1 + \frac{a}{z}$, or 1.052, 978, 0, to the true value of r , which is 1.05. But the third near value of r , to wit, the number 1.050, 096, 4, obtained by the computation of the third expression $1 + \frac{a}{z+x}$, is but a small improvement upon the preceding near value of r , to wit, 1.050, 458, 3, obtained by computing the second expression $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, and therefore seems hardly worth the trouble of obtaining it by the computation of the expression $1 + \frac{a}{z+x}$. Whenever therefore the value of r obtained by means of the second expression $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$ is not thought to be sufficiently exact, I should think it would be more judicious to have recourse to a second process of Mr. Raphson's method of approximation than to the computation of the foregoing third expression $1 + \frac{a}{z+x}$. And, if a general expression for a third near value of r , (derived from it's second near value $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$ by means of Mr. Raphson's method of approximation, in the same manner as the said second near value of it was derived above, in art. 3, from it's first near value $1 + \frac{a}{z}$), be thought to be convenient for the purpose of computing such third near value of r in the present, or any other, numeral example of the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, such general expressions of the said third near value of r in the said equation may be obtained in the following manner.

An Investigation of a General Expression for a Third near Value of r in the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, after we have found the two former near Values of it to be $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, and $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$.

Art. 7. Let c be put for $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, or the first near value of r , investigated above in art. 2; and let d be put for $c - \sqrt[t]{\frac{c-1}{c}}$, or for, it's equal, the second expression $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$, which was investigated in art. 3, and which is somewhat greater than the true value of r . And let w be put for the excess of d above the said true value.

Then will $\frac{a}{z}$ be $(= 1 + \frac{a}{z} - 1) = c - 1$, and r will be $= d - w$; and consequently r^t will be $= \overline{d-w}^t =$, by the binomial theorem, to the series $d^t - t \times d^{t-1} \times w + \&c$; and $\frac{z+a}{z} \times r^t$, or $c \times r^t$, will be $=$ the series $c \times d^t - c \times t \times d^{t-1} \times w + \&c$; and r^{t+1} will be $= \overline{d-w}^{t+1} =$, by the binomial theorem, to the series $d^{t+1} - (t+1) \times d^t \times w + \&c$.

Therefore the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, or $c \times r^t - r^{t+1}$, will be equal to the compound quantity

$$\left\{ \begin{array}{l} c \times d^t - c \times t \times d^{t-1} \times w + \&c \\ - d^{t+1} + (t+1) \times d^t \times w - \&c. \end{array} \right\}$$

But the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, or $c \times r^t - r^{t+1}$, is $(= \frac{a}{z}) = c - 1$.

Therefore

Therefore the compound quantity

$$\left\{ \begin{array}{l} c \times d^t - c \times t \times d^{t-1} \times w + \&c \\ - d^{t+1} + \overline{t+1} \times d^t \times w - \&c \end{array} \right\}$$

will likewise be $= c - 1$.

Therefore, if $\overline{t+1} \times d^t$ is greater than $c \times t \times d^{t-1}$, or if $\overline{t+1} \times d$ is greater than $c \times t$, or if d is greater than $\frac{t}{t+1} \times c$, we shall have $\overline{t+1} \times d^t \times w - c \times t \times d^{t-1} \times w \&c = c - 1 + d^{t+1} - c \times d^t$, or $\overline{t+1} \times d^t - c \times t \times d^{t-1} \times w \&c = c - 1 + d^{t+1} - c \times d^t$, and consequently $w =$ the fraction $\frac{c - 1 + d^{t+1} - c \times d^t}{\overline{t+1} \times d^t - c \times t \times d^{t-1}}$, and consequently $r (= d - w) = d -$ the fraction $\frac{c - 1 + d^{t+1} - c \times d^t}{\overline{t+1} \times d^t - c \times t \times d^{t-1}}$, or $d -$ the fraction $\frac{c - 1 + d^{t+1} - c d^t}{\overline{t+1} \times d^t - t \times c d^{t-1}}$. Q. E. I.

And, if $\overline{t+1} \times d^t$ is less than $c \times t \times d^{t-1}$, or d is less than $\frac{t}{t+1} \times c$, we shall have $c \times d^t - d^{t+1} = c - 1 + c \times t \times d^{t-1} \times w - \overline{t+1} \times d^t \times w$, and $c \times t \times d^{t-1} \times w - \overline{t+1} \times d^t \times w = c \times d^t - d^{t+1} - \overline{c-1}$, or $c \times t \times d^{t-1} - \overline{t+1} \times d^t \times w = c \times d^t - d^{t+1} - \overline{c-1}$, and consequently $w =$ the fraction $\frac{c \times d^t - d^{t+1} - \overline{c-1}}{c \times t \times d^{t-1} - \overline{t+1} \times d^t}$, and r , or $d - w$, $= d -$ the fraction $\frac{c \times d^t - d^{t+1} - \overline{c-1}}{c \times t \times d^{t-1} - \overline{t+1} \times d^t}$, or $d -$ the fraction $\frac{c d^t - d^{t+1} - \overline{c-1}}{t \times c d^{t-1} - \overline{t+1} \times d^t}$. Q. E. I.

Art. 8. We will now apply this last expression of the value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ to the foregoing example of Dr. Halley, that

that we may see whether it will not give us a nearer approximation to the true value of r , (which is 1.05,) than was obtained by the foregoing third expression, $1 + \frac{a}{z+x}$, of the value of r , given us by Dr. Halley.

A Computation of the Value of r in the foregoing Example set forth above in Art. 5,

by means of the Expression $d = \frac{c - 1 + d^{t+1} - cd^t}{t+1 \times d^t - t \times c \times d^{t-1}}$, or the

Expression $d = \frac{cd^t - d^{t+1} - \sqrt{c-1}}{t \times cd^{t-1} - \sqrt{t+1} \times d^t}$, according as $t+1 \times d$ is greater, or less, than $t \times c$.

Here c is = 1.052,978,0, and d , or $\frac{c-1}{t}$, (as it was obtained above in art. 5 by means of the second expression $\frac{z+a}{z} - \frac{a}{z} \times \frac{z}{z+a} \Big)^t$) is = 1.050,458,3; instead of which (in order to lessen the labour of the computation) I shall substitute 1.0504, omitting the three last figures 0.000,058,3.

Now, since d is = 1.0504, and t is = 59, we shall have $d^t = \overline{1.0504}^{59}$, and $d^{t+1} = \overline{1.0504}^{60}$, and $d^{t-1} = \overline{1.0504}^{58}$.

The logarithm of 1.0504 is = 0.021,354,7. Therefore the logarithm of $\overline{1.0504}^{59}$ will be (= $59 \times 0.021,354,7$) = 1.259,927,3; which is the logarithm of the number 18.152,117,1. Therefore d^t , or $\overline{1.0504}^{59}$, is = 18.152,117,1. Therefore d^{t+1} will be (= $d^t \times d = 18.152,117,1 \times 1.0504$) = 19.066,983,8, and d^{t-1} will be (= $\frac{d^t}{d} = \frac{18.152,117,1}{1.0504}$) = 17.281,147,2, and $c \times d^t$ will be (= $1.052,978,0 \times 18.152,117,1$) = 19.113,779,9, and $t+1 \times d^t$ will be (= $60 \times 18.152,117,1$) = 1089.127,026,0, and $t \times c \times d^{t-1}$ will be (= $59 \times 1.052,978,0 \times 17.281,147,2$) = 1073.603,401,1.

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This number is less than 1089.127,026,0, or $\overline{t+1} \times d^t$. Therefore w will be = the fraction $\frac{c-1+d^{t+1}-cd^t}{\overline{t+1} \times d^t - t \times c \times d^{t-1}}$, that is, = to

$$\begin{aligned} & \frac{1.052,978,0 - 1 + 19.066,983,8 - 19.113,779,9}{1089.127,026,0 - 1073.603,401,1} \\ = & \frac{0.052,978,0 + 19.066,983,8 - 19.113,779,9}{15.523,624,9} = \frac{19.119,961,8 - 19.113,779,9}{15.523,624,9} \\ = & \frac{0.006,181,9}{15.523,624,9} = 0.000,398,2. \end{aligned}$$

Therefore r , or $d - w$, will be = 1.050,400,0 - 0.000,398,2 = 1.050,001,8; that is, the third near value of r , or the rate of interest sought, which is obtained by this second process of Mr. Raphson's method of approximation, will be = 1.050,001,8. Q. E. I.

This number 1.050,001,8 is much nearer than the number 1.050,096,4, (which was obtained by means of Dr. Halley's third expression $1 + \frac{a}{z+x}$), to the true value of r , or the rate of interest sought, (which is 1.05,) though that value was near enough to the truth for all useful purposes. But Dr. Halley informs us that, if t was but a small number, that third expression $1 + \frac{a}{z+x}$ would not be near enough. And therefore, upon the whole, I am inclined to prefer this last general expression of a third near value of r , (to wit, the expression $d -$ the fraction $\frac{c-1+d^{t+1}-cd^t}{\overline{t+1} \times d^t - t \times c \times d^{t-1}}$, or the expression $d -$ the fraction $\frac{cd^t - d^{t+1} - \overline{c-1}}{t \times cd^{t-1} - \overline{t+1} \times d^t}$, according as $\overline{t+1} \times d$ is greater, or less, than $t \times c$), that has been obtained by means of a second process of Mr. Raphson's method of approximation, to the third expression $1 + \frac{a}{z+x}$ given us by Dr. Halley.

Art. 9. Dr. Halley, by his giving us the general expression $\frac{z+a}{z} - \frac{a}{z} \times \left[\frac{z}{z+a} \right]^t$ for a second near value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, instead of directing us to make use of the former near value of it, to wit, $1 + \frac{a}{z}$, as a foundation for a process of Mr. Raphson's method of approximation,

mation, (by which we should have obtained the very same number 1.050,458,3 for a second near value of r as was obtained above in art. 5 by computing the expression $\frac{z+a}{z} - \frac{a}{z} \times \frac{z}{z+x} \Big|^t$, seems to have thought that it was easier and more convenient to calculate by means of such a general expression than to apply Mr. Raphson's method of approximation to the particular numeral equation that we wanted to resolve. And Mr. Raphson himself seems to have been of the same opinion. For in his *Analysis Aequationum Universalis* he prefixes to each of the numeral equations which he exhibits as examples of his method, a general *canon*, or *formula*, expressing the second near value of the root sought, and resolves the equation by computing that *canon* or *formula*, instead of investigating the said second near value by a distinct application of the reasonings and principles upon which, in the first pages of his book, he has grounded his method. But I must confess I differ from these eminent writers in this respect. For I have always found it more convenient to resolve the proposed equation by applying to it distinctly the original reasonings and principles upon which Mr. Raphson's method is grounded, than to proceed by means of such general *canon*, or *formula*: and, in particular, it is much *safer*, or less likely to lead us into mistakes, by some omission of a number, or letter, or by a change of a sign $+$ or $-$, or $\times$, than to work by a *canon* or *formula*;—not to mention the greater satisfaction of mind that we receive when we proceed in a plain and distinct manner, (in which we see our way before us, and perceive the reason of every arithmetical operation at the time we perform it,) than when we work in the dark in computing the several members of a tedious and intricate algebræick general expression. And therefore, that my readers may be the better enabled to make a comparison between these two different methods of proceeding, and to judge for themselves which of them deserves to be preferred to the other, I will now investigate the second and third values of r in the foregoing Problem, that resulted from the computation of the two general algebræick expressions above-mentioned, which were obtained by means of Mr. Raphson's method of approximation, to wit, the numbers 1.050,458,3 and 1.050,001,8, by the immediate application of the principles and reasonings of Mr. Raphson's method of approximation to the numeral equation resulting from Dr. Halley's Problem above-stated. This may be done in the manner following.

Art. 10. Since $\frac{a}{z}$, in the foregoing Problem, is $(= \frac{70}{1321.3028}) = 0.052,978,0$, and $1 + \frac{a}{z}$, or $\frac{z+a}{z}$, is $= 1.052,978,0$, and t is $= 59$, and consequently $t + 1$ is $= 60$, the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will, in the Problem proposed above by Dr. Halley, be $1.052,978,0 \times 1^{59} - 1^{60} = 0.052,978,0$; of which we know already (by what is shewn above
 Vol. V. in
 Q q

in art. 2,) that $1 + \frac{a}{x}$, or 1.052,978,0, is a first near value, greater than the truth.

Let w be put for the excess of 1.052,978,0 above r .

Then will r be $= 1.052,978,0 - w$; and consequently r^{59} will be $= \overline{1.052,978,0 - w}^{59} =$, by the binomial theorem, to the series $\overline{1.052,978,0}^{59} - 59 \times \overline{1.052,978,0}^{58} \times w + \&c$, and consequently $1.052,978,0 \times r^{59}$ will be $(= 1.052,978,0 \times \text{the series } \overline{1.052,978,0}^{59} - 59 \times \overline{1.052,978,0}^{58} \times w + \&c) = \text{the series } \overline{1.052,978,0}^{60} - 59 \times \overline{1.052,978,0}^{59} \times w + \&c$; and r^{60} will be $= \overline{1.052,978,0 - w}^{60} =$, by the binomial theorem, to the series $\overline{1.052,978,0}^{60} - 60 \times \overline{1.052,978,0}^{59} \times w + \&c$.

Therefore the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$ will be $=$ the following compound quantity, to wit,

$$\begin{aligned} & \left\{ \begin{array}{l} \text{the series } \overline{1.052,978,0}^{60} - 59 \times \overline{1.052,978,0}^{59} \times w + \&c \\ - \text{the series } \overline{1.052,978,0}^{60} - 60 \times \overline{1.052,978,0}^{59} \times w + \&c \end{array} \right\} \\ = & \left\{ \begin{array}{l} \text{the series } \overline{1.052,978,0}^{60} - 59 \times \overline{1.052,978,0}^{59} \times w + \&c \\ - \overline{1.052,978,0}^{60} + 60 \times \overline{1.052,978,0}^{59} \times w - \&c \end{array} \right\} \\ = & \quad \quad \quad + 1 \times \overline{1.052,978,0}^{59} \times w \quad \&c \end{aligned}$$

or $\overline{1.052,978,0}^{59} \times w \quad \&c$.

But the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$ is $= 0.052,978,0$.

Therefore $\overline{1.052,978,0}^{59} \times w \quad \&c$ will also be $= 0.052,978,0$.

But the logarithm of 1.052,978,0 is $= 0.022,419,3$. Therefore the logarithm of $\overline{1.052,978,0}^{59}$ will be $(= 59 \times 0.022,419,3) = 1.322,738,7$; which is the logarithm of the number 21.025,130,4. Therefore $\overline{1.052,978,0}^{59}$ will be $= 21.025,130,4$.

Therefore $\overline{1.052,978,0}^{59} \times w$ will be $= 21.025,130,4 \times w$; and consequently $21.025,130,4 \times w$ will be $= 0.052,978,0$; and w will be $= \frac{0.052,978,0}{21.025,130,4} = 0.002,519,7$. Therefore r , or $1.052,978,0 - w$, will be $= 1.052,978,0 - 0.002,519,7 = 1.050,458,3$; that is, the second near value of

of r , in the equation $1.052,978,0 \times r^{59} - r^{60} = 0.052,978,0$, resulting from this first process of Mr. Raphson's method of approximation, will be 1.050,458,3. Q. E. I.

This number 1.050,458,3 is exactly the same that we obtained above in art. 5 by computing the expression $\frac{z+a}{z} - \frac{a}{z} \times \frac{z}{z+a}$.

Art. 11. I will now proceed to find a third near value of r in the proposed equation $1.052,978,0 \times r^{59} - r^{60} = 0.052,978,0$ by means of a second process of Mr. Raphson's method of approximation. This may be done in the manner following.

The last near value of r , obtained by the foregoing process of Mr. Raphson's method of approximation, was 1.050,458,3, of which we will reject the three last figures 583, as probably not exact, and retain only the first five figures 1.0504. And these we will, in the first place, substitute instead of r in the compound quantity $1.052,978,0 \times r^{59} - r^{60}$, in order to discover whether the value of the said compound quantity, resulting from the said substitution, will be greater, or less, than 0.052,978,0, or the absolute term of the proposed equation, and thereby to determine whether the number 1.0504 will be less, or greater, than the true value of r in the said equation. For, if the value of the said compound quantity resulting from this substitution is greater than the absolute term 0.052,978,0, we may conclude that the said number 1.050,4 will be less than the true value of r ; but, if it is less than the said absolute term, the said number 1.0504 will be greater than the said true value; as may be shewn in the manner following.

Art. 12. The proposed equation $1.052,978,0 \times r^{59} - r^{60} = 0.052,978,0$ has in truth two roots, of which the lesser is 1. For, if r is = 1, r^{59} will be also = 1, and so will r^{60} ; and consequently the compound quantity $1.052,978,0 \times r^{59} - r^{60}$ (which forms the left-hand side of the equation,) will be $(= 1.052,978,0 \times 1 - 1 = 1.052,978,0 - 1) = 0.052,978,0$, or the absolute term of the equation. But this lesser root of the equation cannot be that which relates to the Problem proposed by Dr. Halley, because in that Problem r represents a pound sterling together with its interest for a year, and therefore must of necessity be greater than 1. Therefore the other, and greater, root of the equation $1.052,978,0 \times r^{59} - r^{60} = 0.052,978,0$ is that which is fitted to solve Dr. Halley's Problem. Now, when in a binomial equation of this form, $Px^m - x^{m+n} = Q$, a quantity greater than the greater root of the equation is substituted instead of x in the binomial quantity $Px^m - x^{m+n}$, (which forms the left-hand side of the equation,) the value of the said binomial quantity resulting from such substitution will be less than it was before, and consequently

Q. q. 2

consequently than the absolute term Q ; and, if a quantity less than the greater root of the equation, but greater than the lower limit of the greater root, (or than the value which x has when the binomial quantity $Px^m - x^{m+n}$ is a *maximum*, or at it's greatest possible magnitude,) be substituted instead of x in the said binomial quantity $Px^m - x^{m+n}$, the value of the said binomial quantity resulting from such substitution will be greater than it was before, and consequently than the absolute term Q ; for the reasons given above in art. 26 and 27 of Note I., pages 262 and 263, to which I refer the reader.

Art. 13. We will therefore now substitute the number 1.0504 instead of r in the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or less, than 0.052,978,0, or the absolute term of the proposed equation.

Now, if r is = 1.0504, the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$ will be = $1.052,978,0 \times \overline{1.0504}^{59} - \overline{1.0504}^{60}$. We must therefore compute the value of the binomial quantity $1.052,978,0 \times \overline{1.0504}^{59} - \overline{1.0504}^{60}$; and, for that purpose, we must find the values of $\overline{1.0504}^{59}$ and $\overline{1.0504}^{60}$, which may be done as follows.

The logarithm of 1.0504 is = 0.021,354,7. Therefore the logarithm of $\overline{1.0504}^{59}$ will be (= $59 \times 0.021,354,7$) = 1.259,927,3; which is the logarithm of the number 18.152,117,1. Therefore $\overline{1.0504}^{59}$ is = 18.152,117,1. Therefore $\overline{1.0504}^{60}$ will be (= $1.0504 \times \overline{1.0504}^{59}$) = $1.0504 \times 18.152,117,1$) = 19.066,983,8.

Therefore the binomial quantity $1.052,978,0 \times \overline{1.0504}^{59} - \overline{1.0504}^{60}$ will be (= $1.052,978,0 \times 18.152,117,1 - 19.066,983,8$) = $19.113,779,9 - 19.066,983,8$) = 0.046,796,1; which is less than 0.052,978,0, or the absolute term of the proposed equation. Therefore we may now conclude that 1.0504 must be greater than the true value of r , or the greater root of the proposed equation.

Art. 14. Let w be put for the unknown excess of 1.0504 above the true value of r .

Then we shall have $r = 1.0504 - w$; and consequently $r^{59} = \overline{1.0504 - w}^{59}$ (=, by the binomial theorem, to the series $\overline{1.0504}^{59} - 59 \times \overline{1.0504}^{58} \times w + \&c$

+ &c = the series $18.152,117,1 - 59 \times \frac{1.0504^{59}}{1.0504} \times w + \&c =$ the series $18.152,117,1 - 59 \times \frac{18.152,117,1}{1.0504} \times w + \&c =$ the series $18.152,117,1 - 59 \times 17.281,147,2 \times w + \&c) =$ the series $18.152,117,1 - 1019.587,684,8 \times w + \&c$; and $1.052,978,0 \times r^{59} (= 1.052,978,0 \times$ the series $18.152,117,1 - 1019.587,684,8 \times w + \&c =$ the series $1.052,978,0 \times 18.152,117,1 - 1.052,978,0 \times 1019.587,684,8 \times w + \&c) =$ the series $19.113,779,9 - 1073.603,401,1 \times w + \&c$. And r^{60} will be $(= \frac{1.0504 - w}{1.0504})^{60} =$, by the binomial theorem, to the series $\frac{1.0504^{60}}{1.0504^{60}} - 60 \times \frac{1.0504^{59}}{1.0504^{60}} \times w + \&c = 19.066,983,8 - 60 \times 18.152,117,1 \times w + \&c) =$ the series $19.066,983,8 - 1089.127,026,0 \times w + \&c$.

Therefore the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$ will be = the following compound quantity, to wit,

$$\left\{ \begin{array}{l} \text{the series } 19.113,779,9 - 1073.603,401,1 \times w + \&c \\ - \text{the series } 19.066,983,8 - 1089.127,026,0 \times w + \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{l} \text{the series } 19.113,779,9 - 1073.603,401,1 \times w + \&c \\ - 19.066,983,8 + 1089.127,026,0 \times w - \&c \end{array} \right\}$$

$$= 0.046,796,1 + 15.523,624,9 \times w \&c.$$

But the binomial quantity $1.052,978,0 \times r^{59} - r^{60}$ is = $0.052,978,0$.

Therefore the compound quantity $0.046,796,1 + 15.523,624,9 \times w$ will also be = $0.052,978,0$. And consequently (subtracting $0.046,796,1$ from both sides,) we shall have $15.523,624,9 \times w = 0.006,181,9$, and $w (= \frac{0.006,181,9}{15.523,624,9}) = 0.000,398,2$. Therefore r , or $1.0504 - w$, will be = $1.0504 - 0.000,398,2 = 1.050,001,8$; that is, the rate of interest sought will be $1.050,001,8$.
Q. E. I.

This number $1.050,001,8$ is the very same that was obtained above in art. 8 by computing the general expression d — the fraction $\frac{c - 1 + d^{t+1} - cd^t}{(t+1) \times d^t - t \times c \times d^{t-1}}$. But the present method of obtaining it appears to me to be simpler and easier, and safer, or less liable to mistakes, and also pleasanter, than the computation of that general expression.

NOTE III.

Article 1. **T**HE third passage in the foregoing Discourse of Dr. Halley which seems to stand in need of explanation, occurs in page 21 of the Introduction to Sherwin's Mathematical Tables, (the 3d Edition, of the year 1741,) and in page 228 of the present Volume, and is as follows.

“If the number of years be small, the aforesaid rule will avail little. In this case it will be requisite to approach the rate thus. Let $\frac{t+1}{2}$ be the index of a root of $\frac{at}{x}$; from which root subtract 1, and call the remainder y : and let $\frac{6}{t-1}$ be called b . I say, that $b - \sqrt{bb - 2by}$ is sufficiently near to $r - 1$; and will be still nearer the truth as the number of years is smaller; and that the error will be always in excess. Hence the rule: Divide the logarithm of $\frac{at}{x}$ by $\frac{t+1}{2}$, &c.”

This rule may be investigated in the following manner.

Art. 2. Let v be put for $r - 1$.

Then will $1 + v$ be $= r$, and $(1 + v)^t$ will be $= r^t$, and consequently $\frac{a}{r-1} - \frac{a}{r^t \times r-1}$ will be $= \frac{a}{v} - \frac{a}{(1+v)^t \times v}$.

But z is $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$.

Therefore z will also be $= \frac{a}{v} - \frac{a}{(1+v)^t \times v}$.

Therefore zv will be $= a - \frac{a}{(1+v)^t}$, and $\frac{zv}{a}$ will be $= 1 - \frac{1}{(1+v)^t} = 1 - (1+v)^{-t}$.

But,

But, by the binomial theorem, $(1+v)^{-t}$, or $\frac{1}{(1+v)^t}$, is* = the series $1 - \frac{t}{1} \times Av + \frac{t+1}{2} \times Bvv - \frac{t+2}{3} \times Cv^3 + \frac{t+3}{4} \times Dv^4 - \&c$ = the series $1 - \frac{t}{1} \times 1 \times v + \frac{t+1}{2} \times \frac{t}{1} \times 1 \times vv - \frac{t+2}{3} \times \frac{t+1}{2} \times \frac{t}{1} \times 1 \times v^3 + \frac{t+3}{4} \times \frac{t+2}{3} \times \frac{t+1}{2} \times \frac{t}{1} \times 1 \times v^4 - \&c = 1 - tv + t \times \frac{t+1}{2} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 - \&c.$

Therefore $\frac{zv}{a}$ (which is equal to $1 - \sqrt{1+v}^{-t}$) will be = $1 -$ the series $1 - tv + t \times \frac{t+1}{2} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 - \&c$ = the series $tv - t \times \frac{t+1}{2} \times vv + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 + \&c.$

Therefore (dividing both sides of the last equation by v , we shall have $\frac{z}{a} =$ the series $t - t \times \frac{t+1}{2} \times v + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$; and (dividing all the terms by t), we shall have $\frac{z}{at} =$ the series $1 - \frac{t+1}{2} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c.$

Art. 3. Now let the fraction $\frac{z}{at}$, (which forms the left-hand side of this last equation,) be raised to the power of which the fraction $\frac{-2}{t+1}$ is the index; and let the series $1 - \frac{t+1}{2} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ (which forms the right-hand side of the same equation) be raised likewise to the same power, of which the index is $\frac{-2}{t+1}$. Then will

* See, in the 3d Volume of the *Scriptores Logarithmici*, the Appendix to the Translation of a part of Mr. James Bernoulli's Treatise *De Arte Conjectandi*, page *124, at the bottom, art. 42.

the said $\frac{-2}{t+1}$ power of the said series be equal to the same power of the said fraction $\frac{z}{at}$.

Now let p be $= \frac{2}{t+1}$.

$$\text{Then will } \left[\frac{z}{at} \right]^{\frac{-2}{t+1}} \text{ be } = \left[\frac{z}{at} \right]^{-p} (= \frac{1}{\left[\frac{z}{at} \right]^p} = \frac{1}{\frac{z^p}{(at)^p}} = 1 \times \frac{(at)^p}{z^p} = 1 \times$$

$\frac{(at)^p}{z^p} = \left[\frac{at}{z} \right]^p) = \left[\frac{at}{z} \right]^{\frac{2}{t+1}}$; that is, the $\frac{-2}{t+1}$ th power of the fraction $\frac{z}{at}$ is equal to the $\frac{+2}{t+1}$ th power of the fraction $\frac{at}{z}$, which is the reciprocal of the fraction $\frac{z}{at}$, or is equal to 1 divided by $\frac{z}{at}$. And therefore, in order to obtain the $\frac{-2}{t+1}$ th power of the fraction $\frac{z}{at}$, we need only find the $\frac{+2}{t+1}$ th power of the fraction $\frac{at}{z}$; which may be done by, first, finding the logarithm of the fraction $\frac{at}{z}$, and then multiplying that logarithm into the fraction $\frac{2}{t+1}$ (which is the index of the power of the said fraction $\frac{at}{z}$ which we want to find,) and then seeking in a table of logarithms the number corresponding to the logarithm which is equal to the product thence arising. For that number will be equal to $\left[\frac{at}{z} \right]^{\frac{2}{t+1}}$, and consequently to $\left[\frac{z}{at} \right]^{-\frac{2}{t+1}}$.

But the raising the series $1 - \frac{t+1}{2} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ to the $\frac{-2}{t+1}$ th power is a business of greater difficulty. And for this purpose we must have recourse to Mr. De Moivre's Multinomial Theorem, which has been set forth above in Note I., page 241, and we must suppose the said theorem to be true, not only in the case of integral powers, but likewise in the more complicated case of fractional powers, and even in the case of powers that are both fractional and negative; though it has never, I believe, been demonstrated in any case but that of integral powers. However, it is probably true in all these cases; and, if it is, it will enable

enable us to find the $\frac{-2}{t+1}$ th power of the said series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ by proceeding in the manner following.

Art. 4. It has been shewn above in page 243, Coroll. 3, that the m th power of the series $1 + bv + cv^2 + dv^3 + ev^4 + \&c$ will be equal to the infinite series

$$\begin{aligned} 1 + \frac{m}{1}bv + \frac{m}{1} \times \frac{m-1}{2} \times b^2v^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3v^3 \\ + \frac{m}{1} \times cv^2 + \frac{m}{1} \times \frac{m-1}{1} \times bcv^3 \\ + \frac{m}{1} \times dv^3 \\ + \&c. \end{aligned}$$

Therefore the m th power of the series $1 - bv + cvv - dv^3 + ev^4 - \&c$ will be equal to the infinite series

$$\begin{aligned} 1 - \frac{m}{1}bv + \frac{m}{1} \times \frac{m-1}{2} \times b^2v^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3v^3 \\ + \frac{m}{1} \times cv^2 - \frac{m}{1} \times \frac{m-1}{1} \times bcv^3 \\ - \frac{m}{1} \times dv^3 \\ + \&c. \end{aligned}$$

Now let the index m be $= \frac{-2}{t+1}$.

Then will $m - 1$ be $(= \frac{-2}{t+1} - 1 = \frac{-2}{t+1} - \frac{1 \times t+1}{t+1} = \frac{-2}{t+1} - \frac{t-1}{t+1})$
 $= \frac{-t-3}{t+1}$; and $m - 2$ will be $(= \frac{-2}{t+1} - 2 = \frac{-2}{t+1} - \frac{2 \times t+1}{t+1} = \frac{-2}{t+1} - \frac{2t-2}{t+1})$
 $= \frac{-2t-4}{t+1}$; and $\frac{m-1}{2}$ will be $= \frac{-t-3}{2t+2}$; and $\frac{m-2}{3}$ will be $= \frac{-2t-4}{3t+3}$; and $\frac{m}{1} \times \frac{m-1}{2}$ will be $(= \frac{-2}{t+1} \times \frac{-t-3}{2t+2} = \frac{-1}{t+1} \times \frac{-t-3}{t+1})$
 $= \frac{t+3}{(t+1)^2}$; and $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ will be $(= \frac{t+3}{(t+1)^2} \times \frac{-2t-4}{3t+3})$

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$$= \frac{t+3 \times -2t-4}{3 \times t+1 \times t+1} = \frac{-2 \times t+3 \times t+2}{3 \times t+1} = -2 \times \frac{t+5t+6}{3 \times t+1} =$$

$$\frac{-2t-10t-12}{3 \times t^3+3t^2+3t+1} = \frac{-2t-10t-12}{3t^3+9t^2+9t+3}; \text{ and } \frac{m}{1} \times \frac{m-1}{1} \text{ will be } (= \frac{-2}{t+1} \times$$

$$\frac{-t-3}{t+1}) = \frac{+2t+6}{t+1}.$$

Therefore the $\frac{-2}{t+1}$ th power of the series $1 - bv + cv^2 - dv^3 + ev^4 - \&c$ will be equal to the infinite series

$$\left\{ \begin{aligned} 1 + \frac{2}{t+1} \times bv + \frac{t+3}{t+1} \times b^2v^2 + \frac{2t+10t+12}{3t^3+9t^2+9t+3} \times b^3v^3 \\ - \frac{2}{t+1} \times cv^2 - \frac{-2t-6}{t+1} \times bcv^3 \\ + \frac{2}{t+1} \times dv^3 \\ \&c. \end{aligned} \right.$$

Art. 5. Now let b be $= \frac{t+1}{2}$.

Then will this last series be converted into the series

$$\left\{ \begin{aligned} 1 + \frac{2}{t+1} \times \frac{t+1}{2} \times v + \frac{t+3}{t+1} \times \frac{t+1}{4} \times vv + \frac{2t+10t+12}{3t^3+9t^2+9t+3} \times \frac{t+1}{8} \times v^3 \\ - \frac{2}{t+1} \times c \times vv - \frac{-2t-6}{t+1} \times \frac{t+1}{2} \times c \times v^3 \\ + \frac{2}{t+1} \times dv^3 \\ \&c, \end{aligned} \right.$$

or into the series

$$\left\{ \begin{aligned} 1 + v + \frac{t+3}{4} \times vv + \frac{2t+10t+12}{3 \times 8} \times v^3 \\ - \frac{2}{t+1} \times c \times vv - \frac{-2t-6}{2 \times t+1} \times c \times v^3 \\ + \frac{2}{t+1} \times d \times v^3 \\ \&c, \end{aligned} \right.$$

or into the series

$$\begin{aligned}
 1 + v + \frac{t+3}{4} \times vv + \frac{t+5t+6}{12} \times v^3 \\
 - \frac{2}{t+1} \times cvv \quad \frac{-t-3}{t+1} \times cv^3 \\
 + \frac{2}{t+1} \times d \times v^3 \\
 \text{\&c.}
 \end{aligned}$$

Now let c be $= \frac{t+1}{2} \times \frac{t+2}{3}$.

Then will the last series be converted into the series

$$\left\{ \begin{aligned}
 1 + v + \frac{t+3}{4} \times vv + \frac{t+5t+6}{12} \times v^3 \\
 - \frac{2}{t+1} \times \frac{t+1}{2} \times \frac{t+2}{3} \times vv \quad \frac{-t-3}{t+1} \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 \\
 + \frac{2}{t+1} \times d \times v^3 \\
 \text{\&c,}
 \end{aligned} \right.$$

or into the series

$$\left\{ \begin{aligned}
 1 + v + \frac{t+3}{4} \times vv + \frac{t+5t+6}{12} \times v^3 \\
 \frac{-t-2}{3} \times vv \quad \frac{-t-3 \times t+2}{6} \times v^3 \\
 + \frac{2}{t+1} \times d \times v^3 \\
 \text{\&c,}
 \end{aligned} \right.$$

or into the series

$$\left\{ \begin{aligned}
 1 + v + \frac{3t+9}{12} \times vv + \frac{t+5t+6}{12} \times v^3 \\
 \frac{-4t-8}{12} \times vv \quad \frac{-t-5t-6}{6} \times v^3 \\
 + \frac{2}{t+1} \times d \times v^3 \\
 \text{\&c,}
 \end{aligned} \right.$$

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or

or into the series

$$\left\{ \begin{aligned} 1 + v & \frac{-t+1}{12} \times vv + \frac{t+5t+6}{12} \times v^3 \\ & \frac{-2t-10t-12}{12} \times v^3 \\ & + \frac{2}{t+1} \times d \times v^3 \end{aligned} \right. \quad \&c,$$

or into the series

$$\left\{ \begin{aligned} 1 + v & - \sqrt{\frac{t-1}{12}} \times vv - \frac{-t-5t-6}{12} \times v^3 \\ & + \frac{2}{t+1} \times d \times v^3 \end{aligned} \right. \quad \&c.$$

Now let d be $= \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4}$.

Then will the last series be converted into the series

$$\left\{ \begin{aligned} 1 + v & - \sqrt{\frac{t-1}{12}} \times vv - \frac{-t-5t-6}{12} \times v^3 \\ & + \frac{2}{t+1} \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 \end{aligned} \right. \quad \&c,$$

or into the series

$$\left\{ \begin{aligned} 1 + v & - \sqrt{\frac{t-1}{12}} \times vv - \frac{-t-5t-6}{12} \times v^3 \\ & + \frac{t+2 \times t+3}{12} \times v^3 \end{aligned} \right. \quad \&c,$$

or into the series

1 +

$$\left\{ \begin{aligned} 1 + v - \sqrt{\frac{t-1}{12}} \times vv &= \frac{-t-5t-6}{12} \times v^3 \\ &+ \frac{t+5t+6}{12} \times v^3 \\ &\&c, \end{aligned} \right.$$

or into the series

$$1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c.$$

Therefore the $\frac{-2}{t+1}$ th power of the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ will be equal to an infinite series of which the three first terms will be $1 + v - \sqrt{\frac{t-1}{12}} \times vv$, and the fourth term will be $0 \times v^3$, or 0, and the following terms will involve the following powers of v , to wit, v^4 , v^5 , v^6 , v^7 , $\&c$, which, (as v is much smaller than 1,) will be much smaller than v and vv .

Art. 6. Therefore (by art. 3,) the series $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \&c$ will be $= \sqrt{\frac{at}{z}}^{\frac{2}{t+1}}$. And consequently (subtracting 1 from both sides,) we shall have the series $v - \sqrt{\frac{t-1}{12}} \times vv + 0 \&c = \sqrt{\frac{at}{z}}^{\frac{2}{t+1}} - 1$. Therefore the two first terms, $v - \sqrt{\frac{t-1}{12}} \times vv$, of this series, will be nearly equal to $\sqrt{\frac{at}{z}}^{\frac{2}{t+1}} - 1$.

Now let y be put $= \sqrt{\frac{at}{z}}^{\frac{2}{t+1}} - 1$.

And we shall then have $v - \sqrt{\frac{t-1}{12}} \times vv$, nearly, $= y$; which is a quadratick equation, of which the root, or unknown quantity, is v .

Now let all the terms of this equation be multiplied into the fraction $\frac{12}{t-1}$, in order to free vv , (the square of the unknown quantity v) from it's co-efficient $\frac{t-1}{12}$. And we shall then have $\frac{12}{t-1} \times v - vv = \frac{12y}{t-1}$.

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Now,

Now, to simplify this last equation, let b be put $= \frac{6}{t-1}$, and consequently $2b$ be $= \frac{12}{t-1}$. And the equation will then become $2bv - vv = 2by$; which is a quadratick equation that has two roots, of which the lesser will be less than b , or half the co-efficient of v , and the greater root will be greater than b . The lesser of these roots is that which will be equal to v , or $r - 1$, in the present investigation, and that which we must therefore now endeavour to determine.

Now in quadratick equations of this form the absolute term can never be greater than the square of half the co-efficient of the unknown quantity. And consequently both sides of this equation may be subtracted from bb , or the square of b , the half of $2b$, the co-efficient of v . Let them be so subtracted. And then we shall have $bb - 2bv + vv = bb - 2by$; and consequently (by extracting the square-roots of both sides,) we shall have $b - v = \sqrt{bb - 2by}$, and (adding v to both sides) $b = v + \sqrt{bb - 2by}$, and, lastly, (subtracting $\sqrt{bb - 2by}$ from both sides,) $v = b - \sqrt{bb - 2by}$, or $b - \sqrt{bb - 2by}^{\frac{1}{2}}$; that is, $r - 1$ will be $= b - \sqrt{bb - 2by}^{\frac{1}{2}}$, which is the expression given for it by Dr. Halley in the passage above cited from his Discourse. Q. E. I.

The substance of this investigation of the expression $b - \sqrt{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley for a near value of v , or $r - 1$, in the foregoing Problem, or in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, when t , or the number of years during which the annuity a is to continue, is but small, (as, for example, less than 40 years,) was communicated to me by my learned friend, Mr. William Morgan, the Actuary of the celebrated Society for making Equitable Assurances on Lives, in Chatham Square near Black Friars' Bridge, London.

Dr. Halley has given us an example of the resolution of an equation of the general form $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ by means of the foregoing expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, which is as follows.

An Example of the Application of the foregoing Expression of Dr. Halley, $b - \sqrt{bb - 2by}$, or $b - \sqrt{bb - 2by}$, to the Resolution of the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ in a particular numeral Equation of that Form, in which t , or the Number of Years during which the Annuity is to continue, is less than 40.

Art. 7. An annuity of 20*l.* per annum, to continue 21 years, is sold for 220*l.* It is required to find the rate of interest allowed the purchaser.

Here a is = 20*l.*, and t is = 21 years, and z is = 220*l.*

Therefore $z + a$ will be (= 220*l.* + 20*l.*) = 240*l.*, and $\frac{z+a}{z}$ will be = $\frac{240}{220} = 1.090,909,0 \&c$, and $\frac{a}{z}$ will be = $\frac{20}{220} = 0.090,909,0 \&c$, and consequently the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will, in this case, become $1.090,909,0 \&c \times r^{21} - r^{22} = 0.090,909,0 \&c$, which therefore must be resolved in order to find the value of r , or the rate of interest sought. Now, according to the foregoing expression of Dr. Halley, $r - 1$ will be nearly equal to $b - \sqrt{bb - 2by}$, and consequently r will be nearly equal to $1 + b - \sqrt{bb - 2by}$, the value of which expression may be found in the manner following.

In the first place, b is = $\frac{6}{t-1}$ (= $\frac{6}{21-1} = \frac{6}{20} = \frac{3}{10}$) = 0.300,000,0, and consequently $2b$ will be = 0.600,000,0, and bb will be = 0.090,000,0.

Secondly, y is = $\frac{at}{z} \left[\frac{2}{t+1} - 1 \right]$ (= $\frac{at}{z} \left[\frac{2}{21+1} - 1 \right] = \frac{at}{z} \left[\frac{2}{22} - 1 \right] = \frac{at}{z} \left[\frac{1}{11} - 1 \right]$
 $- 1 = \frac{20 \times 21}{220} \left[\frac{1}{11} - 1 \right] = \frac{420}{220} \left[\frac{1}{11} - 1 \right] = \frac{42}{22} \left[\frac{1}{11} - 1 \right] = \frac{21}{11} \left[\frac{1}{11} - 1 \right] =$
 $\frac{1}{1.909,090,9} \left[\frac{1}{11} - 1 \right] - 1.$

Now the logarithm of 1.909,090,9 is = 0.280,826,6. Therefore the logarithm of $\frac{1}{1.909,090,9} \left[\frac{1}{11} - 1 \right]$ will be = $\frac{0.280,826,6}{11} = 0.025,529,7$, which is the logarithm.

logarithm of the number 1.060,546,4. Therefore $\overline{1.909,090,9}^{\frac{1}{1r}}$ will be = 1.060,546,4, and consequently $\overline{1.909,090,9}^{\frac{1}{1r}} - 1$ will be = 0.060,546,4; that is, y will be = 0.060,546,4.

Therefore $2by$ will be $(= 0.600,000,0 \times 0.060,546,4) = 0.036,327,84$, and $bb - 2by$ will be $(= 0.090,000,00 - 0.036,327,84) = 0.053,672,16$, and $\sqrt{bb - 2by}$ will be $(= \sqrt{0.053,672,16}) = 0.231,672,5$. Therefore $b - \sqrt{bb - 2by}$, or $b - (bb - 2by)^{\frac{1}{2}}$, will be $(= 0.300,000,0 - 0.231,672,5) = 0.068,327,5$. Therefore $r - 1$ will be, nearly, = 0.068,327,5, and consequently r will be, nearly, = $1 + 0.068,327,5$, or 1.068,327,5.

Q. E. I.

Art. 8. Dr. Halley adds, that this number 1.068,327, found by the expression $1 + b - (bb - 2by)^{\frac{1}{2}}$ for the value of the rate r , is somewhat too great, the true value of r in the present example being 1.06814; but does not inform us how he discovered the said true value to be 1.06814. But "that 1.06814 is the true value of r in this example" will appear by supposing it to be so, and then computing from it the value of z , or the present value of the annuity a , the said annuity itself being supposed to continue, as before, to be 20*l.* *per annum*, and the number of years for which it is granted being also supposed to be the same as before, to wit, 21 years. For we shall find that the present value z of the said annuity for the said number of years will, upon these suppositions, be the same as it was before supposed to be in the foregoing example, to wit, 220*l.*; whence it will follow, *et converso*, that the true value of r in the said example will be 1.06814. This computation of the value of z may be made as follows.

It has been shewn in page 225, that z , or the present value of an annuity a granted for t years, when the rate of interest is r , will be $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$.

Therefore on the suppositions just now mentioned, to wit, that a is = 20*l.*, and t is = 21 years, and that r is = 1.06814,) we shall have $z = \frac{20l.}{0.06814} -$

$$\frac{20l.}{1.06814^{21} \times 0.06814} = \frac{20l.}{0.06814} - \frac{20l.}{0.06814} \times \frac{1}{1.06814^{21}}.$$

$$\text{Now } \frac{20l.}{0.06814} \text{ is } (= \frac{100 \times 20l.}{100 \times 0.06814} = \frac{2000l.}{6.814}) = 293.513,501,6l.$$

And the logarithm of 1.06814 is = 0.028,628,1. Therefore the logarithm of 1.06814^{21} will be $(= 21 \times 0.028,628,1) = 0.601,190,1$, which is the logarithm

logarithm of the number 3.992,000,0. Therefore $\overline{1.06814}^{21}$ will be = 3.992,000,0; and consequently $\frac{20l.}{0.06814} \times \frac{1}{\overline{1.06814}^{21}}$ will be = $\frac{20l.}{0.06814} \times \frac{1}{3.992,000,0}$ ($= 293.513,501,6l. \times \frac{1}{3.992} = \frac{293.513,501,6l.}{3.992}$) = 73.525,426,2l., and $\frac{20l.}{0.06814} - \frac{20l.}{0.06814} \times \frac{1}{\overline{1.06814}^{21}}$ will be ($= 293.513,501,6l. - 73.525,426,2l.$) = 219.988,075,4l., or, very nearly, 220l. Therefore, *è converso*, if z , or the present value of the annuity a , is = 220l., the true value of r will be 1.06814. Q. E. D.

Art. 9. It appears by the foregoing example that the expression $1 + b - \sqrt{bb - 2by}$, or $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley for a near value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, is a very good approximation to it's true value for most purposes when t is not greater than 21 years: for the difference between 1.068,327,5, the near value of r obtained by means of that expression, and 1.06814, it's true value, is too small to be worth attending to. But, if greater exactness were desired, we might obtain a much nearer value of r by denoting the excess of this first value of r , to wit, 1.068,327,5, above it's true value by the letter w , and substituting the binomial quantity $1.068,327,5 - w$ instead of r in the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, but with an omission of all the terms, in the transformed equation resulting from such substitution, that involve any higher powers of the unknown quantity w than it's simple power, or w itself, according to the directions of Mr. Raphson's method of resolving equations by approximation. This may be done in the following manner.

The Investigation of a Second near Value of r in the Equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ by Mr. Raphson's Method of Approximation.

Let w be put for the excess of 1.068,327,5, (or the first near value of y obtained by means of the foregoing expression $1 + b - \sqrt{bb - 2by}$, or $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley,) above the true value of r ; and let 1.068,327,5 - w be substituted instead of r in the proposed equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, omitting all the terms that involve any higher terms of w than the simple power, or w itself.

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Then

Then will r^{21} be $= \overline{1.068,327,5 - w}^{21} =$ (by the binomial theorem) $\overline{1.068,327,5}^{21} - 21 \times \overline{1.068,327,5}^{20} \times w + \&c$; and consequently $1.090,909,0 \times r^{21}$ will be $= 1.090,909,0 \times \overline{1.068,327,5}^{21} - 21 \times 1.090,909,0 \times \overline{1.068,327,5}^{20} \times w + \&c$; and r^{22} will be $= \overline{1.068,327,5 - w}^{22} =$ (by the binomial theorem,) $\overline{1.068,327,5}^{22} - 22 \times \overline{1.068,327,5}^{21} \times w + \&c$.

Therefore the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ will be equal to the following compound quantity, to wit,

$$1.090,909,0 \times \overline{1.068,327,5}^{21} - 21 \times 1.090,909,0 \times \overline{1.068,327,5}^{20} \times w + \&c - \overline{1.068,327,5}^{22} + 22 \times \overline{1.068,327,5}^{21} \times w - \&c.$$

We must therefore now find the values of $\overline{1.068,327,5}^{20}$, $\overline{1.068,327,5}^{21}$, and $\overline{1.068,327,5}^{22}$, which may be done as follows.

The logarithm of $1.068,327,5$ is $= 0.028,704,4$. Therefore the logarithm of $\overline{1.068,327,5}^{21}$ is $(= 21 \times 0.028,704,4) = 0.602,792,4$; which is the logarithm of the number $4.006,751,4$. Therefore $\overline{1.068,327,5}^{21}$ is $= 4.006,751,4$. Therefore $\overline{1.068,327,5}^{22}$ will be $(= \overline{1.068,327,5}^{21} \times 1.068,327,5 = 4.006,751,4 \times 1.068,327,5) = 4.280,522,7$, and $\overline{1.068,327,5}^{20}$ will be $(= \frac{\overline{1.068,327,5}^{21}}{1.068,327,5} = \frac{4.006,751,4}{1.068,327,5}) = 3.750,489,8$.

Therefore the compound quantity

$$\left\{ \begin{array}{l} 1.090,909,0 \times \overline{1.068,327,5}^{21} - 21 \times 1.090,909,0 \times \overline{1.068,327,5}^{20} \times w + \&c \\ - \overline{1.068,327,5}^{22} + 22 \times \overline{1.068,327,5}^{21} \times w - \&c \end{array} \right\}$$

will be =

$$\left\{ \begin{array}{l} 1.090,909,0 \times 4.006,751,4 - 21 \times 1.090,909,0 \times 3.750,489,8 \times w + \&c \\ - 4.280,522,7 + 22 \times 4.006,751,4 \times w - \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 4.371,001,1 - 21 \times 4.091,443,0 \times w + \&c \\ - 4.280,522,7 + 88.148,530,8 \times w - \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{l} 4.371,001,1 - 85.920,303,0 \times w + \&c \\ - 4.280,522,7 + 88.148,530,8 \times w - \&c \end{array} \right\}$$

$$= 0.090,478,4 + 2.228,227,8 \times w \&c.$$

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Therefore

Therefore the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ will be = the compound quantity $0.090,478,4 + 2.228,227,8 \times w$ &c.

But the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ is = $0.090,909,0$.

Therefore the compound quantity $0.090,478,4 + 2.228,227,8 \times w$ &c will also be = $0.090,909,0$.

Therefore $2.228,227,8 \times w$ will be ($= 0.090,909,0 - 0.090,478,4$) = $0.000,430,6$, and consequently w will be $= \frac{0.000,430,6}{2.228,227,8} = 0.000,193,2$.

Therefore r , or $1.068,327,5 - w$, will be ($= 1.068,327,5 - 0.000,193,2$) = $1.068,134,3$; that is, the second near value of r , in the proposed equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, which has been obtained by this process of Mr. Raphson's method of approximation, will be $1.068,134,3$.

Q. E. I.

The difference between this number $1.068,134,3$, or second near value of r , and $1.068,14$, which is it's true value, is $0.000,005,7$, which is less than the 11,954th part of 0.06814 , or the true value of $r - 1$. This is so very small a quantity that it may be totally neglected, and the said second near value of r may be considered, with respect to all useful, or practical, purposes, as being perfectly exact.

Art. 10. In the foregoing example, in which t , or the number of years during which the annuity is to continue, is 21 years, we have seen that the first near value of r , or the rate of interest sought, (which was obtained by means of Dr. Halley's expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$,) was $1.068,327,5$; which is near enough to the true value of r , to wit, 1.06814 , for most purposes: and we have also seen that the second near value of r , (which has been derived from it's first near value, $1.068,327,5$, by only one process of Mr. Raphson's method of approximation,) is $1.068,134,3$; which is so near it's true value, 1.06814 , that it may be considered as being perfectly exact. And hence it seems reasonable to conjecture, that, by the help of only one such process of Mr. Raphson's method of approximation as that which we have just now gone through, grounded upon the first near value of r obtained by means of the expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$ given us above by Dr. Halley, we shall always be able to find the value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ to a sufficient degree of exactness in all cases in which t , or the number of years during which the annuity is to continue, is not greater than 40 years; which are the only cases in which Dr. Halley recommends the use of this expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$ for the resolution of this equation. But, that this

this may not be matter of conjecture, but may be known with certainty, I will now proceed to try the expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} , together with one process of Mr. Raphson's method of approximation grounded upon that expression, in the extreme, or most difficult, case, or when t , or the number of years during which the annuity is to continue, is 40 years. And, to the end that we may know with certainty beforehand what the exact value of r is in the equation we propose to resolve in this manner, and consequently may be able to discover with certainty how near the values of r that will be obtained by means of the expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} and by means of the subsequent process of Mr. Raphson's method of approximation grounded upon the said expression, will approach to the said exact value, I will, first, suppose r to be some known rate of interest, and will compute the value of z , or the present value of a given annuity a for a given number of years, from it; and, having thus obtained the value of z , will afterwards reverse the problem, and suppose z to be equal to the value of it obtained by the former calculation, and derive the value of r from the said value of z by means of the expression $1 + b - \sqrt{bb - 2by}$ ^{$\frac{1}{2}$} and of one process of Mr. Raphson's method of approximation grounded on it. This may be done in the manner following.

Art. 11. Let us therefore, in the first place, suppose r , or the rate of interest, to be = 1.06. And let the annuity a be an annuity of 20*l.* *per annum* that is to continue for a term, t , of 40 years. And let it be required to find z , or the present value of the said annuity.

Upon these suppositions we shall have $r = 1.06$, and $a = 20$ *l.*, and $t = 40$ years.

Therefore z , which is in all cases = $\frac{a}{r-1} - \frac{a}{r^t \times r-1}$, will, in the present case, be = $\frac{20}{0.06} - \frac{20}{1.06^{40} \times 0.06}$ (= $\frac{20}{0.06} - \frac{20}{0.06} \times \frac{1}{1.06^{40}} = \frac{100 \times 20}{100 \times 0.06} - \frac{100 \times 20}{100 \times 0.06} \times \frac{1}{1.06^{40}} = \frac{2000}{6} - \frac{2000}{6} \times \frac{1}{1.06^{40}} = 333.333,333,3 - 333.333,333,3 \times \frac{1}{1.06^{40}} = 333.333,333,3 - \frac{333.333,333,3}{1.06^{40}}$. We must therefore now find the value of 1.06^{40} .

Now the logarithm of 1.06 is = 0.025,305,9. Therefore the logarithm of 1.06^{40} will be (= $40 \times 0.025,305,9$) = 1.012,236,0; which is the logarithm of 10.285,751,2. Therefore 1.06^{40} is = 10.285,751,2.

Therefore

Therefore z will be $(= 333.333,333,3\text{ }l. - \frac{333.333,333,3\text{ }l.}{10.285,751,2} = 333,333,333,3\text{ }l. - 32.407,291,1\text{ }l.) = 300.926,042,2\text{ }l.$; that is, the present value of an annuity of 20*l.* a year for 40 years, when the interest of money is 6 per cent, will be $= 300.926,042,2\text{ }l.$, or 300*l.* 18*s.* 6*d.* Q. E. I.

Art. 12. Now let us reverse the problem, and suppose that an annuity of 20*l.* a year, that is to continue for 40 years, has been sold for the sum of 300*l.* 18*s.* 6*d.*, or 300.926,042,2*l.* And let it be required to determine, from these circumstances, the rate of interest of money that has been allowed to the purchaser of the said annuity when he paid the said sum of 300.926,042,2*l.* for it.

Here a is $= 20\text{ }l.$, as before, and t is $= 40$ years, as before, and z is $= 300.926,042,2\text{ }l.$; and r , or the rate of interest, is the unknown quantity, or root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which is to be found by means of the expression $1 + b - \sqrt[bb - 2by]{\frac{1}{2}}$, given us by Dr. Halley, and of a subsequent process of Mr. Raphson's method of approximation grounded upon that expression.

To compute the expression $1 + b - \sqrt[bb - 2by]{\frac{1}{2}}$ we will, in the first place, determine the value of y .

$$\text{Now } y \text{ is } = \frac{\frac{at}{z}}{t+1} - 1 \left(= \frac{\frac{at}{z}}{40+1} - 1 = \frac{\frac{at}{z}}{41} - 1 = \left(\frac{\frac{20 \times 40}{300.926,042,2}}{41} - 1 = \frac{\frac{800}{300.926,042,2}}{41} - 1 \right) = \frac{2.658,460,5}{41} - 1. \right.$$

We must therefore compute the value of $\frac{2.658,460,5}{41}$; which may be done by means of a table of logarithms as follows.

The logarithm of 2.658,460,5 is $= 0.424,630,2$. Therefore the logarithm of $\frac{2.658,460,5}{41}$ will be $(= \frac{2}{41} \times 0.424,630,2 = \frac{0.849,260,4}{41}) = 0.020,713,7$; which is the logarithm of the number 1.048,852,4. Therefore $\frac{2.658,460,5}{41}$ will be $= 1.048,852,4$; and consequently $\frac{2.658,460,5}{41} - 1$ will be $= 0.048,852,4$, or $\frac{\frac{at}{z}}{t+1} - 1$ will be $= 0.048,852,4$; that is, y will be $= 0.048,852,4$. Q. E. I.

Further,

Further, b is $= \frac{6}{r-1} = \frac{6}{40-1} = \frac{6}{39} = \frac{2}{13} = 0.153,846,1$. Therefore $2b$ will be $(= 2 \times 0.153,846,1) = 0.307,692,2$, and bb will be $(= 0.153,846,1^2) = 0.023,668,622,485,21$, and $2by$ will be $(= 0.307,692,2 \times 0.048,852,4) = 0.015,031,502,431,28$, and $bb - 2by$ will be $(= 0.023,668,622,485,21 - 0.015,031,502,431,28) = 0.008,637,120,053,93$, and $\sqrt{bb - 2by}$ will be $(= \sqrt{0.008,637,120,053,93}) = 0.092,936,1$. Therefore $b - \sqrt{bb - 2by}$ will be $(= 0.153,846,1 - 0.092,936,1) = 0.060,910,0$. Therefore r , or $1 + b - \sqrt{bb - 2by}$, or $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, will be $(= 1 + 0.060,910,0) = 1.060,910,0$; that is, the first near value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, (or $\frac{z+a}{z} \times r^{40} - r^{41} = \frac{a}{z}$, or $\frac{300.926,042,2 + 20}{300.926,042,2} \times r^{40} - r^{41} = \frac{20}{300.926,042,2}$, or $\frac{320.926,042,2}{300.926,042,2} \times r^{40} - r^{41} = \frac{20}{300.926,042,2}$), or $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, which is obtained by means of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley, will be $= 1.060,910,0$. Q. E. I.

This value of r is tolerably near it's true value 1.06. For, according to it, the interest of 100 pounds for one year would be $(= 100 \times 0.060,910,04) = 6.091,004 = 6l. 1s. 9\frac{1}{4}d.$, instead of being exactly 6 pounds. Therefore even in this extreme case it appears that this expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley as a near value of r , is but a little greater than the truth. But, if we make use of it as the ground of another approximation to the true value of r in the method recommended by Mr. Raphson, the second near value of r that will be thereby obtained, will be much more exact and as near to the true value of r as need to be wished. This approximation may be made in the manner following.

Art. 13. We are now to resolve the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, or $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, by Mr. Raphson's method of approximation, after having already discovered by means of Dr. Halley's expression, $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, that 1.060,910,0, or 1.060,91, will be a first near value of r , or the root sought, that will be tolerably near the true value of r , but somewhat too great.

Let us therefore put w for the unknown small excess of 1.06091 above r , and let $1.06091 - w$ be substituted instead of r in the terms of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, but with an omission of all such terms as will involve any higher powers of the unknown quantity w than it's simple power, or w itself.

Then,

Then, since r is $= 1.06091 - w$, we shall have $r^{40} = \overline{1.06091 - w}^{40}$
 $=$, by the binomial theorem, $\overline{1.060,91}^{40} - 40 \times \overline{1.060,91}^{39} \times w + \&c$;
 and consequently $1.066,461,5 \times r^{40}$ will be $(= 1.066,461,5 \times$
 $\overline{1.060,91}^{40} - 40 \times \overline{1.060,91}^{39} \times w + \&c) = 1.066,461,5 \times \overline{1.060,91}^{40}$
 $- 1.066,461,5 \times 40 \times \overline{1.060,91}^{39} \times w + \&c$.

And r^{41} will be $= \overline{1.06091 - w}^{41} =$, by the binomial theorem, $\overline{1.060,91}^{41}$
 $- 41 \times \overline{1.060,91}^{40} \times w + \&c$.

Therefore $1.066,461,5 \times r^{40} - r^{41}$ will be $=$ the following compound quantity, to wit,

$$\left\{ \begin{array}{l} 1.066,461,5 \times \overline{1.060,91}^{40} - 1.066,461,5 \times 40 \times \overline{1.060,91}^{39} \times w + \&c \\ - \overline{1.060,91}^{41} + 41 \times \overline{1.060,91}^{40} \times w - \&c. \end{array} \right\}$$

We must therefore, in the next place, find the values of $\overline{1.060,91}^{39}$, $\overline{1.060,91}^{40}$, and $\overline{1.060,91}^{41}$.

Now the logarithm of 1.06091 is $= 0.025,678,5$. Therefore the logarithm of $\overline{1.060,91}^{40}$ will be $(= 40 \times 0.025,678,5) = 1.027,140,0$; which is the logarithm of $10.644,860,3$. Therefore $\overline{1.060,91}^{40}$ will be $= 10.644,860,3$; and consequently $\overline{1.060,91}^{41}$ will be $(= 1.060,91 \times \overline{1.060,91}^{40} = 1.060,91 \times 10.644,860,3) = 11.293,238,7$, and $\overline{1.060,91}^{39}$ will be $(= \frac{\overline{1.060,91}^{40}}{1.060,91} = \frac{10.644,860,3}{1.060,91}) = 10.033,707,2$.

Therefore $1.066,461,5 \times r^{40} - r^{41}$ (which has been shewn to be equal to the compound quantity

$$1.066,461,5 \times \overline{1.060,91}^{40} - 1.066,461,5 \times 40 \times \overline{1.060,91}^{39} \times w + \&c - \overline{1.061,91}^{41} + 41 \times \overline{1.060,91}^{40} \times w - \&c) \text{ will be } = \text{the compound quantity}$$

$$\left\{ \begin{array}{l} 1.066,461,5 \times 10.644,860,3 - 1.066,461,5 \times 40 \times 10.033,707,2 \times w + \&c \\ - 11.293,238,7 + 41 \times 10.644,860,3 \times w - \&c \end{array} \right\}$$

$=$ the compound quantity

$$\left\{ \begin{array}{l} 11.352,333,7 - 1.066,461,5 \times 401.348,288,0 \times w + \&c \\ - 11.293,238,7 + 436.439,272,3 \times w - \&c \end{array} \right\}$$

$=$

= the compound quantity

$$\left\{ \begin{array}{l} 11.352,333,7 - 428.022,497,2 \times w + \&c \\ - 11.293,238,7 + 436.439,272,3 \times w - \&c \end{array} \right\}$$

$$= 0.059,095,0 + 8.416,775,1 \times w \&c.$$

But the quantity $1.066,461,5 \times r^{40} - r^{41}$ is $= 0.066,461,5$.

Therefore the compound quantity $0.059,095,0 + 8.416,775,1 \times w \&c$ will also be $= 0.066,461,5$. And consequently $8.416,775,1 \times w$ will be $(= 0.066,461,5 - 0.059,095,0) = 0.007,366,5$, and w will be $(= \frac{0.007,366,5}{8.416,775,1}) = 0.000,875,2$. Therefore r , or $1.060,91 - w$, will be $(= 1.060,91 - 0.000,875,2) = 1.060,034,8$; that is, the rate of interest sought will be $= 1.060,034,8$. Q. E. I.

This value of r is exceedingly near to it's true value, which is 1.06. For, according to this value of r , the interest of a hundred pounds for one year would be $(= 100 \times 0.060,034,8\%) = 6.003,48\%$, or 6*l.* 0*s.* 0*d.*, or 6 pounds, and about 3 farthings, which exceeds the true interest, which is 6 pounds, by less than a penny. We may therefore now safely conclude that, when t , or the number of years during which the annuity is to continue, is not greater than 40 years, the value of r , or the rate of interest, or the root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, may be obtained to a great degree of exactness by means of the expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley, and one process of Mr. Raphson's method of approximation grounded on the first value of r obtained by that expression.

Art. 14. Though the foregoing expression, $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, given us by Dr. Halley, and investigated and illustrated by examples in the preceeding part of this Third Note, is a very useful one, on account of it's near approach to the true value of r , or the root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in all those cases of that equation in which Dr. Halley recommends the use of it to his readers; yet the difficulty of investigating it by the application of Mr. De Moivre's Multinomial Theorem to it in the case of fractional and negative powers, and the remoteness and subtlety of the principles on which that investigation is grounded, have induced me to endeavour to find some other expression for a first near value of r that should be derived from simpler and more natural principles. And the result of my inquiries has been the discovery of another expression which is obtained by a course of reasoning much more

more perspicuous and satisfactory than that by which we obtained the foregoing expression of Dr. Halley, and which is near enough to the truth to be, in many cases, highly useful of itself alone, or without any correction of the value of r exhibited by it, and to be in all other cases an excellent groundwork, or basis, for a further approach to the true value of r by a process of Mr. Raphson's method of approximation. This expression may be found in the following manner.

The Investigation of another Expression for a near Value of r in the Equation

$$\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}.$$

Art. 15. It has been shewn above in art. 2, that the fraction $\frac{z}{at}$ is equal to the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \&c.$ Therefore (adding the terms $\frac{t+1}{2} \times v$ and $\frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3$, and the following terms of the series which involve the following odd powers of v , to wit, $v^5, v^7, v^9, v^{11}, \&c.$, which are all marked with the sign $-$, to both sides,) we shall have $\frac{z}{at} + \frac{t+1}{2} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c = 1 + \frac{t+1}{2} \times \frac{t+2}{3} \times vv + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \&c.$ and (subtracting all the terms on the right-hand side of this equation, except the first term 1, that is, all the terms involving vv, v^4 , and the following even powers of v , from both sides,) $\frac{z}{at} + \frac{t+1}{2} \times v - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times vv + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c = 1$, and (subtracting $\frac{z}{at}$ from both sides,) the series $\frac{t+1}{2} \times v - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times vv + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c$

$$\times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c$$

$$\Rightarrow 1 - \frac{z}{at} = \frac{at - z}{at}.$$

Now let both sides of this last equation be multiplied into the fraction $\frac{2}{t+1}$, or the reciprocal of $\frac{t+1}{2}$, the numeral co-efficient of v in the first term of this series. And we shall then have the series $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c = \frac{2at - 2z}{at^2 + at}$, or, (putting $b, c, d, e, f, \&c$ for the several co-efficients of $vv, v^3, v^4, v^5, v^6, \&c$, respectively, and b for the absolute term $\frac{2at - 2z}{at^2 + at}$) the series $v - b \times v^2 + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, or $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+3}{4} \times b \times v^3 - \sqrt{\frac{t+4}{5}} \times c \times v^4 + \frac{t+5}{6} \times d \times v^5 - \sqrt{\frac{t+6}{7}} \times e \times v^6 + \&c = b$. This is the best and simplest form to which this equation can be reduced.

Art. 16. Now in this last equation the unknown quantity v , (being the interest of one pound for one year,) is much less than 1, and the absolute term b being equal to $\frac{2at - 2z}{at^2 + at} = \left[1 - \frac{z}{at} \right] \times \frac{2}{t+1}$, is also less than 1, and consequently the powers both of v and of b , to wit, $v, v^2, v^3, v^4, v^5, v^6, \&c$, and $b, b^2, b^3, b^4, b^5, b^6, \&c$, will form decreasing progressions. Therefore, by what is shewn in page 711 of the 3d Volume of this Collection of Tracts called *Scriptores Logarithmici*, the least root of the said equation may be found by means of a certain general series obtained, in the said Volume of the *Scriptores Logarithmici* in pages 703, 704, 705, &c, --- 710, by Sir Isaac Newton's second method of reverting infinite serieses, which is founded on the clearest and plainest principles of Arithmetick. The five first terms of this general series expressed in the notation of the foregoing equation $v - b \times vv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, will be $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5$; so that v , or the root of the equation $v - b \times vv + cv^3 - d \times v^4 + e \times v^5 - f \times v^6 + \&c = b$ will be, nearly,

nearly, = the said five terms $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5$. Q. E. I.

I will now proceed to apply this series to the determination of the value of v in the particular equation which Dr. Halley has resolved as an example of the utility of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, which he has given us for this purpose.

An Example of the Application of the Series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$ to the Determination of the Value of v in the foregoing Equation $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, when r , or $1 + v$, is the Root of the Numeral Equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, which has been resolved above in Art. 7 by means of Dr. Halley's Expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$.

Art. 17. In that equation the annuity a is 20*l.* per annum, and t , or the number of years during which the annuity is to continue, is 21 years, and z , or the present value of the annuity, is 220*l.* Therefore $z + a$ is ($= 220*l.* + 20*l.*$) = 240*l.*, and $\frac{z+a}{z}$ is ($= \frac{240}{220}$) = 1.090,909,0, and $\frac{a}{z}$ is ($= \frac{20}{220} = \frac{2}{22} = \frac{1}{11}$) = 0.090,909,0, and the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ is converted into the numeral equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$.

Art. 18. Now, since a is = 20, and t is = 21, and z is = 220, we shall have at ($= 20 \times 21$) = 420, and $2at$ ($= 2 \times 420$) = 840, and $2z$ ($= 2 \times 220$) = 440, and consequently $2at - 2z$ ($= 840 - 440$) = 400; and we shall have at^2 ($= a \times t^2 = 20 \times 21^2 = 20 \times 441$) = 8820, and $at^2 + at$ ($= 8820 + 420$) = 9240. Therefore $\frac{2at - 2z}{at^2 + at}$ will be ($= \frac{400}{9240}$) = 0.043,290,0; that is, b will be = 0.043,290,0.

Therefore b^2 will be ($= \overline{0.043,290,0^2}$) = 0.001,874,0, and b^3 will be ($= b^2 \times b = 0.001,874,0 \times 0.043,290,0$) = 0.000,081,1, and b^4 will be ($=$

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($= b^3 \times b = 0.000,081,1 \times 0.043,290,0$) $= 0.000,003,5$, and b^5 will be
 ($= b^4 \times b = 0.000,003,5 \times 0.043,290,0$) $= 0.000,000,15$.

Therefore the five terms $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5$ will be $= 0.043,290,0 + b \times 0.001,874,0 + \overline{2bb - c} \times 0.000,081,1 + \overline{5b^3 - 5bc + d} \times 0.000,003,5 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times 0.000,000,15$.

Further, since b is $= \frac{t+2}{3}$, and c is $= b \times \frac{t+3}{4}$, and d is $= c \times \frac{t+4}{5}$, and e is $= d \times \frac{t+5}{6}$, and f is $= e \times \frac{t+6}{7}$, and t is in this case $= 21$, we shall have

$$b \left(= \frac{t+2}{3} = \frac{21+2}{3} = \frac{23}{3} \right) = 7.666,666,6 \text{ \&c,}$$

$$\text{and } c \left(= b \times \frac{t+3}{4} = b \times \frac{21+3}{4} = b \times \frac{24}{4} = b \times 6 = 7.666,666,6 \text{ \&c} \right. \\ \left. \times 6 \right) = 46.000,000,0,$$

$$\text{and } d \left(= c \times \frac{t+4}{5} = c \times \frac{21+4}{5} = c \times \frac{25}{5} = c \times 5 = 46 \times 5 \right) \\ = 230.000,000,0,$$

$$\text{and } e \left(= d \times \frac{t+5}{6} = d \times \frac{21+5}{6} = d \times \frac{26}{6} = 230 \times \frac{26}{6} = 230 \times \right. \\ \left. 4.333,333,3 \text{ \&c} \right) = 996.666,666,6 \text{ \&c,}$$

$$\text{and } f = e \times \frac{t+6}{7} = e \times \frac{21+6}{7} = e \times \frac{27}{7} = 996.666,666,6 \times 3.857,142,8 \\ = 3844.285,657,3;$$

and consequently the general equation $v - bv^2 + cv^3 - dv^4 + ev^5 - fv^6 + \text{\&c} = b$ will in this case become $v - 7.666,666,6 \text{ \&c} \times vv + 46 \times v^3 - 230 \times v^4 + 996.666,666,6 \times v^5 - 3844.285,657,3 \times v^6 + \text{\&c} = 0.043,290,0$; which is an equation now properly prepared for resolution.

Art. 19. Having thus found the values of b, c, d, e , and f , the co-efficients of vv, v^3, v^4, v^5 , and v^6 in the equation $v - bv^2 + cv^3 - dv^4 + ev^5 - fv^6 + \text{\&c} = b$, and the value of it's absolute term b , we must now compute the terms of the reverted series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \text{\&c}$, which is equal to v , the root of the former equation. This may be done in the following manner.

Since

Since b is $= 7.666,666,6$ &c, we shall have bb ($= \overline{7.666,666,6 \times c}^2$) $= 58.777,777,7$ &c, and b^3 ($= bb \times b = 58.777,777,7 \times 7.666,666,6$) $= 450.629,629,6$, and b^4 ($= b^3 \times b = 450.629,629,6 \times 7.666,666,6$) $= 3404.827,130,2$. Therefore $2bb$ will be ($= 2 \times 58.777,777,7$) $= 117.555,555,5$, and $5b^3$ will be ($= 5 \times 450.629,629,6$) $= 2253.148,148,0$, and $14b^4$ will be ($= 14 \times 3404.827,130,2$) $= 47,667.579,822,8$.

And, since c is $= 46$, we shall have c^2 ($= 46^2$) $= 2116$, and bc ($= 7.666,666,6 \times 46$) $= 352.666,666,6$, and $5bc$ ($= 5 \times 352.666,666,6$) $= 1763.333,333,3$, and bbc ($= 58.777,777,7 \times 46$) $= 2703.777,774,2$, and $2bbc$ ($= 21 \times 2703.777,774,2$) $= 56,779.333,258,2$, and $3c^2$ ($= 3 \times 2116$) $= 6348$.

And, since d is $= 230$, we shall have bd ($= 7.666,666,6 \times 230$) $= 1763.333,318,0$, and $6bd$ ($= 6 \times 1763.333,318,0$) $= 10,580$.

And e is $= 996.666,666,6$.

Therefore $2bb - c$ will be ($= 117.555,555,5 - 46$) $= 71.555,555,5$; and $5b^3 - 5bc$ will be ($= 2253.148,148,0 - 1763.333,333,3$) $= 489.814,814,7$, and $5b^3 - 5bc + d$ will be ($= 489.814,814,7 + 230$) $= 719.814,814,7$; and $3c^2 + 6bd + 14b^4$ will be ($= 6348 + 10,580 + 47,667.579,822,8$) $= 64,595.579,822,8$, and $2bbc + e$ will be $= 56,779.333,258,2 + 996.666,666,6 = 57,776$, and consequently $3c^2 - 2bbc + 6bd + 14b^4 - e$ will be ($= 64,595.579,822,8 - 57,776$) $= 6819.579,822,8$.

Therefore the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 2bbc + 6bd + 14b^4 - e} \times b^5 + \&c$ will be $= b + 7.666,666,6 \times b^2 + 71.555,555,5 \times b^3 + 719.814,814,7 \times b^4 + 6819.579,822,8 \times b^5 + \&c$; in which series it is remarkable that the co-efficients of the powers of b , to wit, 1, 7.666,666,6, 71.555,555,5, 719.814,814,7, and 6819.579,822,8, continually increase in nearly the proportion of 10 to 1. But this increase of these co-efficients will be much over-balanced by the greater decrease of the powers of b into which they are severally multiplied, and consequently the whole terms of the said series will decrease with considerable swiftness, as will now appear by performing these multiplications.

For $7.666,666,6 \times b^2$, the second term of the said series, will be ($= 7.666,666,6 \times 0.001,874,0$) $= 0.014,367,3$, which is much less than the first term b , or 0.043,290,0; and $71.555,555,5 \times b^3$, the third term, will be ($= 71.555,555,5 \times 0.000,081,1$) $= 0.005,803,1$, which is much less than 0.014,367,3, the second term; and $719.814,814,7 \times b^4$, the fourth term, will be ($= 719.814,814,7 \times 0.000,003,5$) $= 0.002,519,3$, which is less than 0.005,803,1, the third term; and $6819.579,822,8 \times b^5$, the fifth term, will be ($= 6819.579,822,8 \times 0.000,000,15$) $= 0.001,022,9$, which is less than 0.002,519,3, the fourth term.

Therefore

Therefore the whole five terms $b + 7.666,666,6 \times b^2 + 71.555,555,5 \times b^3 + 710.814,814,7 \times b^4 + 6819.579,822,8 \times b^5$ will be $= 0.043,290,0 + 0.014,367,3 + 0.005,803,1 + 0.002,519,3 + 0.001,022,9 = 0.067,002,6$. Therefore v will be nearly equal to, but greater than, $0.067,002,6$; and consequently $1 + v$, or r , will be nearly equal to, but greater than, $1 + 0.067,002,6$, or $1.067,002,6$. Q. E. I.

Art. 20. This near value of r falls short of it's true value (which is $1.068,14$) by $0.001,137,4$, which is less than the 939th part of the said true value. It is therefore a very good approximation to the said true value, and sufficient for many useful purposes. And, if greater exactness were required, a single process of Mr. Raphson's method of approximation, grounded upon this number $1.067,002,6$, as a first near value of r in the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, would give us a second near value of r that would be as exact as could be desired for any purpose whatsoever. But the computation of these five terms of the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$, and more especially of the fourth and fifth terms of it, has, I confess, been a very tedious and laborious business; and therefore I consider this method of finding a first near value of r as less convenient than that of Dr. Halley by means of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$.

And it must also be observed that the value of r obtained by Dr. Halley's expression is still nearer the truth than the value of it that has been just now obtained by means of the foregoing series. For that former value was $1.068,327,5$, which exceeds $1.068,14$, or the true value of r , by only $0.000,287,5$, or the 3715th part of the said true value; whereas the latter near value of r was $1.067,002,6$, which falls short of $1.068,14$, or the true value of r , by $0.001,137,4$, or the 939th part of the said true value; which latter difference is almost four times as great as the former difference $0.000,287,5$. So that on this account, as well as on that of it's being less difficult to compute than the latter value, Dr. Halley's expression seems to be preferable to the other.

Art. 21. The latter value of r , obtained by computing the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$, may, however, be rendered considerably more exact than that obtained by the foregoing computation, by an easy operation grounded on a reasonable conjecture, by means of which we shall obtain a tolerably near value of the sum of the remaining terms of the said series that come after the five terms of it that have been above computed. This conjecture may be described as follows.

An Improvement on the foregoing Resolution of the Equation $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, or $v - 7.666,666,6 \times vv + 46 \times v^3 - 230 \times v^4 + 996.666,666,6 \times v^5 - 3884.285,657,3 \times v^6 + \&c = 0.043,290,0$, by means of the Series $b + b \times b^4 + \frac{200 - c}{200 - c} \times b^3 + \frac{5b^3 - 5bc + d}{3c^2 - 21bc + 6bd + 14b^4 - e} \times b^5 + \&c$, or $0.043,290,0 + 0.014,367,3 + 0.005,803,1 + 0.002,519,3 + 0.001,022,9 + \&c$, founded on a Conjectural Estimation of the Sum of the remaining Terms of the said Series after those which have been computed.

Art. 22. If we examine the three last of the five terms $0.043,290,0$, $0.014,367,3$, $0.005,803,1$, $0.002,519,3$, and $0.001,022,9$, we shall find that the last term $0.001,022,9$ is not very different from a third proportional to the two preceeding terms $0.005,803,1$ and $0.002,519,3$. For the said third proportional would be $0.001,093,7$, which is not very much greater than the fifth term, $0.001,022,9$. Therefore it seems reasonable to conjecture that the sixth, seventh, eighth, ninth, and other following terms of this series will not be very different from a set of continued geometrical proportionals to the fourth and fifth terms $0.002,519,3$ and $0.001,022,9$, which have been above computed. And, if this conjecture is true, the sum of the remaining terms of the series which is equal to v , and of which we have computed the first five terms, will not be very different from the sum of all the remaining terms of the geometrical series of which the two first terms are $0.002,519,3$ and $0.001,022,9$; which latter sum may be easily obtained in the following manner.

The sum of all the terms of an infinite series of quantities that decrease in continued geometrical proportion, of which the two first terms are denoted by the letters i and k , is known to be equal to the fraction $\frac{i^2}{i - k}$. Therefore the sum of all the terms of an infinite series of terms decreasing in continued geometrical proportion, of which the two first terms are $0.002,519,3$ and $0.001,022,9$, will be $(= \frac{0.002,519,3^2}{0.002,519,3 - 0.001,022,9} = \frac{0.000,006,346,802,49}{0.001,496,4}) = 0.004,241,4$. Therefore the two last terms, $0.002,519,3$ and $0.001,022,9$, of the series which is equal to v , together with all the remaining terms of the said series, will, according to the aforesaid conjectural supposition, be nearly equal to $0.004,241,4$; and consequently the whole of the said series will be nearly equal

equal to the sum of it's first three terms $0.043,290,0$, $+ 0.014,367,3$ $+ 0.005,803,1$, together with the said number $0.004,241,4$; that is, to the number $0.067,701,8$. And therefore the unknown quantity v , or the root of the equation $v - 7.666,666,6 \times vv + 46 \times v^3 - 230 \times v^4 + 996.666,666,6 \times v^5 - 3844.285,657,3 \times v^6 + \&c = 0.043,290,0$, will be nearly equal to $0.067,701,8$; and $1 + v$, or r , or the root of the proposed equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, will be nearly equal to $1 + 0.067,701,8$, or $1.067,701,8$. Q. E. I.

This number, $1.067,701,8$, is a very near value of r . For it falls short of it's true value, $1.068,14$, by only the small quantity $0.000,438,2$; which is less than the 2437th part of the said true value. So that it appears that the foregoing conjecture concerning the near approach of the sum of the terms of the series that is equal to v , after the fifth term of the said series, to an equality with the sum of the terms of a geometrical series derived from the fourth and fifth terms of the former series, after the two first terms of the said geometrical series, is well-founded.

Art. 23. This conjecture will also be found to hold good, if we derive the geometrical series from the second and third terms of the series which is equal to v , to wit, the terms $0.014,367,3$ and $0.005,803,1$, instead of deriving it from the fourth and fifth terms, $0.002,519,3$ and $0.001,022,9$, of the said series, as was done in the last article. For, if we make $0.014,367,3$ and $0.005,803,1$ the two first terms of an infinite geometrical series, the sum of all

the terms of such series will be $(= \frac{0.014,367,3^2}{0.014,367,3 - 0.005,803,1} = \frac{0.000,206,419,309,29}{0.008,564,2})$

$0.024,102,5$; which, being added to the first term, $0.043,290,0$, of the series that is equal to v , will give us $0.067,392,5$ for a probable near value of the whole series, and consequently of v , and $1 + 0.067,392,5$, or $1.067,392,5$, for a probable near value of $1 + v$, or r , or the root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$. Q. E. I.

This value $1.067,392,5$ is a very good approximation to the true value of r , which is $1.068,14$. For it falls short of it by only the small quantity $0.000,747,5$, which is less than the 1428th part of the said true value, $1.068,14$. And thus by employing this probable and fortunate conjecture concerning the near approach of the sum of the terms of a geometrical series of quantities beginning with the second and third terms of the foregoing series which is equal to v , to an equality with the sum of the second and third and other following terms of the said series itself, we may save ourselves the trouble of computing the fourth term, $5b^3 - 5bc + d \times b^4$, or $0.002,519,3$, and the fifth term, $3c^2 - 21bbc + 6bd + 14b^4 - e \times b^5$, or $0.001,022,9$, of the said series, in which a great part of the labour of the foregoing computation, in art. 18 and 19, consisted.

Art. 24. We may further observe, that 0.067,392,5, or the value of v obtained by computing only the three first terms of the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \&c$, which is equal to v , and adding to them the probable value of the sum of all the remaining terms of the said series derived from the foregoing conjecture, is nearer to the true value of v (which is 1.068,14) than 0.067,002,6, or the value of v obtained in art. 19 by computing the five first terms of the said series without having recourse to the said conjectural estimation of the remaining terms; though it is less near than 0.067,701,8, or the value of v obtained in art. 22 by computing the said five first terms of the said series and by afterwards adding to them the probable sum of all the remaining terms of the series resulting from the said conjectural estimation of them. This shews the great utility of the aforesaid conjecture in enabling us to improve the first value of v , obtained by the mere computation of the first terms of the series.

Art. 25. I will now apply this method of resolving the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, by means of the said series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \&c$ to the numeral equation of the same form, which was resolved above in art. 12 by means of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$ given us by Dr. Halley, and which was afterwards resolved to a greater degree of exactness in art. 13 by a process of Mr. Raphson's method of approximation; in which equation t , or the number of years during which the annuity is to continue, is 40 years, which is the greatest number to which t is ever supposed to be equal in those cases in which Dr. Halley recommends the use of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$. This equation is $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; the annuity a being 20*l.* a year, and t , or the number of years during which it is to continue, being 40 years, and z , or the present value of it, or the price lately paid for it by a purchaser, being 300.926,042,2; whence $\frac{z+a}{z}$ is $(= \frac{320.926,042,2}{300.926,042,2}) = 1.066,461,5$, and $\frac{a}{z}$ is $(= \frac{20}{300.926,042,2}) = 0.066,461,5$, and consequently the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ is converted into the particular, or numeral, equation, $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$.

And in this equation we also know before-hand, (by means of the calculation performed in art. 11,) that the true value of r is 1.06; and therefore we shall be able to determine with certainty to what degree of exactness, or to how many places of decimal figures, the value of r , or $1 + v$, which we shall obtain by means of the said series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \&c$, will exhibit, or agree with, its true value.

Another Example of the Application of the Series $b + b \times b^2 + \sqrt{2bb - c} \times b^3 + \sqrt{5b^3 - 5bc + d} \times b^4 + \sqrt{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$ to the Determination of the Value of v in the Equation $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, when r , or $1 + v$, is the Root of the Numeral Equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, which has been resolved above in Art. 12 by means of Dr. Halley's Expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$.

Art. 26. It is shewn in art. 15, that, if v be put $= r - 1$, or $1 + v$ be $= r$, in the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, we shall thence obtain the equation $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times v^4 + \frac{t+2}{3} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times v^5 - \sqrt{\frac{t+2}{3}} \times \frac{t+3}{4} \times \frac{t+4}{5} \times \frac{t+5}{6} \times \frac{t+6}{7} \times v^6 + \&c = \frac{2at - 2z}{at^2 + at}$, or (putting b, c, d, e , and f , for the co-efficients of vv, v^3, v^4, v^5, v^6 , respectively, and b for the absolute term $\frac{2at - 2z}{at^2 + at}$;) the equation $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, or $v - \sqrt{\frac{t+2}{3}} \times vv + \frac{t+2}{3} \times b \times v^3 - \sqrt{\frac{t+2}{3}} \times c \times v^4 + \frac{t+2}{3} \times d \times v^5 - \sqrt{\frac{t+2}{3}} \times e \times v^6 + \&c = b$.

Further, it was shewn in art. 16 that $\frac{2at - 2z}{at^2 + at}$, or b , must always be less than 1, and consequently that b^2, b^3, b^4, b^5 , and the following powers of b , must all be less than b itself, and must form a decreasing progression of terms; and it was shewn also that v must be less than 1, and consequently that vv will be less than v , and that v^3 will be less than vv , and v^4 than v^3 , and every following power of v less than that which immediately precedes it, or that the powers of v , to wit, $v, vv, v^3, v^4, v^5, v^6$, &c will also form a decreasing progression. And hence it follows, (from what is shewn in the 3d Volume of this Collection of Tracts, called *Scriptores Logarithmici*, page 711,) that this equation may be resolved by means of a certain series there set forth, which is investigated in pages 703, 704, 705, &c - - - 710 of the said Volume, by Sir

Sir Isaac Newton's second method of reverting infinite serieses, which is founded on the clearest and plainest principles of Arithmetick, and consequently that v , or the root of the equation $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c = b$, will be equal to an infinite series of which the five first terms are $b + b \times b^3 +$

$\frac{2bb - c}{5} \times b^3 + \frac{5b^3 - 5bc + d}{4} \times b^4 + \frac{3c^2 - 2bbc + 6bd + 14b^4 - e}{3} \times b^5.$

We must therefore now determine the numeral values of $b, c, d, e, f, \&c$ and of $b, b^2, b^3, b^4, b^5, \&c$, or of so many of these quantities as we shall think fit to make use of in the resolution of the equation $v - bvv + cv^3 - dv^4 + ev^5$

$$- fv^6 + \&c = b, \text{ or } v - \frac{t+2}{3} \times vv + \frac{t+3}{4} \times b \times v^3 - \frac{t+4}{5} \times c \times v^5 + \frac{t+5}{6} \times d \times v^5 - \frac{t+6}{7} \times e \times v^6 + \&c = b.$$

Art. 27. Now in this equation t is = 40. Therefore $\frac{t+2}{3}$ will be = $\frac{42}{3} = 14$, and $\frac{t+3}{4}$ will be = $\frac{43}{4}$, and $\frac{t+4}{5}$ will be = $\frac{44}{5}$, and $\frac{t+5}{6}$ will be = $\frac{45}{6}$, and $\frac{t+6}{7}$ will be = $\frac{46}{7}$.

Therefore we shall have $b (= \frac{42}{3}) = 14$, and $c (= b \times \frac{t+3}{4} = 14 \times \frac{43}{4} = 7 \times \frac{43}{2} = \frac{301}{2}) = 150.5$, and $d (= c \times \frac{t+4}{5} = 150.5 \times \frac{44}{5} = 30.1 \times 44 = 1324.4$, and $e (= d \times \frac{t+5}{6} = 1324.4 \times \frac{45}{6} = \frac{59,598}{6}) = 9933$; and $f (= e \times \frac{t+6}{7} = 9933 \times \frac{46}{7} = \frac{456,918}{7}) = 65,274$.

Therefore the series $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c$ will be = $v - 14vv + 150.5 \times v^3 - 1324.4 \times v^4 + 9933 \times v^5 - 65,274 \times v^6 + \&c$.

We must next find the value of b .

Now b is = the fraction $\frac{2at - 2z}{at^2 + at}$. But a is = 20 l ., and t is = 40 years, and z is = 300.926,042,2 l .

Therefore at will be (= 20 $\times$ 40) = 800, and $2at$ will be (= 2 $\times$ at) = 1600, and $2z$ will be (= 2 $\times$ 300.926,042,2) = 601.852,084,4, and $2at - 2z$ will be (= 1600 - 601.852,084,4) = 998.147,915,6; and at^2 will be (= 20 $\times$ 40² = 20 $\times$ 1600) = 32,000, and $at^2 + at$ will be (= 32,000 + 800) = 32,800. Therefore the fraction $\frac{2at - 2z}{at^2 + at}$ will be (= the fraction $\frac{998.147,915,6}{32,800}$) = 0.030,431,3; that is, b will be = 0.030,431,3.

U u 2

Therefore

Therefore the last equation $v - 14 \times vv + 150.5 \times v^3 - 1324.4 \times v^4 + 9933 \times v^5 - 65,274 \times v^6 + \&c = b$ will become $v - 14 \times vv + 150.5 \times v^3 - 1324.4 \times v^4 + 9933 \times v^5 - 65,274 \times v^6 + \&c = 0.030,431,3$; which is an equation properly prepared for a resolution by means of the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$, of which I will now proceed to compute the four first terms $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4$.

Art. 28. Since b is $= 0.030,431,3$, we shall have $b^2 (= \overline{0.030,431,3^2}) = 0.000,926,064$, &c, and $b^3 (= b^2 \times b = 0.000,926,064 \times 0.030,431,3) = 0.000,028,181$, and $b^4 (= b^3 \times b = 0.000,028,181 \times 0.030,431,3) = 0.000,000,857,584$. Therefore the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \&c$ will be $= 0.030,431,3 + b \times 0.000,926,064 + \overline{2bb - c} \times 0.000,028,181 + \overline{5b^3 - 5bc + d} \times 0.000,000,857,584 + \&c$.

Now, since b is $= 14$, we shall have $bb (= \overline{14^2}) = 196$, and $b^3 (= bb \times b = 196 \times 14) = 2744$, and consequently $2bb (= 2 \times 196) = 392$, and $5b^3 (= 5 \times 2744) = 13,720$.

And, because c is $= 150.5$, we shall have $bc (= 14 \times 150.5) = 2107$, and $5bc (= 5 \times 2107) = 10,535$.

Therefore $2bb - c$ will be $(= 392 - 150.5) = 241.5$, and $5b^3 - 5bc$ will be $(= 13,720 - 10,535) = 3185$; and, because d is $= 1324.4$, we shall have $5b^3 - 5bc + d (= 3185 + 1324.4) = 4509.4$.

Therefore the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \&c$ will be $= b + 14 \times b^2 + 241.5 \times b^3 + 4509.4 \times b^4 + \&c = 0.030,431,3 + 14 \times 0.000,926,064 + 241.5 \times 0.000,028,181 + 4509.4 \times 0.000,000,857,584 + \&c = 0.030,431,3 + 0.012,964,896 + 0.006,805,711 + 0.003,867,189 + \&c = 0.054,069,096 + \&c$, or (neglecting the two last figures of these two terms,) $0.054,069,0$ &c. Therefore this number $0.054,069,0$ will be nearly equal to, but less than, the true value of v in the equation $v - 14vv + 150.5 \times v^3 - 1324.4 \times v^4 + 9933 \times v^5 - 65,274 \times v^6 + \&c = 0.030,431,3$. Q. E. I.

Therefore $1 + v$, or r , or the root of the equation $1.066,461,5 \times r^{10} - r^{11} = 0.066,461,5$, (from which the equation $v - 14vv + 150.5 \times v^3 - 1324.4 \times v^4 + 9933 \times v^5 - 65,274 \times v^6 + \&c = 0.030,431,3$ was derived,) will be nearly equal to, but less than, $1 + 0.054,069,0$, or $1.054,069,0$.

Art. 29.

Art. 29. The difference of this number, 1.054,069,0, from the true value of r in the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, (which is 1.06,) is 0.005,931,0; which is somewhat less than the 178th part of the said true value. This difference is certainly considerable; but yet it is less than I expected to find it in this extreme and most difficult case of the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in which t is = 40, or is of the greatest magnitude which Dr. Halley supposes it ever to have in the cases for which he recommends the use of his expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$. But even in this case, if we have recourse to the conjectural method, above-mentioned in art. 22, of estimating the sum of all the terms of the series $b + b \times b^2 + \sqrt{2bb - c} \times b^3 + \sqrt{5b^3 - 5bc + d} \times b^4 + \&c$ after the said four first terms, and add the probable value of the said sum to the sum of the said four terms that have been already computed, it will give us a very excellent second near value of v , that will approach to it's true value to a degree that seems very surprizing. For, if we suppose the third and fourth terms of the series $b + b \times b^2 + \sqrt{2bb - c} \times b^3 + \sqrt{5b^3 - 5bc + d} \times b^4 + \&c$, to wit, the terms 0.006,805,711 and 0.003,867,189, or (neglecting the two last figures of these numbers,) 0.006,805,7 and 0.003,867,2, to be the two first terms of an infinite series of terms in continued geometrical proportion, the sum of all the terms of such geometrical series of quantities will be $(= \frac{0.006,805,7 \times 0.006,805,7}{0.006,805,7 - 0.003,867,2} = \frac{0.000,046,317,552,49}{0.002,938,5})$ = 0.015,762,3; which being added to the two first terms of the said series $b + b \times b^2 + \sqrt{2bb - c} \times b^3 + \sqrt{5b^3 - 5bc + d} \times b^4 + \&c$, which is equal to v , to wit, the two terms b and $b \times b^2$, or 0.030,431,3 and 0.012,964,9, will make the probable value of the whole of the said series to be 0.059,158,5. Therefore v will be, nearly, = 0.059,158,5, and consequently $1 + v$, or r , or the root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, will be, nearly, = $1 + 0.059,158,5$, or 1.059,158,5; which falls short of 1.06, or the true value of r , in that equation, by only the small quantity 0.000,841,5, which is less than the 1259th part of the said true value.

Art. 30. According to this near value of r , to wit, 1.059,158,5, the interest of a hundred pounds for a year would be $(= 100 \times 0.059,158,5) = 5.915,85\text{ }l.$, or 5*l.* 18*s.* 3*d.*, instead of being exactly 6 pounds. This degree of exactness seems very great for this extreme and most difficult case, in which t is = 40, or of the greatest magnitude that Dr. Halley supposes it ever to have in those cases for which he recommends the use of his expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$. And it equals the degree of exactness of that expression of Dr. Halley in the same case. For the near value of r in this equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, that resulted from the computation of this

this expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$ made above in art. 12, is 1.060,910,0, which exceeds it's true value 1.06 by the quantity 0.000,910,0, which is a little greater than the quantity 0.000,841,5, by which the foregoing near value of r , 1.059,158,5, falls short of the said true value.

An arithmetical mean between 1.060,910,0, (the near value of r in this equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, obtained above in art. 12 by means of Dr. Halley's expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$), and 1.059,158,5, (the near value of r just now obtained by means of the four first terms of the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \&c$, together with the above-mentioned conjectural estimation of the sum of all the following terms of the said series,) will be $(= \frac{1.060,910,0 + 1.059,158,5}{2} = \frac{2.120,068,5}{2}) = 1.060,034,2$, which is extremely near to 1.06, or the true value of r .

Art. 31. If the number 1.059,158,5, obtained in art. 29 for the value of r in the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$ by means of the series $b + b \times b^2 + \overline{2bb - c} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \&c$, be made the ground-work of a further approach to the true value of r by Mr. Raphson's method of approximation, one process of that method will be sufficient to procure a second near value of r that shall be as near to it's true value 1.06 as need ever be desired for any purpose whatsoever. Such a process may be performed as follows.

Let w be the quantity by which the number 1.059,158,5 falls short of the true value of r .

Then will r be $= 1.059,158,5 + w$, and consequently r^{40} will be $= \overline{1.059,158,5 + w}^{40} =$, by the binomial theorem, to the series $\overline{1.059,158,5}^{40} + 40 \times \overline{1.059,158,5}^{39} \times w + \&c$, and r^{41} will be $= \overline{1.059,158,5 + w}^{41} =$, by the binomial theorem, to the series $\overline{1.059,158,5}^{41} + 41 \times \overline{1.059,158,5}^{40} \times w + \&c$.

We must therefore find the values of $\overline{1.059,158,5}^{39}$, $\overline{1.059,158,5}^{40}$, and $\overline{1.059,158,5}^{41}$.

Now the logarithm of 1.059,158,5 is $= 0.024,960,9$. Therefore the logarithm of $\overline{1.059,158,5}^{40}$ will be $(= 40 \times 0.024,960,9) = 0.998,436,0$; which is the logarithm of the number 9.964,052,3. Therefore $\overline{1.059,158,5}^{40}$ is

is = 9.964,052,3, and consequently $\overline{1.059,158,5}^{41}$ will be ($= \overline{1.059,158,5}^{40} \times 1.059,158,5 = 9.964,052,3 \times 1.059,158,5$) = 10.553,510,6, and $\overline{1.059,158,5}^{39}$ will be ($= \frac{\overline{1.059,158,5}^{40}}{1.059,158,5} = \frac{9.964,052,3}{1.059,158,5}$) = 9.407,517,6.

Therefore r^{40} will be ($=$ the series $\overline{1.059,158,5}^{40} + 40 \times \overline{1.059,158,5}^{39} \times w + \&c$ = the series 9.964,052,3 + 40 $\times$ 9.407,517,6 $\times w$ + $\&c$) = the series 9.964,052,3 + 376.300,704,0 $\times w$ + $\&c$; and r^{41} will be ($=$ the series $\overline{1.059,158,5}^{41} + 41 \times \overline{1.059,158,5}^{40} \times w + \&c$ = the series 10.553,510,6 + 41 $\times$ 9.964,052,3 $\times w$ + $\&c$) = 10.553,510,6 + 408.526,144,3 $\times w$ + $\&c$.

Therefore $1.066,461,5 \times r^{40}$ will be ($= 1.066,461,5 \times$ the series 9.964,052,3 + 376.300,704,0 $\times w$ + $\&c$ = $1.066,461,5 \times 9.964,052,3$ + $1.066,461,5 \times 376.300,704,0 \times w$ + $\&c$) = 10.626,278,1 + 401.310,213,2 $\times w$ + $\&c$. And consequently the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ will be = the compound quantity

$$\left\{ \begin{array}{l} 10.626,278,1 + 401.310,213,2 \times w + \&c \\ - 10.553,510,6 - 408.526,144,3 \times w - \&c \end{array} \right\}$$

$$= 0.072,767,5 - 7.215,931,1 \times w - \&c.$$

But the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ is = 0.066,461,5.

Therefore the compound quantity $0.072,767,5 - 7.215,931,1 \times w - \&c$ will also be = 0.066,461,5.

Therefore $0.072,767,5 - \&c$ will be = $0.066,461,5 + 7.215,931,1 \times w$, and $7.215,931,1 \times w$ will be ($= 0.072,767,5 - 0.066,461,5 - \&c$) = $0.006,306,0 - \&c$, and consequently w will be = $\frac{0.006,306,0}{7.215,931,1} - \&c = 0.000,873,8, - \&c$.

Therefore r , or $1.059,158,5 + w$, will be = $1.059,158,5 + 0.000,873,8 - \&c = 1.060,032,3 - \&c$, or the second near value of r , obtained by this one process of Mr. Raphson's method of approximation, will be 1.060,032,3.

Q. E. I.

This number 1.060,032,3 is so nearly equal to 1.06, or the true value of r , that, according to it, the interest of 100 pounds for one year would be ($= 100 \times 0.060,032,3$) = 6.003,23 l ., or 6 l . os. 0 $\frac{3}{4}$ d ., or 6 pounds and about 3 farthings, instead of being exactly 6 pounds. And this, I presume, is as great a degree of exactness as need ever be desired for any purpose whatsoever.

Art. 32.

Art. 32. It appears therefore that even in this extreme and most difficult case of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in which t , or the number of years during which the annuity is to continue, is 40 years, (which is the greatest number of years for which the annuity is ever supposed to be granted in those cases of the said equation for which Dr. Halley recommends the use of his expression $1 + b - \sqrt{bb - 2by}$ to obtain a near value of r ,) a near value of r , as near to it's true value as the near value of it obtained by that expression of Dr. Halley, may also be obtained by the other method set forth in the foregoing articles, which is derived from Sir Isaac Newton's second method of reverting infinite serieses; which second method of reversion is founded on the clearest and plainest principles of Arithmetick. And it appears likewise that the value of r , obtained in this extreme case by this latter method, will, if made the ground-work of a further approach to the true value of r by Mr. Raphson's method of approximation, enable us, by only a single process of the said method, to find a second near value of r that shall be as exact as ever can be desired. And hence it will follow, *a fortiori*, that in all easier cases of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, or when t , or the number of years for which the annuity a is granted, is less than 40 years, the same method of proceeding will give us the value of r , or the root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, to a greater degree of exactness than in this extreme case, and consequently that, if it is accompanied by only one process of Mr. Raphson's method of approximation, it will enable us to find a second near value of r that will be nearer to it's true value than 1.060,032,3, or the value of r obtained above in art. 31, is to 1.06, or 1.060,000,0, it's true value, in the numeral equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, and which therefore will be as near to the true value of r as need ever be desired for any purposes whatsoever.

Art. 33. If the above-mentioned conjectural method of estimating the sum of the remaining terms of the series $b + b \times b^2 + \overline{2bb - d} \times b^3 + \overline{5b^3 - 5bc + d} \times b^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + 8c$ after a few of the first terms of it have been computed, (which in the two foregoing examples has proved so successful) should be thought worthy to be adopted; and, if, in consequence of adopting it, we should compute only the four first of the said five terms of the said series, the labour of calculation required in the application of this method would be considerably diminished by dropping the computation of the fifth and most complicated term $\overline{3c^2 - 21bbc + 6bd + 14b^4 - e}$

$\times b^t$, and, perhaps, would thereby become not much greater than that of computing Dr. Halley's expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, in which the finding the value of y or $\left(\frac{at}{z}\right)^{\frac{2}{t-1}}$ is rather a troublesome operation. And in this case, perhaps, this second method of obtaining the value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, when t is not greater than 40, may be justly reckoned as convenient as Dr. Halley's method of obtaining it by means of the said expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, with which it has been shewn to be equally, if not more, exact. But this must be left to the decision of the reader.

A short and convenient Direction for the Application of this last Method of resolving the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Art. 34. If this second method of obtaining a near value of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, when t is not greater than 40 years, should be thought fit to be adopted, the following short direction for the application of it may be of use.

Compute the quantities at and at^2 , and divide the quantity $2at - 2z$ by the quantity $at^2 + at$, and let the quotient be called b . And let $r - 1$ be called v , and $\frac{t+2}{3}$ be called b , and $\frac{t+3}{4} \times b$ be called c , and $\frac{t+4}{5} \times c$ be called d , and $\frac{t+5}{6} \times d$ be called e . Then will the series $v - bvv + cv^3 - dv^4 + ev^5 - fv^6 + \&c$ be $= b$, and, *vice versa*, v will be $=$ the series $b + b \times b^2 + \sqrt{2bb - c} \times b^3 + \sqrt{5b^3 - 5bc + d} \times b^4 + \sqrt{3c^2 - 21bbc + 6bd + 14b^4 - e} \times b^5 + \&c$. Compute the four first terms of this last series, and let the third term $\sqrt{2bb - c} \times b^3$ be called i , and the fourth term $\sqrt{5b^3 - 5bc + d} \times b^4$ be called k . Divide i^2 , or the square of the third term i , by $i - k$, or the excess of the third term i above the fourth term k , and add the quotient thence resulting, or the value of the fraction $\frac{i^2}{i - k}$, to the two first terms $b + b \times b^2$. And the sum $b + b \times b^2 + \frac{i^2}{i - k}$ will be very nearly equal to v , or $r - 1$; and consequently $1 + b + b \times b^2 + \frac{i^2}{i - k}$ will be very nearly $= r$.

Q. E. I.

NOTE IV.

THE last passage in the foregoing Tract of Dr. Halley on Compound Interest that seems to want explanation, is in page 229 of the present Volume, and in page 22 of the Introduction to Sherwin's Mathematical Tables, (the 3d Edition, published in the year 1741,) and is as follows.

"Lastly, by way of Corollary to the former; let it be required to find the interest allowed the purchaser when he pays a sum $= z$ for an annuity a wherein he has already a term t , to have it prolonged for a certain time equal to x .

EXAMPLE.

"An annuity of 20*l.* *per annum* that is already granted for a term of years of which 21 are still to come, may, for the sum of 40 pounds paid down, be prolonged for 10 years more, or to 31 years. What is the rate of interest required?"

"Put $T = 2t + x + 1$; and $\frac{1}{2}T$ shall be the index of a root of $\frac{ax}{z}$. Let $\sqrt[\frac{1}{2}T]{\frac{ax}{z}}$ be $= 1 + y$, and $\frac{6T + 6}{xx}$ be $= b$. I say, $r - 1$ is very near to $b - \sqrt{bb - 2by}^{\frac{1}{2}}$."

Dr. Halley has not given us any investigation of this expression, though it seems very much to require one. I shall therefore now endeavour to supply this defect by stating, in as full and clear a manner as I can, an investigation of it, or, rather, of an expression nearly equal to it and bearing a great resemblance to it, which has been communicated to me by my learned and ingenious friend Mr. Morgan, the Actuary of the Society for Equitable Assurances on Lives in Chatham Place near Black Friars' Bridge.

An Investigation of an Expression bearing a great Resemblance, and nearly equal, to the foregoing Expression $b - \overline{bb} - 2by)^{\frac{1}{2}}$ given by Dr. Halley for a near Value of $r - 1$ in the foregoing Corollary concerning a Sum of Money paid for the Prolongation of a given Annuity for an additional Number of Years.

Article 1. The value of an annuity of 1*l.* per annum for t years is known to be $= \frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$. Therefore the value of an annuity of 1*l.* per annum for $t + x$ years will be $= \frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$. Therefore the excess of the value of an annuity of 1*l.* per annum for $t + x$ years above the value of the like annuity for only t years will be equal to the excess of the sum of $\frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$ above the sum of $\frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$, that is, to the sum of $\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$. But the excess of the value of an annuity of 1*l.* a year for $t + x$ years above the value of an annuity of 1*l.* a year for only t years is equal to the value of an annuity of 1*l.* a year to commence at the end of t years, or to be added to an annuity of 1*l.* a year already granted for a term of t years. Therefore the value of such additional annuity of 1*l.* a year for x years to be added to a like annuity of 1*l.* a year for t years already granted will be $= \frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$. And consequently the value of an annuity of a pounds a year for x years to be added to a like annuity of a pounds a year for t years already granted, will be a times the former value, or will be $= a \times \left(\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1} \right) = \frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$; that is, according to Dr. Halley's notation, z will be $= \frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$. Therefore $\frac{z}{a}$ will be $= \frac{1}{r^t \times r-1} - \frac{1}{r^{t+x} \times r-1}$, and $\frac{z \times r-1}{a}$ will be $= \frac{1}{r^t} - \frac{1}{r^{t+x}}$.

X x 2

Art. 2.

Art. 2. Let v be put $= r - 1$.

Then will r be $= 1 + v$, and $r^t = \overline{1+v}^t$, and $r^{t+x} = \overline{1+v}^{t+x}$, and $\frac{z \times \overline{r-1}}{a}$ will be $= \frac{zv}{a}$.

Therefore $\frac{zv}{a}$ will be $= \frac{1}{\overline{1+v}^t} - \frac{1}{\overline{1+v}^{t+x}}$.

But $\frac{1}{\overline{1+v}^t}$ is $= \overline{1+v}^{-t}$, by the binomial theorem, to the series $1 - t \times v + t \times \frac{t+1}{2} \times vv - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 + \&c$; and $\overline{1+v}^{-(t+x)}$ is, by the same theorem, $=$ to the series $1 - [t+x] \times v + [t+x] \times \frac{t+x+1}{2} \times vv - [t+x] \times \frac{t+x+1}{2} \times \frac{t+x+2}{3} \times v^3 + [t+x] \times \frac{t+x+1}{2} \times \frac{t+x+2}{3} \times \frac{t+x+3}{4} \times v^4 + \&c$. And consequently $\overline{1+v}^{-t} - \overline{1+v}^{-(t+x)}$ will be equal to the excess of the former of these series above the latter, that is, to the series $[t+x]v - t \times v - [t+x] \times \frac{t+x+1}{2} \times vv + t \times \frac{t+1}{2} \times vv + [t+x] \times \frac{t+x+1}{2} \times \frac{t+x+2}{3} \times v^3 - t \times \frac{t+1}{2} \times \frac{t+2}{3} \times v^3 - [t+x] \times \frac{t+x+1}{2} \times \frac{t+x+2}{3} \times \frac{t+x+3}{4} \times v^4 + t \times \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^4 + \&c =$ the series

$$zv - \left\{ \begin{array}{ccc} t^2 & + & t \\ + & 2tx & + x \\ + & x^2 & \end{array} \right\} \times vv + \frac{t^2+t}{2} \times vv$$

$$+ \left\{ \begin{array}{ccc} t^3 & + & 3t^2 & + & 2t \\ + & 3t^2x & + & 6tx & + & 2x \\ + & 3tx^2 & + & 3x^2 & \\ + & x^3 & \end{array} \right\} \times v^3 - \frac{t^3+3t^2+2t}{6} \times v^3$$

$$- \left\{ \begin{array}{l} t^4 + 6t^3 + 11t^2 + 6t \\ + 4t^3x + 6x^3 + 11x^2 + 6x \\ + 6t^2x^2 + 18t^2x + 22tx \\ + 4tx^3 + 18tx^2 \\ + x^4 \end{array} \right\} \times v^4 + \frac{t^4 + 6t^3 + 11t^2 + 6t}{24} \times v^4$$

+ &c = the series

$$xv - \left\{ \begin{array}{l} 2tx + x \\ + x^2 \end{array} \right\} \times vv + \left\{ \begin{array}{l} 3t^2x + 6tx + 2x \\ + 3tx^2 + 3x^2 \\ + x^3 \end{array} \right\} \times v^3$$

$$- \left\{ \begin{array}{l} 4t^3x + 6x^3 + 11x^2 + 6x \\ + 6t^2x^2 + 18t^2x + 22tx \\ + 4tx^3 + 18tx^2 \\ + x^4 \end{array} \right\} \times v^4 + \&c.$$

Therefore $\frac{xv}{a}$ will be = to the said series

$$xv - \left\{ \begin{array}{l} 2tx + x \\ + x^2 \end{array} \right\} \times vv + \left\{ \begin{array}{l} 3t^2x + 6tx + 2x \\ + 3tx^2 + 3x^2 \\ + x^3 \end{array} \right\} \times v^3$$

$$- \left\{ \begin{array}{l} 4t^3x + 6x^3 + 11x^2 + 6x \\ + 6t^2x^2 + 18t^2x + 22tx \\ + 4tx^3 + 18tx^2 \\ + x^4 \end{array} \right\} \times v^4 + \&c; \text{ and consequently}$$

(dividing all the terms of the equation by v , we shall have $\frac{x}{a} =$ the series

$$x - \left\{ \begin{array}{l} 2tx + x \\ + x^2 \end{array} \right\} \times v + \left\{ \begin{array}{l} 3t^2x + 6tx + 2x \\ + 3tx^2 + 3x^2 \\ + x^3 \end{array} \right\} \times vv$$

$$- \left\{ \begin{array}{l} 4t^3x + 6x^3 + 11tx^2 + 6x \\ + 6t^2x^2 + 18t^2x + 22tx \\ + 4tx^3 + 18tx^2 \\ + x^4 \end{array} \right\} \times v^3 + \&c; \text{ and (dividing all} \\ 24$$

the terms by x), we shall have $\frac{z}{ax} =$ the series

$$1 - \left\{ \begin{array}{l} 2t + 1 \\ + x \end{array} \right\} \times v + \left\{ \begin{array}{l} 3t^2 + 6t + 2 \\ + 3tx + 3x \\ + x^2 \end{array} \right\} \times vv \\ 2 \qquad \qquad \qquad 6$$

$$- \left\{ \begin{array}{l} 4t^3 + 6x^3 + 11tx + 6 \\ + 6t^2x + 18t^2 + 22t \\ + 4tx^2 + 18tx \\ + x^3 \end{array} \right\} \times v^3 + \&c. \\ 24$$

Art. 3. Now, to simplify the co-efficients of the powers of v in this series, let T be put $= 2t + x + 1$, or the numerator of the co-efficient of v .

Then will TT be $= 4tt + 4tx + 4t + xx + 2x + 1$.

Therefore $\frac{2t + x + 1}{2}$, or the co-efficient of v in the foregoing series, will be $= \frac{T}{2}$; and $\left\{ \begin{array}{l} 3t^2 + 6t + 2 \\ + 3tx + 3x \\ + x^2 \end{array} \right\} \times \frac{1}{6}$, or the co-efficient of vv , will be $(=$

$$\left\{ \begin{array}{l} 4tt + 4tx + 4t + x^2 + 2x + 1 \\ - tt - tx + 2t + x + 1 \end{array} \right\} \times \frac{1}{6} = \frac{TT - tt - tx + 2t + x + 1}{6}$$

$= \frac{TT - tt - tx + T}{6} = \frac{TT + T - tt - tx}{6}$; and therefore the three first terms of the foregoing series will be $1 - \frac{T}{2} \times v + \frac{TT + T - tt - tx}{6} \times vv$, or

$$1 - \frac{T}{2} \times v + \frac{T \times T + 1 - (tt + tx)}{6} \times vv.$$

Therefore

Therefore $\frac{z}{ax}$ will be equal to the series

$$1 - \frac{T}{2} \times v + \frac{T \times \overline{T+1} - \overline{tt+tx}}{6} \times vv - \frac{\left\{ \begin{array}{l} 4t^3 + 6x^2 + 11tx + 6 \\ + 6t^2x + 18t^2 + 22t \\ + 4tx^2 + 18tx \\ + x^3 \end{array} \right\}}{24} \times v^3$$

+ &c, or (omitting the last term,) to the series

$$1 - \frac{T}{2} \times v + \frac{T \times \overline{T+1} - \overline{tt+tx}}{6} \times vv - \&c.$$

Therefore, if we raise the fraction $\frac{z}{ax}$ to any power whatsoever, (whether integral or fractional, affirmative or negative,) and raise the said series $1 - \frac{T}{2} \times v + \frac{T \times \overline{T+1} - \overline{tt+tx}}{6} \times vv - \&c$ to the same power by means of Mr. De Moivre's multinomial theorem, the said power of the said series will be equal to the same power of the said fraction $\frac{z}{ax}$.

Art. 4. Now let both the said fraction and the said series be raised to the power of which the fraction $\frac{-2}{T}$ is the index.

Then will the $\frac{-2}{T}$ th power of the said series $1 - \frac{T}{2} \times v + \frac{T \times \overline{T+1} - \overline{tt+tx}}{6} \times vv - \&c$ be equal to the $\frac{-2}{T}$ th power of the said fraction $\frac{z}{ax}$.

Now the $\frac{-2}{T}$ th power of the fraction $\frac{z}{ax}$, or the quantity $\left[\frac{z}{ax} \right]^{-\frac{2}{T}}$, is

$$\left(= \frac{1}{\left[\frac{z}{ax} \right] + \frac{2}{T}} = \frac{1}{\frac{z}{ax} + \frac{2}{T}} = 1 \times \frac{\overline{ax} + \frac{2}{T}}{z + \frac{2}{T}} = 1 \times \frac{\overline{ax}}{z} \left[+ \frac{2}{T} \right] = \frac{\overline{ax}}{z} \left[+ \frac{2}{T} \right], \right.$$

and therefore may be found by means of a Table of Logarithms, by, first, finding in such table the logarithm of the fraction $\frac{ax}{z}$ (which is the reciprocal of the fraction $\frac{z}{ax}$, or is equal to $\frac{1}{\frac{z}{ax}}$, or the quotient of the division of 1

by

by the fraction $\frac{z}{ax}$, and then multiplying the said logarithm into the fraction $\frac{2}{T}$, and, lastly, looking-out in the Table of Logarithms the number to which the product of the said multiplication corresponds, or of which it is the logarithm. For that number will be the $\frac{+2}{T}$ th power of the fraction $\frac{ax}{z}$, or the $\frac{-2}{T}$ th power of the fraction $\frac{z}{ax}$.

And the $\frac{-2}{T}$ th power of the series $1 - \frac{T}{2} \times v + \frac{T \times \overline{T+1} - \sqrt{u+ix}}{6} \times vv - \&c$ may be found by the help of Mr. De Moivre's multinomial theorem in the manner following.

Art. 5. It has been shewn above in page 243, Coroll. 3, that the m th power of the series $1 + bv + cvv + dv^3 + ev^4 + \&c$ will be equal to the infinite series

$$\begin{aligned} 1 + \frac{m}{1} \times bv + \frac{m}{1} \times \frac{m-1}{2} \times b^2 v^2 + \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3 v^3 \\ + \frac{m}{1} \times cv^2 + \frac{m}{1} \times \frac{m-1}{1} \times bcv^3 \\ + \frac{m}{1} \times dv^3 \\ + \&c. \end{aligned}$$

Therefore the m th power of the series $1 - bv + cvv - dv^3 + ev^4 - \&c$ will be equal to the infinite series

$$\begin{aligned} 1 - \frac{m}{1} \times bv + \frac{m}{1} \times \frac{m-1}{2} \times b^2 v^2 - \frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times b^3 v^3 \\ + \frac{m}{1} \times cv^2 - \frac{m}{1} \times \frac{m-1}{1} \times bcv^3 \\ - \frac{m}{1} \times dv^3 \\ + \&c. \end{aligned}$$

Now let the index m be $= \frac{-2}{T}$.

Then will $m - 1$ be $(= \frac{-2}{T} - 1 = \frac{-2}{T} - \frac{T}{T}) = \frac{-2-T}{T}$, and $m - 2$ will be $(= \frac{-2}{T} - 2 = \frac{-2}{T} - \frac{2T}{T}) = \frac{-2-2T}{T}$; and $\frac{m-1}{2}$ will be $= \frac{-2-T}{2T}$, and

and $\frac{m-2}{3}$ will be $= \frac{-2-2T}{3T}$; and $\frac{m}{1} \times \frac{m-1}{2}$ will be $(= \frac{-2}{T} \times \frac{-2-T}{2T})$
 $= \frac{+4+2T}{2T^2} = \frac{+2+T}{T^2}$; and $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3}$ will be $(= \frac{+2+T}{T^2} \times$
 $\frac{m-2}{3} = \frac{+2+T}{T^2} \times \frac{-2-2T}{3T}) = \frac{-4-6T-TT}{3T^3}$; and $\frac{m}{1} \times \frac{m-1}{1}$ will be
 $= \frac{-2}{T} \times \frac{-2-T}{T} = \frac{+4+2T}{T^2}$.

Therefore the $\frac{-2}{T}$ th power of the series $1 - bv + cv^2 - dv^3 + ev^4 - \&c$
will be equal to the infinite series

$$1 + \frac{2}{T} \times bv + \frac{2+T}{T^2} \times b^2v^2 + \frac{4+6T+TT}{3T^3} \times b^3v^3 \\
- \frac{2}{T} \times cv^2 - \frac{-4-2T}{T^2} \times bcv^3 \\
+ \frac{2}{T} \times dv^3 \\
+ \&c.$$

Art. 6. Now let b be $= \frac{T}{2}$.

Then will this last series be converted into the series

$$1 + \frac{2}{T} \times \frac{T}{2} \times v + \frac{2+T}{T^2} \times \frac{T^2}{4} \times v^2 + \frac{4+6T+TT}{3T^3} \times \frac{T^3}{8} \times v^3 \\
- \frac{2}{T} \times cv^2 - \frac{-4-2T}{T^2} \times \frac{T}{2} \times c \times v^3 \\
+ \frac{2}{T} \times d \times v^3 \\
+ \&c,$$

or into the series

$$1 + v + \frac{T+2}{4} \times v^2 + \frac{TT+6T+4}{24} \times v^3 \\
- \frac{2}{T} \times cv^2 - \frac{-T-2}{T} \times c \times v^3 \\
+ \frac{2}{T} \times d \times v^3 \\
+ \&c.$$

Art. 7. Now let c be $= \frac{T \times \overline{T+1} - \sqrt{tt+tx}}{6}$.

Then will the last series be converted into the series

$$\begin{aligned} 1 + v &+ \frac{T+2}{4} \times v^2 && + \frac{TT+6T+4}{24} \times v^3 \\ &- \frac{2}{T} \times \frac{T \times \overline{T+1} - \sqrt{tt+tx}}{6} \times v^2 && - \frac{T-2}{T} \times \frac{T \times \overline{T+1} - \sqrt{tt+tx}}{6} \times v^3 \\ &&& + \frac{2}{T} \times d \times v^3 \\ &&& + \&c, \end{aligned}$$

or (neglecting the fourth term, of which the co-efficient is so complicated that the computation of it would be extremely difficult,) into the series

$$1 + v + \frac{T+2}{4} \times v^2 - \frac{T \times \overline{T+1} + tt + tx}{3T} \times v^2 \&c, \text{ or into the series}$$

$$\begin{aligned} 1 + v &+ \frac{T+2}{4} \times vv \\ &- \left[\frac{T+1}{3} \times vv \right. \\ &\quad \left. + \frac{tt+tx}{3T} \times vv \&c, \text{ or into the series} \right] \end{aligned}$$

$$\begin{aligned} 1 + v &+ \frac{3T+6}{12} \times vv \\ &- \frac{4T-4}{12} \times vv \\ &+ \frac{tt+tx}{3T} \times vv \&c, \text{ or into the series} \end{aligned}$$

$$\begin{aligned} 1 + v &- \frac{T+2}{12} \times vv \\ &+ \frac{tt+tx}{3T} \times vv \&c, \text{ or into the series} \end{aligned}$$

$$\begin{aligned} 1 + v &- \frac{3T^2+6T}{36T} \times vv \\ &+ \frac{12tt+12tx}{36T} \times vv \&c, \text{ or into the series} \end{aligned}$$

1 +

$$1 + v - \frac{-3T^2 + 6T + 12t + 12tx}{36T} \times vv \text{ \&c, or into the series}$$

$$1 + v - \frac{-T^2 + 2T + 4t + 4tx}{12T} \times vv \text{ \&c, or into the series}$$

$$1 + v - \frac{-4t - 4tx - 4t^2 - x^2 - 2x - 1}{12T} \times vv \\ + \frac{4t + 2x + 2}{12T} \times vv \\ + \frac{4t + 4tx}{12T} \times vv \text{ \&c, or into the series}$$

$$1 + v - \frac{-x^2 + 1}{12T} \times vv \text{ \&c, or into the series}$$

$$1 + v - \sqrt{\frac{x^2 - 1}{12T}} \times vv \text{ \&c.}$$

Therefore this last series $1 + v - \sqrt{\frac{x^2 - 1}{12T}} \times vv \text{ \&c}$ will be the $\frac{-2}{T}$ th power of the series $1 - \frac{T}{2} \times v + \frac{T \times T + 1 - \sqrt{4t + tx}}{6} \times vv - \text{\&c}$ which is equal to the fraction $\frac{z}{ax}$. And consequently this last series $1 + v - \sqrt{\frac{x^2 - 1}{12T}} \times vv \text{ \&c}$ will be equal to the $\frac{-2}{T}$ th power of the said fraction $\frac{z}{ax}$, or to the $\frac{+2}{T}$ th power of it's reciprocal, the fraction $\frac{ax}{z}$.

Art. 8. Since the series $1 + v - \sqrt{\frac{xx - 1}{12T}} \times vv \text{ \&c}$ is $= \left(\frac{ax}{z}\right)^{\frac{2}{T}}$, let 1 be subtracted from both sides of the equation. And we shall then have the series $v - \sqrt{\frac{xx - 1}{12T}} \times vv \text{ \&c} = \left(\frac{ax}{z}\right)^{\frac{2}{T}} - 1$; and consequently the two first terms of the said series, to wit, the terms $v - \sqrt{\frac{xx - 1}{12T}} \times vv$, will be nearly equal to $\left(\frac{ax}{z}\right)^{\frac{2}{T}} - 1$. Let y be put $= \left(\frac{ax}{z}\right)^{\frac{2}{T}} - 1$, or the absolute term of the quadratick equation $v - \sqrt{\frac{xx - 1}{12T}} \times vv$; and we shall then have $v - \sqrt{\frac{xx - 1}{12T}} \times vv = y$.

Y y 2

Multiply

Multiply all the terms of this equation into the fraction $\frac{12T}{xx-1}$, or the reciprocal of the co-efficient of vv , or the square of the unknown quantity v , in order to free the said square from it's co-efficient. And we shall then have $\frac{12T}{xx-1} \times v - vv = \frac{12T}{xx-1} \times y$. Now, to simplify the equation, let b be put $= \frac{6T}{xx-1}$, or half the co-efficient of v . And we shall then have $2bv - vv = 2by$; which is a quadratick equation properly prepared for resolution. Subtract both sides of this equation from bb , or the square of half the co-efficient of v , than which square they are always necessarily less. And we shall then have $bb - 2bv + vv = bb - 2by$. Therefore, extracting the square-roots of both sides, and seeking the lesser of the two values of v , which will be less than b , we shall have $b - v = \sqrt{bb - 2by}$, and (adding v to both sides,) $b = v + \sqrt{bb - 2by}$, and (subtracting $\sqrt{bb - 2by}$ from both sides,) $v = b - \sqrt{bb - 2by}$, or $v = b - \overline{bb - 2by}^{\frac{1}{2}}$. Therefore $r - 1$ (which is $= v$) will be $= b - \overline{bb - 2by}^{\frac{1}{2}}$.

A SCHOLIUM.

Art. 9. This expression for $r - 1$ is not exactly the same with that given by Dr. Halley in the above-cited passage of his Discourse on Compound Interest. For he makes $b = \frac{6T+6}{xx}$ instead of $\frac{6T}{xx-1}$, as it is in the foregoing investigation communicated to me by Mr. Morgan; and Mr. Morgan says, that he does not know whence this difference arises. But he adds, that it is not of much consequence; because both the rules are very accurate: and of the two he thinks his own rule (in which b is $= \frac{6T}{xx-1}$), the most accurate. And he is therefore inclined to suspect that Dr. Halley may have made some small mistake in his investigation; or that he may, perhaps, have proceeded in a different manner from Mr. Morgan in making his investigation, and by this means have obtained a different value for b . But he adds that, be that as it may, there need not be required a more correct rule than either of the two preceeding ones, as he has found from repeated trials in cases where an inaccuracy was most likely to take place.

Art. 10. I will now proceed to apply the foregoing, or Mr. Morgan's, expression to the solution of the Problem proposed by Dr. Halley, as an example of the utility of his expression. This Problem is as follows.

A PROBLEM.

An annuity of 20*l.* a year that is already granted for a term of years of which 21 are still to come, may, for the sum of 40 pounds, be prolonged for 10 years more, or to 31 years. What is the rate of interest required?

Here *a*, or the annuity granted, is 20*l. per annum*; and *t*, or the number of years for which it has been already granted, and which are still to come before the annuity ceases, is 21 years; and *x*, or the additional number of years during which the annuity is to be continued beyond the said original term of 21 years, is 10 years. And *z*, or the sum of money to be paid in order to procure the said prolongation of the annuity, is 40 pounds.

Therefore *ax* will be = 20*l.* × 10 = 200*l.*, and $\frac{z}{ax}$ will be = $\frac{40}{200}$ (= $\frac{2}{10}$) = 0.2, and $\frac{ax}{z}$ will be = $\frac{200}{40}$ = 5.

Further, *T*, or $2t + x + 1$, will be (= $2 \times 21 + 10 + 1 = 42 + 10 + 1$) = 53, and consequently the fraction $\frac{2}{T}$ will be = $\frac{2}{53}$, and $\left(\frac{ax}{z}\right)^{\frac{2}{T}}$ will be = $5^{\frac{2}{53}}$.

Now the logarithm of 5 is = 0.698,970,0. Therefore the logarithm of $5^{\frac{2}{53}}$ will be = $\frac{2}{53} \times 0.698,970,0$ (= $\frac{2 \times 0.698,970,0}{53} = \frac{1.397,940,0}{53}$) = 0.026,376,2; which is the logarithm of the number 1.062,615,6. Therefore $5^{\frac{2}{53}}$, or $\left(\frac{ax}{z}\right)^{\frac{2}{T}}$, will be = 1.062,615,6. Therefore $\left(\frac{ax}{z}\right)^{\frac{2}{T}} - 1$ will be (= 1.062,615,6 - 1) = 0.062,615,6; that is, *y* will be = 0.062,615,6.

Further, since *T* is = 53, we shall have 6*T* (= 6×53) = 318, and $\frac{6T}{xx - 1}$ (= $\frac{318}{100 - 1} = \frac{318}{99} = \frac{106}{33}$) = 3.212,121,2; that is, *b* will be = 3.212,

3 212,121,2. Therefore by will be $(= 3.212,121,2 \times 0.062,615,6) = 0.201,128,896,210,72$, and $2by$ will be $(= 2 \times 0.201,128,896,210,72) = 0.402,257,792,421,44$; and bb will be $(= 3.212,121,2)^2 = 10.317,722,603,489,44$. Therefore $bb - 2by$ will be $(= 10.317,722,603,489,44 - 0.402,257,792,421,44) = 9.915,464,811,068,00$, and consequently $\sqrt{bb - 2by}$ will be $(= \sqrt{9.915,464,811,068,00}) = 3.148,883,1$, and $b - \sqrt{bb - 2by}$ will be $= 3.212,121,2 - 3.148,883,1 = 0.063,238,1$. Therefore $r - 1$ will be nearly $= 0.063,238,1$, and consequently r will be $= 1.063,238,1$.

Q. E. I.

This number, 1.063,238,1, is very nearly equal to, but a little greater than, the value of r found by Dr. Halley, which is 1.063,233. So that Dr. Halley's and Mr. Morgan's expressions seem to be equally useful in practice. But, as we know the investigation of Mr. Morgan's expression, (which has been given at length in the foregoing articles,) and are ignorant of the manner in which Dr. Halley obtained his expression, (in which b is made equal to $\frac{6T+6}{xx}$ instead of $\frac{6T}{xx-1}$), I should be inclined to make use of Mr. Morgan's expression in preference to Dr. Halley's.

Art. 11. The number 1.063,233, assigned by Dr. Halley, as the value of r in this equation, may be computed in the following manner.

Since $6T$ is $= 318$, we shall have $6T + 6 (= 318 + 6) = 324$, and $\frac{6T+6}{xx} (= \frac{324}{100}) = 3.24$; that is, b will be $= 3.24$. Therefore bb will be $(= 3.24^2) = 10.4976$, and by will be $(= 3.24 \times 0.062,615,6) = 0.202,874,544$, and $2by$ will be $(= 2 \times 0.202,874,544) = 0.405,749,088$, and $bb - 2by$ will be $(= 10.497,600,000 - 0.405,749,088) = 10.091,850,912$. Therefore $\sqrt{bb - 2by}$ will be $(= \sqrt{10.091,850,912}) = 3.176,767,3$, and $b - \sqrt{bb - 2by}$, or $b - \overline{bb - 2by}^{\frac{1}{2}}$, will be $(= 3.240,000,0 - 3.176,767,3) = 0.063,232,7$, or (very nearly) 0.063,233, as Dr. Halley makes it; and consequently r , or $1 + b - \sqrt{bb - 2by}$, or $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, will be $= 1 + 0.063,233$, or 1.063,233. Q. E. I.

Art. 12. I will now proceed to try the exactness of the number 1.063,238,1, obtained above by means of Mr. Morgan's expression $1 + b - \sqrt{bb - 2by}$, upon a supposition that b is $= \frac{6T}{xx-1}$, by supposing r to be equal to it, and computing, upon that supposition, the value of x , or the price to be paid for the

the prolongation of an annuity of 20 pounds *per annum*, already granted for a term of 21 years, for an additional term of 10 years. For, if that price shall be very nearly equal to 40 pounds, which was the sum supposed to be paid for the said prolongation in the former Problem, it will follow that the value of r , obtained in the solution of that Problem by means of the expression $1 + b - \sqrt{bb - 2by}$, or $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, upon a supposition that b is $= \frac{6T}{xx - 1}$, will be very nearly equal to it's true value. This trial may be made in the following manner.

A Trial of the Degree of Exactness of the Number 1.063,238,1, obtained above in Art. 10 for the Value of r in the foregoing Problem by means of the Expression $b - \sqrt{bb - 2by}^{\frac{1}{2}}$ investigated in the foregoing Articles, in which Expression b is $= \frac{6T}{xx - 1}$.

Art. 13. It has been shewn above in art. 1 of this Note, that z will be $= \frac{a}{r^t \times r - 1} - \frac{a}{r^{t+x} \times r - 1}$. We must therefore compute this expression $\frac{a}{r^t \times r - 1}$, upon a supposition that a , or the annuity granted, is 20*l.* *per annum*, and that t , or the number of years of the first term for which it was granted, that are still to come, is 21 years, and that x , or the additional number of years for which it is to be prolonged, is 10 years, and that r , or the rate of interest of money, is 1.063,238,1.

Now, upon these suppositions, z , or the price to be paid for the said prolongation of the annuity, (being in all cases equal to the quantity $\frac{a}{r^t \times r - 1} - \frac{a}{r^{t+x} \times r - 1}$), will be $= \frac{20*l.*}{r^{21} \times r - 1} - \frac{20*l.*}{r^{21+10} \times r - 1}$ ($= \frac{20*l.*}{r^{21} \times r - 1} - \frac{20*l.*}{r^{31} \times r - 1}$)

$$= \frac{20*l.*}{r^{21} \times 1.063,238,1 - 1} - \frac{20*l.*}{r^{31} \times 1.063,238,1 - 1} = \frac{20*l.*}{r^{21} \times 0.063,238,1} - \frac{20*l.*}{r^{31} \times 0.063,238,1}$$

$$= \frac{20*l.*}{0.063,238,1} \times \frac{1}{r^{21}} - \frac{20*l.*}{0.063,238,1} \times \frac{1}{r^{31}} = \frac{100 \times 20*l.*}{100 \times 0.063,238,1} \times \frac{1}{r^{21}} - \frac{100 \times 20*l.*}{100 \times 0.063,238,1} \times \frac{1}{r^{31}} = \frac{2000*l.*}{6.323,81} \times \frac{1}{r^{21}} - \frac{2000*l.*}{6.323,81} \times \frac{1}{r^{31}} = 316.265,036,4*l.*$$

X

$$\times \frac{1}{r^{21}} = 316.265,036,4l. \times \frac{1}{r^{31}} = \frac{316.265,036,4l.}{r^{21}} - \frac{316.265,036,4l.}{r^{31}} =$$

$$\frac{316.265,036,4l.}{1.063,238,1^{21}} - \frac{316.265,036,4l.}{1.063,238,1^{31}}.$$

We must therefore, in the next place, find the values of $1.063,238,1^{21}$ and $1.063,238,1^{31}$; which may be done as follows.

The logarithm of $1.063,238,1$ is $= 0.026,630,5$. Therefore the logarithm of $1.063,238,1^{21}$ will be $(= 21 \times 0.026,630,5) = 0.559,240,5$, and the logarithm of $1.063,238,1^{31}$ will be $(= 31 \times 0.026,630,5) = 0.825,545,5$.

But $0.559,240,5$ is the logarithm of $3.624,436,6$, and $0.825,545,5$ is the logarithm of $6.691,840,0$. Therefore $1.063,238,1^{21}$ will be $= 3.624,436,6$, and $1.063,238,1^{31}$ will be $= 6.691,840,0$.

Therefore $\frac{316.265,036,4l.}{1.063,238,1^{21}}$ will be $(= \frac{316.265,036,4l.}{3.624,436,6}) = 87.259,089,1l.$, and $\frac{316.265,036,4l.}{1.063,238,1^{31}}$ will be $(= \frac{316.265,036,4l.}{6.691,840,0}) = 47.261,296,8l.$ And z will be $=$

$87.259,089,1l. - 47.261,296,8l. = 39.998,792,3l.$, or $39l. 19s. 10\frac{1}{4}d.$; that is, the price to be paid for the prolongation of the annuity for 10 years more, in addition to the 21 years of the first term which are still to come, will, if r is $= 1.063,238,1$, or the interest of money is $6.323,81l.$, or $6l. 6s. 5d.$ per cent, be $39l. 19s. 10\frac{1}{4}d.$, instead of being exactly $40l.$, as it was supposed to be in the foregoing Problem. This is a very near approach to the former value of z , (or the price paid for the prolongation of the annuity,) and proves that the value of r , obtained in the foregoing Problem by the computation of Mr.

Morgan's expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, in which b is $= \frac{6T}{xx - 1}$, to wit,

the number $1.063,238,1$, is very nearly equal to, but somewhat greater than, the true value of r in the said Problem, in which the price of the prolongation of the annuity was supposed to be exactly 40 pounds; because the smaller is the price paid for a given annuity for a given number of years, the higher must be the rate of the interest of money allowed to the purchaser of it. But the difference between these two prices, or values of z , is so trifling that it proves that, at least, in this particular example, or with these values of t and x , or the numbers of years for which the annuity is granted and prolonged, the said ex-

pression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, given by Mr. Morgan's investigation above-mentioned, is a very useful approximation to r , or the rate of interest sought.

Art. 14. If the foregoing General Problem, for the solution of which Dr. Halley has given us the expression $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$ mentioned in the passage which is the subject of this Note, be reduced to an equation, that equation will be a trinomial equation, or will contain three different terms involving three different powers of the unknown quantity r , and will be as follows; to wit, $\frac{a}{z} \times r^x + r^{t+x} + r^{t+x+1} = \frac{a}{z}$; as may be shewn in the following manner.

It has been shewn in art. 1, that z is $= \frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$. Therefore (multiplying both sides into $r-1$,) we shall have $rz - z = \frac{a}{r^t} - \frac{a}{r^{t+x}}$, and (multiplying all the terms into r^t ,) we shall have $r^{t+1} \times z - r^t \times z = a - \frac{a \times r^t}{r^{t+x}}$ ($= a - \frac{ar^t}{r^t \times r^x}$) $= a - \frac{a}{r^x}$, and (multiplying all the terms into r^x ,) we shall have $r^{t+x+1} \times z - r^{t+x} \times z = a \times r^x - a$, and (adding a to both sides,) $r^{t+x+1} \times z - r^{t+x} \times z + a = a \times r^x$, and (adding $r^{t+x} \times z$ to both sides,) $r^{t+x+1} \times z + a = a \times r^x + z \times r^{t+x}$, and (subtracting $r^{t+x+1} \times z$ from both sides,) $a = a \times r^x + z \times r^{t+x} - z \times r^{t+x+1}$, and (dividing all the terms by z ,) $\frac{a}{z} = \frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1}$, or (in the more usual way of ranging the terms, with the known quantity, or absolute term, on the right hand,) $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$; in which equation a , z , t , and x , are supposed to be known quantities, and r to be the only unknown quantity. Q. E. D.

If the near value of r in this trinomial equation, obtained by means of the foregoing expression $b - \sqrt{bb - 2by}^{\frac{1}{2}}$ given by Mr. Morgan's investigation above stated, is not thought to be sufficiently exact, it will, at least, make an excellent first near value of r , to be used as the ground-work of a further approach to it's true value by Mr. Raphson's method of approximation; and one process of that method will give us a second near value of r that will be as near the truth as ever need to be desired. Of this I will now give an example in the more exact solution of the foregoing Problem of Dr. Halley, by making the

number 1.063,238,1 (which was obtained in art. 10 by means of the expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$, upon a supposition that b is $= \frac{6T}{xx - 1}$), the ground-work of a process of Mr. Raphson's method of approximation to the true value of r in the aforefaid trinomial equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$. This process may be performed in the following manner.

Art. 15. In the foregoing Problem of Dr. Halley the annuity a is 20*l.* *per annum*; and t , the number of years that are still to come of the term for which the said annuity was originally granted, is 21 years; and x , or the number of years during which the said annuity is to be continued beyond the first term for which it was granted, is 10 years; and z , or the price to be paid for such prolongation of the annuity, is 40 pounds. Therefore the foregoing general equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$ will, in this particular case, be $\frac{20}{40} \times r^{10} + r^{21+10} - r^{21+10+1} = \frac{20}{40}$, or $\frac{1}{2} \times r^{10} + r^{31} - r^{32} = \frac{1}{2}$, or $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Now it has been shewn in art. 13 that the true value of r in this equation must be less than the number 1.063,238,1. Therefore, if w be put for the unknown difference between 1.063,238,1 and the said true value, we shall have $r = 1.063,238,1 - w$, and consequently $r^{10} = \overline{1.063,238,1 - w}^{10} =$, by the binomial theorem, to the series $\overline{1.063,238,1}^{10} - 10 \times \overline{1.063,238,1}^9 \times w + \&c$, and $r^{31} = \overline{1.063,238,1 - w}^{31} =$, by the binomial theorem, to the series $\overline{1.063,238,1}^{31} - 31 \times \overline{1.063,238,1}^{30} \times w + \&c$, and $r^{32} = \overline{1.063,238,1 - w}^{32} = \overline{1.063,238,1}^{32} =$, by the binomial theorem, to the series $\overline{1.063,238,1}^{32} - 32 \times \overline{1.063,238,1}^{31} \times w + \&c$.

We must therefore, in the next place, seek the values of $\overline{1.063,238,1}^9$, $\overline{1.063,238,1}^{10}$, $\overline{1.063,238,1}^{30}$, $\overline{1.063,238,1}^{31}$, and $\overline{1.063,238,1}^{32}$.

Now the logarithm of 1.063,238,1 is $= 0.026,630,5$. Therefore the logarithm of $\overline{1.063,238,1}^{10}$ will be $(= 10 \times 0.026,630,5) = 0.266,305,0$; which is the logarithm of the number 1.846,311,5. Therefore $\overline{1.063,238,1}^{10}$ will

will be $= 1.846,311,5$. And consequently $\overline{1.063,238,1}^9$ will be $(= \frac{\overline{1.063,238,1}^{10}}{\overline{1.063,238,1}} = \frac{1.846,311,5}{1.063,238,1}) = 1.736,498,6$, and $10 \times \overline{1.063,238,1}^9 \times w$ will be $(= 10 \times 1.736,498,6) = 17.364,986,0 \times w$. Therefore r^{10} will be $(= \overline{1.063,238,1}^{10} - 10 \times \overline{1.063,238,1}^9 \times w + \&c) = 1.846,311,5 - 17.364,986,0 \times w + \&c$.

Further, the logarithm of $\overline{1.063,238,1}^{31}$ will be $(= 31 \times 0.026,630,5) = 0.825,545,5$; which is the logarithm of $6.691,840,0$. Therefore $\overline{1.063,238,1}^{31}$ will be $= 6.691,840,0$, and consequently $\overline{1.063,238,1}^{30}$ will be $(= \frac{\overline{1.063,238,1}^{31}}{\overline{1.063,238,1}} = \frac{6.691,840,0}{1.063,238,1}) = 6.293,830,1$, and $\overline{1.063,238,1}^{32}$ will be $(= \overline{1.063,238,1}^{31} \times \overline{1.063,238,1} = 6.691,840,0 \times 1.063,238,1) = 7.115,019,2$. Therefore $31 \times \overline{1.063,238,1}^{30} \times w$ will be $(= 31 \times 6.293,830,1 \times w) = 195.108,733,1 \times w$, and $32 \times \overline{1.063,238,1}^{31} \times w$ will be $(= 32 \times 6.691,840,0 \times w) = 214.138,880,0 \times w$.

Therefore r^{31} will be $(= \overline{1.063,238,1}^{31} - 31 \times \overline{1.063,238,1}^{30} \times w + \&c) = 6.691,840,0 - 195.108,733,1 \times w + \&c$, and r^{32} will be $(= \overline{1.063,238,1}^{32} - 32 \times \overline{1.063,238,1}^{31} \times w + \&c) = 7.115,019,2 - 214.138,880,0 \times w + \&c$.

Therefore the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be equal to the compound quantity

$$\begin{aligned} & \left\{ \begin{array}{l} 0.500,000,0 \times 1.846,311,5 - 0.500,000,0 \times 17.364,986,0 \times w + \&c \\ \quad + 6.691,840,0 - 195.108,733,1 \times w + \&c \\ \quad - 7.115,019,2 + 214.138,880,0 \times w - \&c \end{array} \right\} \\ &= \left\{ \begin{array}{l} 0.923,155,7 - 8.682,493,0 \times w + \&c \\ \quad + 6.691,840,0 - 195.108,733,1 \times w + \&c \\ \quad - 7.115,019,2 + 214.138,880,0 \times w - \&c \end{array} \right\} \\ &= \left\{ \begin{array}{l} 7.614,995,7 - 203.791,226,1 \times w + \&c \\ \quad - 7.115,019,2 + 214.138,880,0 \times w + \&c \end{array} \right\} \\ &= 0.499,976,5 + 10.347,653,9 \times w + \&c. \end{aligned}$$

But the said trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ is = $0.500,000,0$.

Therefore the compound quantity $0.499,976,5 + 10.347,653,9 \times w$ &c will also be = $0.500,000,0$.

Therefore (subtracting $0.499,976,5$ from both sides,) we shall have $10.347,653,9 \times w$ (= $0.500,000,0 - 0.499,976,5$) = $0.000,023,5$, and w (= $\frac{0.000,023,5}{10.347,653,9}$) = $0.000,002,2$. Therefore r , or $1.063,238,1 - w$, will be (= $1.063,238,1 - 0.000,002,2$) = $1.063,235,9$. Q. E. I.

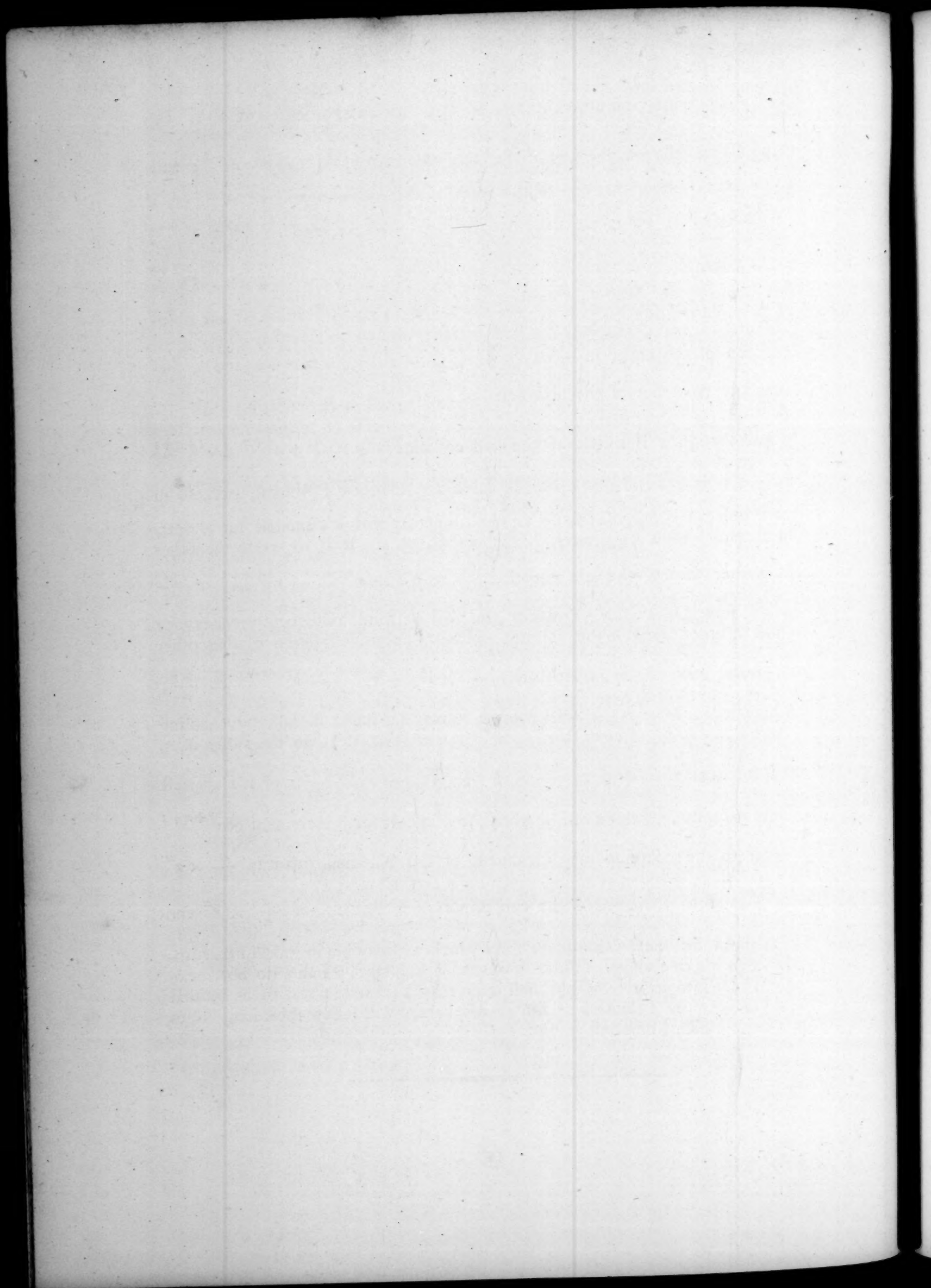
This number $1.063,235,9$, (found for a second near value of r , or the rate of interest sought, by this process of Mr. Raphson's method of approximation,) is probably true in all it's figures. And it agrees with the first near value of it, $1.063,238,1$, found by Mr. Morgan's expression $1 + b - \sqrt{bb - 2by}$, with b taken = $\frac{6T}{xx - 1}$, and likewise with the number $1.063,233$, found by Dr. Halley's expression $1 + b - \sqrt{bb - 2by}$, with b taken = $\frac{6T+6}{xx}$, in the first six figures $1.063,23$, and therefore proves that each of those numbers is true in those first six figures, and consequently that either of those expressions will be an excellent approximation to the true value of r , or the rate of interest, in this last Problem of Dr. Halley, agreeably to Mr. Morgan's declarations mentioned above in art. 9, page 348.

This number $1.063,235,9$, (which may be considered as the exact value of r), is less than the number $1.063,238,1$, found by means of Mr. Morgan's expression, and greater than the number $1.063,232,7$, or $1.063,233$, found by means of Dr. Halley's expression, and is a very little greater than $1.063,235,4$, the arithmetical mean between them.

Art. 16. Before I conclude this Fourth Note, which relates to the value of r in the general equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$ resulting from the last Problem of Dr. Halley concerning the prolongation of an annuity for a given number of years beyond the original term for which it was granted, I will just make two observations concerning the said equation, to wit, in the first place, that this is an equation of such a form as to admit of having two different roots, or, in the language of modern Algebraists, two different real and affirmative roots, there being one change of the signs $+$ and $-$ amongst the terms on the left-hand side of the equation, and the highest power of the unknown quantity r , to wit, r^{t+x+1} , having the sign $-$ prefixed to it, or being

being subtracted from the sum of the other two terms $\frac{a}{z} \times r^x + r^{t+x}$; and, in the second place, that one of the roots of this equation will be 1. For, if we suppose r to be 1, we shall have $r^x (= 1^x) = 1$, and $r^{t+x} (= 1^{t+x}) = 1$, and $r^{t+x+1} (= 1^{t+x+1})$ also = 1, and consequently $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} (= \frac{a}{z} \times 1 + 1 - 1 = \frac{a}{z} \times 1) = \frac{a}{z}$, which is the absolute term of the equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$; and consequently 1 is a root of the said equation. Q. E. D.

But, though 1 is a root of the said equation $\frac{a}{z} \times r^x + r^{t+x} + r^{t+x+1} = \frac{a}{z}$, it is not the root that will solve Dr. Halley's Problem; because in that Problem r is supposed to be 1 pound together with it's interest for a year, and therefore must be greater than 1 pound, or 1. It is therefore the other and greater root of the said equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$ that will solve Dr. Halley's Problem, and that we must consequently endeavour to find in order to effect that solution. But in our investigation of that other and greater root of the said equation $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$, the knowledge that 1 is the lesser root of it may be made usefull, by enabling us to find a quantity greater than 1, and greater likewise than the value of r , when the trinomial quantity $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1}$ is of it's greatest possible magnitude, (which will be when the infinitely small increment of r^{t+x+1} is equal to the contemporary increment of the binomial quantity $\frac{a}{z} \times r^x + r^{t+x}$, or to the sum of the contemporary increments of $\frac{a}{z} \times r^x$ and r^{t+x} ;) for a first near value of r , from which we may begin our further approaches to it's true value. This, however, is a subject which I do not mean to go further into at present, but shall reserve for a separate tract, to be intitled *An Appendix* to this Discourse of Dr. Halley, which will follow these long Notes upon it, to which I here put an end.



AN
A P P E N D I X
TO THE
FOREGOING DISCOURSE OF DR. HALLEY
ON
COMPOUND INTEREST;

Shewing how the Equations in which r , or the Rate of Interest of Money, allowed in Bargains concerning Annuities for Terms of Years, is the unknown Quantity, may be resolved in a clear and satisfactory Manner without having Recourse to the Algebraick Expressions which he has given us for that Purpose.

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CURSITOR BARON OF THE EXCHEQUER.

Article 1. **T**HE difficult passages of the foregoing Discourse of Dr. Halley on Compound Interest, which have been the subject of the foregoing Notes, relate to the solution of three Problems concerning annuities for terms of years, in which the rate of interest of money allowed in the bargains concerning these annuities is the unknown quantity which we are required to determine from our knowledge of the other circumstances relating to them. Each of these Problems may be reduced to an equation, of which r , or the said rate of interest, is the unknown quantity, or root; and the solution of the Problem depends on, or may be effected by, the resolution of the said equation resulting from it: so that the resolution of the equation resulting from each of these Problems may be considered as the same thing with the solution of the Problem from which it is derived.

Art. 2.

Art. 2. These equations rise to a very high order, to wit, in some cases to the order denoted by the number t , or the number of years during which the annuity is supposed to continue, and in other cases to the order denoted by the number $t + 1$, or the former number increased by an unit. And this circumstance makes them somewhat difficult to resolve: and therefore Dr. Halley has given us certain Algebraick expressions for the roots of these equations, which will approach pretty nearly to their true values, and save us the trouble of resolving these high equations in the ordinary methods in order to obtain them. But he has omitted to give us any investigations of these expressions; and those which Mr. Morgan (the learned and ingenious Actuary of the celebrated Society for granting Equitable Assurances on Lives in Chatham Square near Black Friars' Bridge,) has found-out and communicated to me, and which I have set forth in the foregoing Notes in as full and clear a manner as I could, are so tedious and intricate, and are derived from such remote and subtle principles, that, I confess, they afford me but an imperfect degree of satisfaction, notwithstanding the expressions obtained by means of them are found to be tolerably near the truth and are consequently very useful in practice. I shall therefore now endeavour to point-out some other methods of resolving these high equations involving r , or the unknown rate of interest, which shall be as exact as those expressions of Dr. Halley, and nearly as easy to reduce to practice, and that shall depend on much clearer and simpler principles than those by which those expressions have been obtained in the investigations of them given above in the foregoing Notes. And for this purpose I will here state a-new the aforesaid three Problems in Dr. Halley's Discourse which have given occasion to the foregoing Notes, and the equations resulting from them, and which it will be our object to resolve.

Art. 3. The first of these Problems is that which Dr. Halley calls Problem IV of the Problems concerning the amount of a given annuity; and it occurs above in the lower part of page 223, in these words.

PROBLEM IV.

The annuity (a), time (t), and amount (z), being given; to find (r), the rate.

Here Dr. Halley says that, in order to find r , the former equation is reduced to $\frac{z}{a} - 1 = \frac{z}{a} \times r - r^t$, or to $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or, in our present case, to $16.8614r = 17.8614r - r^t$, or to $17.8614 \times r - r^t = 16.8614$; which is so affected as not readily to be resolved by the general method

method for the resolution of equations, unless we can first approach it by some other means; for which purpose take the following rule, which will suffice where great exactness is not required.

These are the words of Dr. Halley. But the Problem may be stated more fully in the following manner.

PROBLEM IV.

If an annuity of a pounds a year has been granted a certain number of years ago, but has not yet been paid to the annuitant to whom it had been granted, the receipt of it having been forborn by the annuitant, in consideration of an agreement made with the grantor of the annuity, that, when he pays the annuitant the arrears of the said annuity in a single payment, he will allow him compound interest upon all the arrears of the annuity that shall be then due; and, if the said single payment, which is now become due to the annuitant in consequence of his having totally forborn to receive any part of the said annuity untill the present time, (which single payment is called *the amount* of the said annuity,) is equal to a known, or given, sum of money called z ; and the number of years during which the annuity shall have been due, but not paid, is called t ; and r be the value of 1*l.* together with it's interest for one year according to the rate of interest, (hitherto unknown,) which makes the said amount of the annuity be equal to z ; it is required from this knowledge of a , t , and z , to determine the value of r , or the rate of interest.

Art. 4. This Problem may be reduced to the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by the following train of reasoning.

It is shewn above in page 222 that z is $= \frac{ar^t - a}{r - 1}$. Therefore $\overline{r - 1} \times z$ will be $= ar^t - a = a \times \overline{r^t - 1}$, and consequently a will be $= z \times \frac{r - 1}{r^t - 1}$. Therefore $\frac{a}{a}$ will be $= \frac{z}{a} \times \frac{r - 1}{r^t - 1}$; that is, 1 will be $= \frac{z}{a} \times \frac{r - 1}{r^t - 1}$; and consequently $1 \times \overline{r^t - 1}$ will be $= \frac{z}{a} \times \overline{r - 1}$, or $r^t - 1$ will be $= \frac{z}{a} \times \overline{r - 1}$ ($= \frac{z}{a} \times r - \frac{z}{a} \times 1$) $= \frac{z}{a} \times r - \frac{z}{a}$. Therefore (adding $\frac{z}{a}$ to both sides,) we shall have $r^t - 1 + \frac{z}{a} = \frac{z}{a} \times r$, and (subtracting r^t

from both sides,) $\frac{z}{a} - 1 = \frac{z}{a} \times r - r^t$, or (in the more usual way of ranging the terms, with the known quantity, or absolute term, on the right hand,) $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. Q. E. D.

We must therefore now endeavour to resolve the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Art. 5. Now it is easy to observe that 1 will be a root of this equation. For, if r be supposed to be $= 1$, r^t will also be $= 1$, and consequently $\frac{z}{a} \times r - r^t$ will be $= \frac{z}{a} \times 1 - 1$, which is $= \frac{z}{a} - 1$, or the absolute term of the equation. And consequently 1 will be $= r$, or will be a root of this equation. Q. E. D.

* But, though 1 is a root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, yet it cannot be *that* root of the equation which will solve the foregoing Problem; because in that Problem r is supposed to be the sum of one pound and it's interest for a year, and therefore must necessarily be greater than 1. Therefore the value of r that will solve the foregoing Problem must be the greater of the two roots of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which is an equation of such a form as to admit of two roots, or (in the language of modern Algebraists,) of two real and affirmative roots.

Art. 6. The observation made in the foregoing article, "that 1 is equal to one of the roots of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$," is not without some degree of use in our present inquiry, notwithstanding it is not the root which will solve the Problem. For it will assist us in finding another quantity greater than itself that shall approach pretty nearly to the other, or greater, root of the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, of which we are in search. But that we may be able to apply it to this purpose, it will be convenient, first, to make a few remarks on the number of roots in a binomial equation of the general form $Px - x^m = Q$ and on the limits of their magnitudes; which may afterwards be transferred to the roots of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which comes under that more general form.

Of

Of the Number of Roots in the Binomial Equation $Px - x^m = Q$, and the Limits of their Magnitudes.

Art. 7. Now, if P and Q are known quantities, and x an unknown quantity, and m is any whole number whatsoever, and $Px - x^m = Q$, it is evident, in the first place, that x^m will be less than Px , and consequently that $\frac{x^m}{x}$, or x^{m-1} , will be less than P , and that x will be less than $\sqrt[m-1]{P}$, or than $P^{\frac{1}{m-1}}$.

Secondly, if x be supposed to increase continually from 0 *ad infinitum*, it is evident that, while x is less than 1, x^m will be less than x ; and, when x is equal to 1, x^m will be equal to 1, or to x ; and, when x becomes greater than 1, x^m will also be greater than 1 and greater than x ; and that, when x becomes greater than $\sqrt[m-1]{P}$, or $P^{\frac{1}{m-1}}$, x^{m-1} will become greater than P , and x^m will become greater than Px , and the difference between Px and x^m will be no longer $Px - x^m$, but $x^m - Px$; and, as x increases further from $P^{\frac{1}{m-1}}$ *ad infinitum*, the binomial quantity $x^m - Px$ will increase continually from 0 *ad infinitum*, so as to become equal to any quantity, how small, or how great, soever.

Thirdly, if the greatest magnitude which x ever attains in the equation $Px - x^m = Q$, namely, the quantity $P^{\frac{1}{m-1}}$, be divided into some very great number of small and equal parts, as, for example, into ten thousand million such parts, and each of these small and equal parts be denoted by $\dot{x}$, or the letter x with a point placed over it, it is evident that, while x , in it's increase from 0 to it's said greatest magnitude $P^{\frac{1}{m-1}}$, increases from x to $x + \dot{x}$, or receives the small increment $\dot{x}$, the term Px will increase at the same time from Px to $P \times \overline{x + \dot{x}}$, or to $Px + P\dot{x}$, or will receive the small increment $P\dot{x}$, and x^m will increase at the same time from x^m to $\overline{x + \dot{x}}^m$, or $x^m + mx^{m-1}\dot{x} + \&c$, or will receive the small increment $mx^{m-1} \times \dot{x}$.

Fourthly, while $m \times x^{m-1}$ is less than P , the increment, $mx^{m-1}\dot{x}$, of x^m will be less than $P\dot{x}$, the contemporary increment of Px ; and, when $m \times x^{m-1}$ is greater than P , the said increment of x^m will be greater than the contemporary increment of Px . Therefore, while $m \times x^{m-1}$ is less than P , the binomial quantity $Px - x^m$, or the excess of Px above x^m , will continually increase; and, when mx^{m-1} is greater than P , the said binomial quantity will continually decrease, till it becomes equal to 0 when x^m is equal to Px , or x^{m-1}

is equal to P , or x is equal to $\sqrt[m-1]{P}$ or $P^{\frac{1}{m-1}}$. Therefore the said binomial quantity $Px - x^m$ will, first, increase from 0 to it's greatest magnitude, while mx^{m-1} increases from 0 to P , or while x^{m-1} increases from 0 to $\frac{P}{m}$, or while x increases from 0 to $\left(\frac{P}{m}\right)^{\frac{1}{m-1}}$; and it will then decrease from it's greatest magnitude to 0, while x increases further from $\left(\frac{P}{m}\right)^{\frac{1}{m-1}}$ to $P^{\frac{1}{m-1}}$. And consequently the greatest magnitude of the said binomial quantity $Px - x^m$ will be that which it has when x is equal to $\left(\frac{P}{m}\right)^{\frac{1}{m-1}}$.

But, when x is $= \left(\frac{P}{m}\right)^{\frac{1}{m-1}}$, Px will be $= P \times \left(\frac{P}{m}\right)^{\frac{1}{m-1}}$ ($= m \times \frac{P}{m} \times \left(\frac{P}{m}\right)^{\frac{1}{m-1}}$ $= m \times \frac{P}{m} \times \left(\frac{P}{m}\right)^{\frac{m-1}{m-1}} \times \left(\frac{P}{m}\right)^{\frac{1}{m-1}}$ $= m \times \frac{P}{m} \times \left(\frac{P}{m}\right)^{\frac{m-1+1}{m-1}}$ $= m \times \frac{P}{m} \times \left(\frac{P}{m}\right)^{\frac{m}{m-1}}$;

and consequently the binomial quantity $Px - x^m$ will be $= m \times \frac{P}{m} \times \left(\frac{P}{m}\right)^{\frac{m}{m-1}}$

$- \left(\frac{P}{m}\right)^{\frac{m}{m-1}} = (m-1) \times \left(\frac{P}{m}\right)^{\frac{m}{m-1}}$. Therefore the greatest possible magnitude of the binomial quantity $Px - x^m$ in the equation $Px - x^m = Q$ will be

$$(m-1) \times \left(\frac{P}{m}\right)^{\frac{m}{m-1}}. \quad Q. E. I.$$

Art. 8. And hence it follows that, if Q , or the absolute term of the binomial equation $Px - x^m = Q$, is equal to $\overline{m-1} \times \frac{P}{m}^{\frac{m}{m-1}}$, the said equation will have but one root, which will be $\frac{P}{m}^{\frac{1}{m-1}}$; but, if the absolute term Q is less than $\overline{m-1} \times \frac{P}{m}^{\frac{m}{m-1}}$, the said equation will have two roots, to wit, one root that will be less than $\frac{P}{m}^{\frac{1}{m-1}}$, and another root that will be greater than $\frac{P}{m}^{\frac{1}{m-1}}$, but less than $P^{\frac{1}{m-1}}$.

Art. 9. If $\frac{P}{m}^{\frac{1}{m-1}}$, or the middle value of x , which it has when the binomial quantity $Px - x^m$ has attained it's greatest possible magnitude, be denoted by the letter M , and the excess of this middle value of x above the lesser root of the equation $Px - x^m = Q$ be denoted by the letter d , the greater root of the said equation will be less than $M + d$. For, if x be taken $= M + d$, the value of $Px - x^m$ thence arising will be less than Q , or it's value when x is $= M - d$, as may be shewn in the manner following

If x is $= M - d$, we shall have $Px (= P \times \overline{M-d}) = PM - Pd$; and we shall have $x^m = \overline{M-d}^m$. Therefore $Px - x^m$ will be $= PM - Pd - \overline{M-d}^m$.

Therefore Q , or the absolute term of the equation $Px - x^m = Q$, will be $= PM - Pd - \overline{M-d}^m$.

Now let x be supposed to be $= M + d$.

Then we shall have $Px (= P \times \overline{M+d}) = P \times M + P \times d$; and we shall have $x^m = \overline{M+d}^m$. Therefore $Px - x^m$ will be $= PM + Pd - \overline{M+d}^m$.

We

We must therefore compare together the former compound quantity $PM - Pd - \overline{M-d}^m$ (which is equal to the absolute term Q ;) with this latter compound quantity $PM + Pd - \overline{M+d}^m$, in order to discover which of the two is the greater.

Now the former compound quantity $PM - Pd - \overline{M-d}^m$ will be greater than the latter compound quantity $PM + Pd - \overline{M+d}^m$ if $PM - \overline{M-d}^m$ is greater than $PM + 2Pd - \overline{M+d}^m$, or if $PM + \overline{M+d}^m - \overline{M-d}^m$ is greater than $PM + 2Pd$, or if $\overline{M+d}^m - \overline{M-d}^m$ is greater than $2Pd$. We must therefore inquire whether $\overline{M+d}^m - \overline{M-d}^m$ is greater or less than $2Pd$.

Now $\overline{M+d}^m$ is, by the binomial theorem, = to the series $M^m + m \times M^{m-1} \times d + m \times \frac{m-1}{2} \times M^{m-2} \times d^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times M^{m-3} \times d^3 + \&c$, or (if we put $B = m$, and $C = M \times \frac{m-1}{2}$, and $D = m \times \frac{m-1}{2} \times \frac{m-2}{3}$;) = to the series $M^m + B \times M^{m-1} \times d + C \times M^{m-2} \times d^2 + D \times M^{m-3} \times d^3 + \&c$; and $\overline{M-d}^m$ is, by the residual theorem, = to the series $M^m - B \times M^{m-1} \times d + C \times M^{m-2} \times d^2 - D \times M^{m-3} \times d^3 + \&c$, which consists of the same terms with the former series, but with the sign $-$, or the sign of subtraction, prefixed to it's second, fourth, and other following even terms. Therefore $\overline{M+d}^m - \overline{M-d}^m$ will be equal to the excess of the former series above the latter, and consequently to the series $2B \times M^{m-1} \times d + 2D \times M^{m-3} \times d^3 + \&c$, the terms of which are respectively double of the second, fourth, and other following even terms of the first series. We must therefore compare this series $2B \times M^{m-1} \times d + 2D \times M^{m-3} \times d^3 + \&c$ with $2Pd$, to discover which of the two is the greater quantity.

Now the series $2B \times M^{m-1} \times d + 2D \times M^{m-3} \times d^3 + \&c$ will be greater than $2Pd$ if the series $B \times M^{m-1} + d \times M^{m-3} \times d^2 + \&c$ (arising from

from a division of the terms of the last series by $2d$) is greater than P . We must therefore inquire whether the series $B \times M^{m-1} + D \times M^{m-3} \times d^2 + \&c$ will be greater, or less, than P .

Now B is $= m$, and M is $= \sqrt[m]{\frac{P}{m}}$. Therefore M^{m-1} will be $= \frac{P}{m}$, and $B \times M^{m-1}$ will be $(= B \times \frac{P}{m} = m \times \frac{P}{m}) = P$, that is, the first term $B \times M^{m-1}$ of that series will be equal to P , and consequently the first term together with those that follow it (which are all marked with the sign $+$, or added to the first term,) that is, the whole of the said series, must be greater than P .

Therefore the compound quantity $PM - Pd - (M-d)^m$, (which is equal to the absolute term Q .) will be greater than the compound quantity $PM + Pd - (M+d)^m$. Q. E. D.

Therefore, while x increafes from M , or $\sqrt[m]{\frac{P}{m}}$, to $M + d$, the binomial quantity $Px - x^m$ will decrease from it's greatest magnitude to a quantity less than Q or the absolute term of the equation $Px - x^m = Q$; and consequently the value of x at the instant of time at which the said binomial quantity $Px - x^m$ is equal to Q , must be somewhat less than $M + d$, or $\sqrt[m]{\frac{P}{m}}$.

Art. 10. Though $M + d$ will be somewhat greater than the greater root of the equation $Px - x^m = Q$, yet, for the most part, the difference between them will be but small, and consequently the number obtained by computing

the value of $M + d$, or $\sqrt[m]{\frac{P}{m}} + d$, will be near enough to the true value of this greater root to be a very convenient ground-work, or basis, of a further approach to it's true value by Mr. Raphson's method of approximation, of which two successive processes would be sufficient in most cases to give us the value of the said greater root to a very great degree of exactness. But it will be still more convenient to make one more approach to the true value of the said greater root before we have recourse to Mr. Raphson's method of approximation,

mation, by employing *the Differential Method* of approximation, described in my Appendix to Dr. Halley's Tract on this subject (pages 97, 98, 99, &c - - - 108,) for this purpose. This may be done in the manner following.

Let $M + d$, or $\left[\frac{P}{m}\right]^{\frac{1}{m-1}} + d$, be substituted instead of x in the binomial quantity $Px - x^m$; and let the value of the said binomial quantity resulting from such substitution be called D .

Secondly, let another quantity less than $M + d$ by a 200th part of it, or equal to $M + d - \frac{M + d}{200}$, be substituted instead of x in the binomial quantity $Px - x^m$; and let the value of the said binomial quantity resulting from such substitution be called E . Then we shall have three different values of x that are nearly equal to each other, to wit, $M + d$, $M + d - \frac{M + d}{200}$, and the true value of the greater root of the equation $Px - x^m = Q$, and likewise three values of the binomial quantity $Px - x^m$ corresponding to the said three values of x , to wit, D , E , and Q , which are, all three, known quantities, as are likewise two of the former three quantities to which they correspond, to wit, the two quantities $M + d$ and $M + d - \frac{M + d}{200}$; so that x is the only quantity of all the fix that is unknown. We must, therefore, according to the directions of that differential method of approximation, make the following proportion for the discovery of a nearer value of x ; to wit, As the difference of D and E (which are the values of the binomial quantity $Px - x^m$ corresponding to $M + d$ and $M + d - \frac{M + d}{200}$ respectively,) is to the difference of D and Q (which are the values of the said binomial quantity corresponding to $M + d$ and x , or the greater root of the equation $Px - x^m$, respectively,) so, very nearly, will be the difference of $M + d$ and $M + d - \frac{M + d}{200}$, that is, the small quantity $\frac{M + d}{200}$, to the difference of $M + d$ and x ; or as $E - D$ is to $Q - D$, so, very nearly, will $\frac{M + d}{200}$ be to $M + d - x$; or, if we put y for $M + d$, as $E - D$ is to $Q - D$, so, very nearly, will $\frac{y}{200}$ be to $y - x$;

from

from which proportion we shall have $y - x$, very nearly, $= \frac{\frac{y}{200} \times \overline{Q-D}}{E-D}$, and

consequently (adding x to both sides,) y , nearly, $= \frac{\frac{y}{200} \times \overline{Q-D}}{E-D} + x$, and

(subtracting $\frac{\frac{y}{200} \times \overline{Q-D}}{E-D}$ from both sides,) x , nearly, $= y - \frac{\frac{y}{200} \times \overline{Q-D}}{E-D}$ the fraction

$$\frac{\frac{y}{200} \times \overline{Q-D}}{E-D}$$

This quantity will be much nearer to the true value of x , or the greater root of the equation $Px - x^m = Q$ than $M + d$, or y , was, and will, in many cases, be as near to the said true value as need to be desired. But, if it is not thought to be sufficiently exact, it will, at least, be an excellent ground-work of a further approach to the said true value by Mr. Raphson's method of approximation, of which a single process will give us another value of the said greater root of the equation $Px - x^m = Q$ that may be considered as perfectly exact. And this, I apprehend, will be found a very convenient and satisfactory method of investigating the value of the greater root of this equation $Px - x^m = Q$ when we already know the value of its lesser root.

Art. II. If it should happen (which, perhaps, may in some cases be possible, that $M + d$ will be considerably greater than the greater root of the equation $Px - x^m = Q$, and consequently that it will not be very fit to be chosen for the first near value of that greater root, or the ground of our approach to a second near value of it either by Mr. Raphson's method of approximation, or by the aforesaid differential method of approximation, such over-great magnitude of it will be discovered by the substitution of it instead of x in the binomial quantity $Px - x^m$; for the value of the said binomial quantity, resulting from such substitution, (which value we have above called D ;) will, in such case, be much less than Q . And in this case we must take some quantity less than $M + d$, that we shall conjecture to be nearly equal to the true value of the said greater root; and, calling the said lesser quantity $M + e$, we must substitute $M + e$ instead of x in the binomial quantity $Px - x^m$, and call the result of the substitution E ; and afterwards we must substitute $M + e - \sqrt[200]{\frac{M+e}{200}}$ instead of x in the said binomial quantity $Px - x^m$, and call the result of the said substitution F ; and then, putting $y = M + e$, (instead of making it $= M + d$, as before,) we must say, As $F - E$ is to $Q - E$, so, very nearly, will

$M + e - \sqrt{M + e - \frac{M + e}{200}}$, or the small quantity $\frac{M + e}{200}$, be to $M + e - x$;
 or, as $F - E$ is to $Q - E$, so, very nearly, will $y - \sqrt{y - \frac{y}{200}}$, or the small
 quantity $\frac{y}{200}$, be to $y - x$; whence we shall have $y - x = \frac{\frac{y}{200} \times \overline{Q - E}}{F - E}$, and
 (adding x to both sides,) $y = \frac{\frac{y}{200} \times \overline{Q - E}}{F - E} + x$, and (subtracting the fraction
 $\frac{\frac{y}{200} \times \overline{Q - E}}{F - E}$ from both sides, $x = y - \text{the fraction } \frac{\frac{y}{200} \times \overline{Q - E}}{F - E}$; which will
 therefore be a second near value of x , or the greater root of the equation
 $Px - x^m = Q$, which will be much more exact than the first near value y , or
 $M + e$. And afterwards, if greater exactness be required, we may, by one
 process of Mr. Raphson's method of approximation, obtain a third near value
 of x , or the said greater root, that will be as exact as can be desired.

The Application of the foregoing Conclusions to the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Art. 12. I now return to the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, of which
 we know the lesser root to be $= 1$, and will apply the conclusions obtained in
 the preceeding articles concerning the roots of the binomial equation $Px - x^m$
 $= Q$, to the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, of which we wish to find
 the greater root.

Now in this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ the letter t answers to the
 letter m in the equation $Px - x^m = Q$, and $\frac{z}{a}$ answers to P , and $\frac{z}{a} - 1$
 answers to Q , and r answers to the unknown quantity x .

Therefore $\frac{P}{m}$ will be $= \frac{\frac{z}{a}}{t} = \frac{z}{a \times t}$, or $\frac{z}{at}$, and $\left(\frac{P}{m}\right)^{\frac{1}{m-1}}$, or M , will
 be $= \left(\frac{z}{at}\right)^{\frac{1}{t-1}}$. Therefore the greater root of the equation $\frac{z}{a} \times r - r^t =$
 $\frac{z}{a} - 1$ will be greater than $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$.

Further,

Further, the excess of $\sqrt[m]{\frac{P}{m}}$, or $\sqrt[t]{\frac{z}{at}}$, or M , above the lesser root of the equation (which we know to be 1,) is $= \sqrt[t]{\frac{z}{at}} - 1$; that is, d is $= \sqrt[t]{\frac{z}{at}} - 1$. Therefore $M + d$, or $\sqrt[m]{\frac{P}{m}} + d$, will be $(= \sqrt[t]{\frac{z}{at}} + \sqrt[t]{\frac{z}{at}} - 1) = 2 \times \sqrt[t]{\frac{z}{at}} - 1$. Let this quantity be called y .

Having thus obtained y , or the first near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, we must substitute it instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, and put D for the value of the said binomial quantity resulting from such substitution; and then we must subtract from y a 200th part of it, or $\frac{y}{200}$, and substitute the remainder, or $y - \frac{y}{200}$, instead of r in the said binomial quantity $\frac{z}{a} \times r - r^t$, and call the result of such substitution E . Then we must make the following proportion; As $E - D$ is to $Q - D$, or to $\frac{z}{a} - 1 - D$, so, very nearly, will $y - \sqrt{\frac{y}{200}}$, or the small quantity $\frac{y}{200}$, be to $y - r$. And we shall thereby have $y - r =$

$\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, and consequently (adding r to both sides,) $y = r +$ the

fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, and (subtracting $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$ from both

sides,) $r = y -$ the fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$; which will be a very good

approximation to the true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, that will be much nearer than $M + d$, or y , to the true value of the said greater root.

Q. E. I.

And, if we make this second near value of r the ground-work of a further approach to the true value of the said quantity by Mr. Raphson's method of approximation, a single process of that method will give us a third near value of it that will be as near the truth as any one can desire.

Art. 13. I will now proceed to try the accuracy of this expression y — the fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, just now given for a second near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by applying it to the resolution of the numeral equation $17.8614 \times r - r^{12.5} = 16.8614$, which Dr. Halley has brought as an example of the utility of his expression $1 + \sqrt[2]{{vv + 2by}} - b$ for finding the value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

An Example of the Resolution of the Numeral Equation $17.8614 \times r - r^{12.5} = 16.8614$, (which comes under the General Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$), by means of the foregoing Expression, $r = y$ — the Fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$ given in the preceeding Article.

This numeral equation $17.8614 \times r - r^{12.5} = 16.8614$ is derived from the general equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by supposing the annuity a to be 34*l.* 8*s.*, or 34.4*l.* *per annum*; and t , or the number of years during which the receipt of the annuity has been forborn, to be 12 years and a half; and z , or the amount of the annuity (which is now to be paid to the annuitant in a single payment, with compound interest upon it, in consequence of his having forborn to demand the said annuity during the whole of the said term of 12 years and a half,) to be 614.4327*l.* And it is required to find r , the rate of interest which corresponds to that amount.

Here $\frac{z}{a}$ is $(= \frac{614.4327}{34.4}) = 17.8614$, and $\frac{z}{a} - 1$ is $(= 17.8614 - 1) = 16.8614$, and t is $= 12.5$. Therefore at will be $(= 34.4 \times 12.5) = 430$,
and

and $\frac{z}{at}$ will be $(= \frac{614.43272}{430}) = 1.428,913,2$, and $\sqrt[t-1]{\frac{z}{at}}$ will be $(= \sqrt[11.5]{1.428,913,2}) = \sqrt[11.5]{1.428,913,2}^{12.5-1} = \sqrt[11.5]{1.428,913,2}^{11.5}$; which root of 1.428,913,2 may be most readily found by means of a Table of Logarithms.

Now the logarithm of 1.428,913,2 is $= 0.155,005,8$.

Therefore the logarithm of $\sqrt[11.5]{1.428,913,2}$ will be $= \frac{0.155,005,8}{11.5} = 0.013,478,7$; which is the logarithm of the number 1.031,522,5. Therefore $\sqrt[11.5]{1.428,913,2}$ will be $= 1.031,522,5$, or $\sqrt[t-1]{\frac{z}{at}}$, or M, will be $= 1.031,522,5$.

Therefore $2 \times \sqrt[t-1]{\frac{z}{at}}$ will be $(= 2 \times 1.031,522,5) = 2.063,045,0$, and

$2 \times \sqrt[t-1]{\frac{z}{at}} - 1$ will be $(= 2.063,045,0 - 1) = 1.063,045,0$; that is, $M + d$, or y , will be $= 1.063,045,0$. Q. E. I.

This is a pretty good approximation to the true value of r , which is 1.06. But the next approximation, which we shall obtain by computing the expression

$y - \frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, will be much nearer to the said true value. This expression may be computed as follows.

Art. 14. Since y is $= 1.063,045,0$, we shall have $\frac{y}{200} (= \frac{1.063,045,0}{200}) = 0.005,315,2$, and $y - \frac{y}{200} (= 1.063,045,0 - 0.005,315,2) = 1.057,729,8$.

These values of y and $y - \frac{y}{200}$ must now be substituted instead of r in the binomial quantity $17.8614 \times r - r^{12.5}$, and the results of these substitutions will be equal to D and E in the foregoing expression.

Now, if r be $= y$, or 1.063,045,0, we shall have $17.8614 \times r (= 17.8614 \times 1.063,045,0) = 18.987,471,963$, and $r^{12.5} = \sqrt[12.5]{1.063,045,0}^{12.5}$, and consequently $17.8614 \times r - r^{12.5} = 18.987,471,9 - \sqrt[12.5]{1.063,045,0}^{12.5}$.

We must therefore endeavour to find the value of $\sqrt[12.5]{1.063,045,0}^{12.5}$.

Now

Now the logarithm of 1.063,045,0 is = 0.026,551,6. Therefore the logarithm of $\overline{1.063,045,0}^{12.5}$ will be ($= 12.5 \times 0.026,551,6$) = 0.331,895,0; which is the logarithm of the number 2.147,311,3. Therefore $\overline{1.063,045,0}^{12.5}$ will be = 2.147,311,3. And consequently $17.8614 \times r - r^{12.5}$ will be ($= 18.987,471,9 - 2.147,311,3$) = 16.840,160,6. Therefore D, or the result of the substitution of y , or 1.063,045,0, instead of r , in the binomial quantity $17.8614r - r^{12.5}$, will be = 16.840,160,6. Q. E. I.

We must next substitute $y - \frac{y}{200}$, or 1.057,729,8, instead of r , in the binomial quantity $17.8614 \times r - r^{12.5}$, and call the result E.

Now, if r is = 1.057,729,8, we shall have $17.8614 \times r$ ($= 17.8614 \times 1.057,729,8$) = 18.892,535,049,72, and $r^{12.5} = \overline{1.057,729,8}^{12.5}$, and consequently $17.8614 \times r - r^{12.5} = 18.892,535,049,72 - \overline{1.057,729,8}^{12.5}$.

We must therefore endeavour to find the value of $\overline{1.057,729,8}^{12.5}$.

Now the logarithm of 1.057,729,8 is = 0.024,374,7. Therefore the logarithm of $\overline{1.057,729,8}^{12.5}$ will be ($= 12.5 \times 0.024,374,7$) = 0.304,683,7; which is the logarithm of the number 2.016,896,7. Therefore $\overline{1.057,729,8}^{12.5}$ will be = 2.016,896,7. And consequently $17.8614 \times r - r^{12.5}$ will be ($= 18.892,535,049,72 - 2.016,896,7$) = 16.875,638,3.

Therefore $E - D$ will be ($= 16.875,638,3 - 16.840,160,6$) = 0.035,477,7.

Further, since $\frac{z}{a} - 1$ is = 16.8614, and D is = 16.840,160,6, we shall have $\frac{z}{a} - 1 - D$ ($= 16.8614 - 16.840,160,6$) = 0.021,239,4, and consequently $\frac{y}{200} \times \frac{z}{a} - 1 - D$ ($= \frac{y}{200} \times 0.021,239,4$ = 0.005,315,2

$\times 0.021,239,4$) = 0.000,112,891,658,88, and $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$ ($= \frac{0.000,112,891,658,88}{0.035,477,7}$) = 0.003,182,0. Therefore $y -$ the fraction

$\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$ will be ($= 1.063,045,0 - 0.003,182,0$ = 1.059,863,0; that is, the greater root of the proposed equation $17.8614 \times r - r^{12.5} = 16.8614$ will be nearly = 1.059,863,0. Q. E. I.

This

This second near value of r is very near the true value of it, (which is 1.06,) the difference being only 0.000,137,0, or the 7737th part of the said true value. This difference is so small that the interest of a hundred pounds for a year, estimated according to this value of r , would be equal to 6 pounds wanting only ($100 \times 0.000,137,04$, or 0.013,704, or 240×0.1370 pence, or 3.2830 pence, or) 3 pence, farthing, instead of being exactly equal to 6 pounds.

This is a very great degree of exactness, and almost equal to that of the value of r , or this greater root of the equation $17.8614 \times r - r^{12.5} = 16.8614$, obtained by Dr. Halley's expression $1 + \sqrt[12.5]{bb + 2by} - b$, which was 1.059,990,9. But, if greater exactness should be required, it may be obtained by a single process of Mr. Raphson's method of approximation, grounded on this second near value of r , just now obtained, to wit, 1.059,863,0. Such a process will be as follows.

Art. 15. Since 1.059,863,0 has been found to be a near value of r in the proposed equation $17.8614 \times r - r^{12.5} = 16.8614$, we must begin our process by substituting this near value of r instead of r in the binomial quantity $17.8614 \times r - r^{12.5}$, in order to discover whether the result of such substitution will be greater, or less, than the absolute term 16.8614, and consequently whether the said number 1.059,863,0 will be less, or greater, than the true value of r , or the greater root of the said equation.

Now, if r be supposed to be $= 1.059,863,0$, we shall have $17.8614 \times r (= 17.8614 \times 1.059,863,0) = 18.930,636,988,2$, and consequently $17.8614 \times r - r^{12.5} = 18.930,636,988,2 - 1.059,863,0^{12.5}$.

We must therefore find the value of $1.059,863,0^{12.5}$; which may be done as follows.

The logarithm of 1.059,863,0 is $= 0.025,248,7$. Therefore the logarithm of $1.059,863,0^{12.5}$ will be $(= 12.5 \times 0.025,248,7) = 0.315,608,7$; which is the logarithm of the number 2.068,277,1. Therefore $1.059,863,0^{12.5}$ will be $= 2.068,277,1$.

Therefore $17.8614 \times r - r^{12.5}$ will be $(= 18.930,636,9 - 2.068,277,1) = 16.862,359,8$. This number is very nearly equal to, but a little greater than, 16.8614, or the absolute term of the proposed equation $17.8614 \times r - r^{12.5} = 16.8614$; and therefore the number 1.059,863,0 must be very nearly equal to, but somewhat less than, the true value of r , or the greater root of the said equation.

Q. E. I.

Now let w be put for the excess of the true value of r , or the greater root of this equation, above the number 1.059,863,0.

Then

Then will r be $= 1.059,863,0 + w$, and consequently $17.8614 \times r$ will be
 $(= 17.8614 \times \overline{1.059,863,0 + w}) = 17.8614 \times 1.059,863,0 + 17.8614 \times w$
 $= 18.930,636,9 + 17.8614 \times w$.

Further, $r^{12.5}$ will be $= \overline{1.059,863,0 + w}^{12.5}$ ($=$, by the binomial theorem,
to the series $\overline{1.059,863,0}^{12.5} + 12.5 \times \overline{1.059,863,0}^{11.5} \times w + \&c =$
 $\overline{1.059,863,0}^{12.5} + 12.5 \times \frac{\overline{1.059,863,0}^{12.5}}{1.059,863,0} \times w + \&c = 2.068,277,1 + 12.5$
 $\times \frac{2.068,277,1}{1.059,863,0} \times w + \&c = 2.068,277,1 + 12.5 \times 1.951,457,0 \times w$
 $+ \&c) = 2.068,277,1 + 24.393,212,5 \times w + \&c$.

Therefore the binomial quantity $17.8614 \times r - r^{12.5}$ will be equal to the
compound quantity

$$\left\{ \begin{array}{l} 18.930,636,9 + 17.8614 \times w \\ - 2.068,277,1 - 24.393,212,5 \times w - \&c \end{array} \right\}$$

$$= 16.862,359,8 - 6.531,812,5 \times w - \&c.$$

But the binomial quantity $17.8614 \times r - r^{12.5}$ is $= 16.8614$.

Therefore the compound quantity $16.862,359,8 - 6.531,812,5 \times w - \&c$
will also be $= 16.8614$.

Therefore (adding $6.531,812,5 \times w$ to both sides,) we shall have $16.862,$
 $359,8 - \&c = 16.8614 + 6.531,812,5 \times w$, and (subtracting 16.8614
from both sides,) $6.531,812,5 \times w (= 16.862,359,8 - 16.8614 - \&c)$
 $= 0.000,959,8 - \&c$, and consequently $w (= \frac{0.000,959,8}{6.531,812,5} - \&c) = 0.000,$
 $146,9 - \&c$. Therefore $1.059,863,0 + w$ will be $= 1.059,863,0 +$
 $0.000,146,9 - \&c = 1.060,009,9 - \&c$; that is, this third near value of r ,
or the greater root of the proposed equation $17.8614 \times r - r^{12.5} = 16.8614$,
will be $= 1.060,009,9 - \&c$, or somewhat less than $1.060,009,9$. Q. E. I.

This number, $1.060,009,9$, is so near the true value of r (which is 1.06 ,
or $1.060,000,0$), that the interest of a hundred pounds for a year, computed
according to this value of r , would be equal to $(100 \times 0.060,009,9\%$, or to
 $6.000,99\%$, or to 6 pounds and $0.000,99$ parts of a pound, or 6 pounds and
 $960 \times 0.000,99$ farthings, or 6 pounds and $0.950,40$ parts of a farthing,
or) 6 pounds and less than a single farthing, instead of being exactly equal
to 6 pounds. This number $1.060,009,9$ exhibits the value of r , or the
greater root of the proposed equation $17.8614r - r^{12.5} = 16.8614$, to nearly
the same degree of exactness as the number $1.059,990,9$, obtained for it by
Dr.

Dr. Halley's expression $1 + \sqrt[3]{bb + 2by} - b$; that number being less than the true value of r (which is 1.06,) by the small fraction 0.000,009,1, and the number 1.060,009,9, just now obtained, being greater than the said true value by the small fraction 0.000,009,9, which differs from the fraction 0.000,009,1 by 0.000,000,8, which is less than 0.000,001, or one millionth part of an unit.

Another Example of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by means of the foregoing Expressions $M + d$, or y , and $y - \text{the Fraction}$

$$\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$$

, when t , or the Number of Years during which the Annuity has been forborn, is 70 Years.

Art. 16. I will now try the foregoing method of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the expressions $M + d$, or y , and $y - \text{the fraction}$ $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, or expressions similar to them, in a case in which t is a very great number of years.

Let us therefore suppose the annuity to be, as before, an annuity of 34.4*l.*, or 34*l.* 8*s.* per annum, but the number of years during which the receipt of it has been forborn, to be 70, instead of 12 $\frac{1}{2}$, or 12.5, years, and the amount of the said annuity at the end of the said 70 years to be 33,297.095,903,04*l.* And let it be required, from the knowledge of these three quantities, to find r , or the rate of interest corresponding to the said amount 33,297.095,903,04*l.*

Now in this Problem the true value of r is 1.06; because the amount 33,297.095,903*l.* was derived above, in pages 265, 266, from a supposition that r was = 1.06. And in reversing the Problem, and computing the value of r by means of Dr. Halley's expression $1 + \sqrt[3]{bb + 2by} - b$, it was found to be = 1.058,712,7, in page 267; which number is tolerably near the true value of r , or 1.06, and shews that this expression of Dr. Halley is very useful even in this difficult case of so great a number of years as 70. By comparing the values of r that we shall obtain in the following computation with these two numbers 1.06 and 1.058,712,7, we shall be able to judge of the usefulness of the expressions by means of which those values will be obtained.

Now in this Problem a is = 34.4/, and t is = 70, and z is = 33,297.095,903,04/, and consequently $\frac{z}{a}$ is (= $\frac{33,297.095,903,04}{34.4}$) = 967.648,136,7, and $\frac{z}{a} - 1$ is = 966.648,136,7. Therefore the general equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ becomes in this case $967.648,136,7 \times r - r^{70} = 966.648,136,7$.

The Investigation of a first near Value of r , or the greater Root of the Equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, by means of the lesser Root of the said Equation, which is known to be = 1.

Art. 17. Since $\frac{z}{a}$ is = 967.648,136,7, and t is = 70, we shall have $\frac{z}{at}$ (= $\frac{z}{a} = \frac{967.648,136,7}{70}$) = 13.823,544,8, and $\sqrt[t-1]{\frac{z}{at}}$ (= $\sqrt[70-1]{\frac{z}{at}}$ = $\sqrt[69]{\frac{z}{at}}$) = $\sqrt[69]{13.823,544,8}$. We must therefore find the value of $\sqrt[69]{13.823,544,8}$.

Now the logarithm of 13.823,544,8 is 1.140,619,4. Therefore the logarithm of $\sqrt[69]{13.823,544,8}$ will be (= $\frac{1.140,619,4}{69}$) = 0.016,530,7; which is the logarithm of the number 1.038,797,1. Therefore $\sqrt[69]{13.823,544,8}$ will be = 1.038,797,1; that is, $\sqrt[t-1]{\frac{z}{at}}$, or M , will be = 1.038,797,1.

Therefore $2 \times \sqrt[t-1]{\frac{z}{at}}$ will be (= $2 \times 1.038,797,1$) = 2.077,594,2, and $2 \times \sqrt[t-1]{\frac{z}{at}} - 1$ will be (= $2.077,594,2 - 1.000,000,0$) = 1.077,594,2; that is, $M + d$ will be = 1.077,594,2.

Art. 18.

Art. 18. This quantity $M + d$, or 1.077,594,2, we know to be greater than the true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or $967.648,136,7 \times r - r^{70} = 966.648,136,7$; but we do not know whether the difference will be considerable or not, and consequently whether this quantity will be fit to be made use of as a first near value of r in

the following expression $y = \frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, or any similar expression, by which we may endeavour to obtain a second and much nearer value of it. In order therefore to be able to judge, whether the excess of $M + d$, or 1.077,594,2, above the true value of r will be small, or great, we will substitute 1.077,594,2 instead of r in the binomial quantity $967.648,136,7 \times r - r^{70}$, and compare the result of this substitution with 966.648,136,7, or the absolute term of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, which is equal to the true value of the said binomial quantity, or it's value in the foregoing equation.

Now, if r is = 1.077,594,2, we shall have $967.648,136,7 \times r (= 967.648,136,7 \times 1.077,594,2) = 1042.732,019,748,727,14$, and $r^{70} = 1.077,594,2^{70}$. We must therefore find the value of $1.077,594,2^{70}$.

The logarithm of 1.077,594,2 is = 0.032,455,2. Therefore the logarithm of $1.077,594,2^{70}$ will be $(= 70 \times 0.032,455,2) = 2.271,864,0$; which is the logarithm of the number 187.01. Therefore $1.077,594,2^{70}$ is = 187.010,000,0; and consequently the binomial quantity $967.648,136,7 \times r - r^{70}$ will be $(= 1042.732,019,7 - 187.010,000,0) = 855.722,019,7$. This number is much less than 966.648,136,7, or the value of the binomial quantity $967.648,136,7 \times r - r^{70}$, when r is equal to the true value of the greater root of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$. Therefore 1.077,594,2 must be considerably greater than the true value of that greater root, and consequently will not be very fit to be made use of as it's first near value, or the basis of the following approximation. And therefore it will be expedient in this case to choose some number less than 1.077,594,2 for our first near value of r , that is to be called y , and employed in the next expression

$y = \frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, or other expression of the same kind that we shall have occasion to compute. Now we know that this greater root (though it is less than $M + d$, or 1.077,594,2, and from this trial it appears to be considerably less than that number,) yet will be greater than M , (or

$\frac{1}{at} \left| \frac{1}{t-1} \right|$, or $13.823,544,8^{\frac{1}{65}}$,) or 1.038,797,1. And therefore, since it is

greater than 1.038,797,1, and a good deal less than 1.077,594,2, we will now suppose it to be nearly equal to an arithmetical mean between those two numbers, and consequently to be nearly $= \left(\frac{1.077,594,2 + 1.038,797,1}{2} \right)$, or $\frac{2.116,391,3}{2}$, or) 1.058,195,6, and will try the effect of this conjecture by substituting 1.058,195,6 instead of r in the binomial quantity $967.648,136,7 \times r - r^{70}$, in order to discover whether the value of the said binomial quantity, resulting from such substitution, will be nearly equal to the absolute term 966.648,136,7, or not, and consequently whether 1.058,195,6 will be nearly equal to the true value of r , or the greater root of the proposed equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, or not.

Now, if r is $= 1.058,195,6$, we shall have $967.648,136,7 \times r (= 967.648,136,7 \times 1.058,195,6) = 1023.961,000,604,138,52$, and r^{70} will be $= 1.058,195,6^{70}$. We must therefore find the value of $1.058,195,6^{70}$.

The logarithm of 1.058,195,6 is $= 0.024,565,9$. Therefore the logarithm of $1.058,195,6^{70}$ will be $(= 70 \times 0.024,565,9) = 1.719,613,0$; which is the logarithm of the number 52.434,000,0. Therefore $1.058,195,6^{70}$ will be $= 52.434,000,0$; and consequently the binomial quantity $967.648,136,7 \times r - r^{70}$ will be $(= 1023.961,000,6 - 52.434,000,0) = 971.527,000,6$.

This number 971.527,000,6 is nearly equal to, but a little greater than, 966.648,136,7, or the absolute term of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$. Therefore 1.058,195,6 will be nearly equal to, but a little less than, the true value of the greater root of the said equation, and will be near enough to the said true value to be fit to be taken for a first near value of the said greater root, and employed as such in the next expression by which we are to make a further approach to the true value of the said greater root.

The Investigation of a second near Value of r , or the greater Root of the Equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, by the Differential Method of Approximation.

Art. 19. We will therefore put $y = 1.058,195,6$, and $D = 971.527,000,6$, or the value of the binomial quantity $967.648,136,7 \times r - r^{70}$ resulting from the substitution of 1.058,195,6 in it's terms instead of r . And, as 1.058,195,6 is not greater than the true value of r , (as $M + d$, or 1.077,594,2, was,) but less than the said true value, we will add $\frac{y}{200}$ to y , (instead of subtracting $\frac{y}{200}$ from y , as in the last example,) and substitute the sum $y + \frac{y}{200}$ instead of r in the binomial quantity $967.648,136,7 \times r - r^{70}$, and call the value of the said binomial

binomial quantity resulting from such substitution E. This may be done in the following manner.

Since y is $= 1.058,195,6$, we shall have $\frac{y}{200} (= \frac{1.058,195,6}{200}) = 0.005,290,9$, and consequently $y + \frac{y}{200} (= 1.058,195,6 + 0.005,290,9) = 1.063,486,5$.

We must therefore substitute $1.063,486,5$, instead of r , in the binomial quantity $967.648,136,7 \times r - r^{70}$, and call the value of the said quantity resulting from such substitution E.

Now, if r be supposed to be $= 1.063,486,5$, we shall have $967.648,136,7 \times r (= 967.648,136,7 \times 1.063,486,5) = 1029.080,730,130,604,55$; and r^{70} will be $= 1.063,486,5^{70}$. We must therefore find the value of $1.063,486,5^{70}$.

The logarithm of $1.063,486,5$ is $= 0.026,731,9$. Therefore the logarithm of $1.063,486,5^{70}$ will be $(= 70 \times 0.026,731,9) = 1.871,233,0$; which is the logarithm of the number $74.341,793,1$. Therefore $1.063,486,5^{70}$ will be $= 74.341,793,1$; and consequently the binomial quantity $967.648,136,7 \times r - r^{70}$ will be $(= 1029.080,730,1 - 74.341,793,1) = 954.738,937,0$; that is, E will be $= 954.738,937,0$.

This number E, or $954.738,937,0$, is less than $966.648,136,7$, or the absolute term of the proposed equation; and therefore $1.063,486,5$, or the value of r corresponding to it, will be greater than the true value of r , or the greater root of that equation.

Here therefore we have three contiguous values of r , to wit, 1st, the number $1.058,195,6$, or y , (which is less than the true value of r , or the greater root of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$;) and, 2ndly, the said true value of r , or the greater root of the said equation, and, 3dly, the number $1.063,486,5$, or $y + \frac{y}{200}$, (which is greater than the said true value of r , or the greater root of the said equation,) and we have also the three corresponding values of the binomial quantity $967.648,136,7 \times r - r^{70}$, to wit, 1st, the number, or value, $971.527,000,6$, or D, corresponding to $1.058,195,6$, or y , and, 2ndly, the absolute term $966.648,136,7$, or $\frac{z}{a} - 1$, corresponding to the true value of r , or the said greater root of the equation, and, 3dly, the number, or value, $954.738,937,0$, or E, corresponding to $1.063,486,5$, or $y + \frac{y}{200}$; and consequently we may form the following proportion for the discovery of a second near value of r , or the greater root of the said equation: As $D - E$ is to $D - \left[\frac{z}{a} - 1 \right]$, or as $971.527,000,6 - 954.738,$

954.738,937,0 is to 971.527,000,6 — 966.648,136,7, so, very nearly, will $y + \frac{y}{200} - y$, or $\frac{y}{200}$, or $\frac{1.058,195,6}{200}$, or 0.005,290,9, be to $r - y$; or, as 16.788,063,6 is to 4.878,863,9, so, very nearly, will 0.005,290,9 be to $r - y$, or to $r - 1.058,195,6$. Therefore $r - 1.058,195,6$ will be = the fraction $\frac{4.878,863,9 \times 0.005,290,9}{16.788,063,6} (= \frac{0.025,813,581,008,51}{16.788,063,6}) = 0.001,537,6$; and consequently r will be $(= 0.001,537,6 + 1.058,195,6) = 1.059,733,2$; that is, the second near value of r , or the greater root of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, obtained by this application of the differential method of approximation, will be 1.059,733,2. Q. E. I.

This number, 1.059,733,2, is a good deal nearer to the true value of r in this equation (which is 1.06,) than the number obtained for it in page 267, by means of Dr. Halley's expression $1 + \sqrt[3]{bb + 2by}^{\frac{1}{3}} - b$, which was 1.058,712,7. And, if this number 1.059,733,2 is not thought to be sufficiently exact, a single process of Mr. Raphson's method of approximation, grounded upon this number, will give us another near value of r , that will be as exact as need be desired for any purposes whatsoever. Such a process will be as follows.

The Investigation of a third near Value of r , or the greater Root of the Equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, by Mr. Raphson's Method of Approximation.

Art. 20. The first operation of this process must be to substitute the number 1.059,733,2 instead of r in the binomial quantity $967.648,136,7 \times r - r^{70}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or less, than 966.648,136,7, or the absolute term of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, and consequently whether the said number 1.059,733,2 will be less, or greater, than the true value of r , or the greater root of the said equation.

Now, if we suppose r to be = 1.059,733,2, we shall have $967.648,136,7 \times r (= 967.648,136,7 \times 1.059,733,2) = 1025.448,856,379,128,44$, and $r^{70} = 1.059,733,2^{70}$. We must therefore find the value of $1.059,733,2^{70}$.

The logarithm of 1.059,733,2 is = 0.025,196,5. Therefore the logarithm of $1.059,733,2^{70}$ will be $(= 70 \times 0.025,196,5) = 1.763,755,0$; which is the logarithm of the number 58.043,693,3.

Therefore

Therefore $\overline{1.059,733,2}^{70}$ is $= 58.043,693,3$. And consequently the binomial quantity $967.648,136,7 \times r - r^{70}$ will be $(= 1025.448,856,3 - 58.043,693,3) = 967.405,163,0$. This number is greater than $966.648,136,7$, or the absolute term of the proposed equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$; and therefore the number $1.059,733,2$ must be less than the true value of r , or the greater root of the said equation. Q. E. I.

Art. 21. Let w be equal to the small excess of r above the number $1.059,733,2$.

Then we shall have $r = 1.059,733,2 + w$, and consequently $967.648,136,7 \times r (= 967.648,136,7 \times \overline{1.059,733,2} + w = 967.648,136,7 \times 1.059,733,2 + 967.648,136,7 \times w) = 1025.448,856,379,128,44 + 967.648,136,7 \times w$. And r^{70} will be $= \overline{1.059,733,2 + w}^{70}$ ($=$, by the binomial theorem, to the series $\overline{1.059,733,2}^{70} + 70 \times \overline{1.059,733,2}^{69} \times w + \&c = \overline{1.059,733,2}^{70} + 70 \times \frac{\overline{1.059,733,2}^{70}}{1.059,733,2} \times w + \&c = 58.043,693,3 + 70 \times \frac{58.043,693,3}{1.059,733,2} \times w + \&c = 58.043,693,3 + 70 \times 54.771,987,2 \times w + \&c) = 58.043,693,3 + 3834.039,104,0 \times w + \&c$.

Therefore the binomial quantity $967.648,136,7 \times r - r^{70}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1025.448,856,3 + 967.648,136,7 \times w \\ - 58.043,693,3 - 3834.039,104,0 \times w - \&c \end{array} \right\} \\ = 967.405,163,0 - 2866.390,967,3 \times w - \&c.$$

But that binomial quantity is $= 966.648,136,7$.

Therefore the compound quantity $967.405,163,0 - 2866.390,967,3 \times w - \&c$ will also be $= 966.648,136,7$. And consequently (adding $2866.390,967,3 \times w$ to both sides,) we shall have $967.405,163,0 = 966.648,136,7 + 2866.390,967,3 \times w$, and (subtracting $966.648,136,7$ from both sides,) we shall have $2866.390,967,3 \times w (= 967.405,163,0 - 966.648,136,7) = 0.757,026,3$, and consequently $w (= \frac{0.757,026,3}{2866.390,967,3}) = 0.000,264,1$.

Therefore r , or $1.059,733,2 + w$, will be $(= 1.059,733,2 + 0.000,264,1) 1.059,997,3$; that is, the third near value of r , or the greater root of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, obtained by this process of Mr. Raphson's method of approximation, will be $= 1.059,997,3$.

Q. E. I.

This

This number, 1.059,997,3, is so near to the true value of r (which is 1.06,) that the interest of a hundred pounds for a year, computed according to this estimation of r , would be $(= 100 \times 0.059,997,3^l. = 5.999,73^l., \text{ or } 5^l. 19s. + 0.049,73^l. = 5^l. 19s. + 20 \times 0.049,73s. = 5^l. 19s. + 0.994,60 \text{ of a shilling, or } 5^l. 19s., \text{ and more than } \frac{99}{100} \text{ parts of another shilling, instead of being exactly equal to 6 pounds. The difference of these sums is less than half a farthing. The equation therefore is now resolved with great exactness.}$

Observations on the foregoing Method of resolving the Equation

$$\frac{z}{a} \times r - r^t = \frac{z}{a} - 1.$$

Art. 22. The principle of this method of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is to make use of our knowledge "that the lesser root of it is equal to 1" as a means of discovering the greater root of it, which is the object of our pursuit. This is done by observing, first, that the middle value of r in an equation of this form, $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, (which is the higher limit of it's lesser root and the lower

limit of it's greater root,) is $= \sqrt[t]{\frac{z}{a}}$, or $\sqrt[t]{\frac{z}{at}}$, and, secondly, that, if

this middle value of r be called M , and it's excess above 1, or the lesser root

of the equation, be called d , that is, if d be $= \sqrt[t]{\frac{z}{at}} - 1$, the greater root of the equation will always be less than $M + d$; so that the said greater root will always be less than $M + d$, but greater than M . And we observed, in the 3d place, that $M + d$, though greater than the greater root of the equation, would often exceed it by only a very small quantity, and therefore would be sufficiently near to it's true value to be a very convenient first near value of it, or to be fit to be employed in a second process of approximation to it's true value either by the differential method of approximation or by Mr. Raphson's method. And we further observed that we might always discover whether $M + d$ was, or was not, near enough to the true value of the said greater root to be fit to be so employed, by substituting it instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, and comparing the value of the said binomial quantity resulting

resulting from such substitution with $\frac{z}{a} - 1$, or the absolute term of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. For, if the value of the said binomial quantity resulting from such substitution was only a little less than the absolute term $\frac{z}{a} - 1$, we might conclude that $M + d$ would be only a little greater than the true value of the said greater root of the equation, and would therefore be fit to be employed as a first near value of it in a second process of approximation to it's true value: but, if the value of the said binomial quantity resulting from such substitution was very much less than the absolute term $\frac{z}{a} - 1$, we might conclude that $M + d$ would be a good deal greater than the true value of the said greater root of the equation, and that it would in that case be expedient to find, by a reasonable conjecture, another quantity, less than $M + d$, but greater than M , for our first near value of r , or the said greater root of the equation, which we are to employ in a second process of approximation to it's true value. And in this latter case it will sometimes be convenient to take the arithmetical mean between $M + d$ and M for the said first near value of the said greater root, as we have seen in the second of the two foregoing examples, or the resolution of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$.

Now, when we have thus obtained our first near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, either by making it equal to $M + d$, if that quantity is pretty near the truth, (as we did in the first of the two foregoing examples, or in the resolution of the equation $17.8014 \times r - r^{12.5} = 16.8614$.) or by making it equal to some quantity less than $M + d$, as, for example, to an arithmetical mean between $M + d$ and M (as we did in the second of the foregoing examples,) if $M + d$ is found to be too great, we may find a second near value of r , or the said greater root of the equation, that will be very near to it's true value, by a different method of proceeding from that above-described in the foregoing articles, (which was the differential method of approximation;) to wit, by the resolution of a certain quadratick equation, that may be deduced from the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, though not without a good deal of computation both algebraical and arithmetical, which makes this method, in my opinion, much less convenient than the said former method already described. But, that the reader may judge of this matter for himself, I will now proceed to lay before him this second way of finding such second near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by means of a quadratick equation; which may be described as follows.

Another Method of resolving the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by the Resolution of a Quadratick Equation.

Art. 23. Let $M + d$, or some quantity less than $M + d$, but greater than the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, be taken as a first near value of r , or the said greater root, by means of which we are to find a second and much nearer value of the said greater root. Call this quantity b , and it's excess above the true value of r , or the said greater root, w .

Then will r be $= b - w = b \times \overline{1 - \frac{w}{b}}$, and r^t will be $= \overline{b - w}^t = b^t \times \overline{1 - \frac{w}{b}}^t =$ (by the binomial theorem,) $b^t \times$ the series $1 - t \times \frac{w}{b} + t \times \frac{t-1}{2} \times \frac{w^2}{b^2} - \&c.$

Now let x be $= \frac{w}{b}$.

Then we shall have $r = b \times \overline{1 - x}$, and $r^t = b^t \times$ the series $1 - t \times x + t \times \frac{t-1}{2} \times x^2 - \&c =$ the series $b^t - t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 - \&c.$ And $\frac{z}{a} \times r$ will be $= \frac{z}{a} \times b \times \overline{1 - x} = \frac{bz}{a} - \frac{bzx}{a}.$

Therefore the binomial quantity $\frac{z}{a} \times r - r^t$ will be $=$ the compound quantity

$\frac{bz}{a} - \frac{bzx}{a} \times x -$ the series $b^t - t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 - \&c$
 $=$ the compound quantity

$$\left\{ \begin{array}{l} \frac{bz}{a} - \frac{bzx}{a} \times x \\ - b^t + t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c. \end{array} \right\}$$

But

But the binomial quantity $\frac{z}{a} \times r - r^t$ is $= \frac{z}{a} - 1$.

Therefore the compound quantity

$$\left\{ \begin{array}{l} \frac{bz}{a} - \frac{bz}{a} \times x \\ - b^t + t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c \end{array} \right\}$$

will also be $= \frac{z}{a} - 1$.

To simplify the notation, put $e = \frac{z}{a}$, and $f = \frac{z}{a} - 1$.

And we shall then have the compound quantity

$$\left\{ \begin{array}{l} be - be \times x \\ - b^t + t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c \end{array} \right\} = f.$$

Now, because b is supposed to be greater than r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, it follows that $\frac{z}{a} \times b - b^t$ will be less than $\frac{z}{a} \times r - r^t$, or than (it's equal) the absolute term $\frac{z}{a} - 1$; that is, $eb - b^t$ in the last equation will be less than f , and therefore may be subtracted from both sides of the equation. Let it be so subtracted. And we shall then have $t \times b^t \times x - ebx - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = f + b^t - eb$, or $t \times b^t - eb \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = f + b^t - eb$. Therefore (multiplying both sides into the fraction $\frac{2}{t \times t-1 \times b^t}$, or the reciprocal of $t \times \frac{t-1}{2} \times b^t$, the co-efficient of x^2 , in order to free x^2 from it's said co-efficient,) we shall have $\frac{2tb^t - 2eb}{t^2b^t - tb^t} \times x - x^2 + \&c = \frac{2f + 2b^t - 2eb}{t^2b^t - tb^t}$.

Now let G be $= \frac{tb^t - eb}{t^2b^t - tb^t}$, and H be $= \frac{2f + 2b^t - 2eb}{t^2b^t - tb^t}$.

And we shall then have $2Gx - x^2 + \&c = H$, and $2Gx - x^2 = H - \&c$.

Now let both sides of this equation be subtracted from GG , than which they are necessarily less.

And we shall then have $GG - 2Gx + x^2 = GG - H + \&c$. Therefore $G - x$ will be $= \sqrt{GG - H + \&c}$, and G will be $= x + \sqrt{GG - H + \&c}$, and x will be $= G - \sqrt{GG - H + \&c}$, or x will be somewhat less than $G - \sqrt{GG - H}$. Therefore w , or $b \times x$, will be somewhat less than $b \times \sqrt{G - \sqrt{GG - H}}$, and r , or $b - w$, will be $= b - b \times \sqrt{G - \sqrt{GG - H}} + \&c$, or somewhat greater than $b - b \times \sqrt{G - \sqrt{GG - H}}$; that is, the second near value of r , or the greater root of the proposed equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, will be nearly equal to, but somewhat greater than, $b - b \times \sqrt{G - \sqrt{GG - H}}$. Q. E. I.

Art. 24. In the foregoing article the near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which was denoted by b , was supposed to be greater than the true value of r , or that greater root, though less than $M + d$. But it may sometimes be less than the said true value, as well as less than $M + d$; and then the manner of proceeding, in order to find a second near value of r by the resolution of a quadratick equation, will be as follows.

Let w be put for the excess of the true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, above b , it's first and known near value; and let x be $= \frac{w}{b}$.

Then we shall have $r = b + w = b \times 1 + \frac{w}{b} = b \times 1 + x = b + bx$.

And r^t will be $= \overline{b \times 1 + x}^t = b^t \times \overline{1 + x}^t =$, by the binomial theorem, $b^t \times$ the series $1 + t \times x + t \times \frac{t-1}{2} \times x^2 + \&c =$ the series $b^t + t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$; and $\frac{z}{a} \times r$ will be $= \frac{z}{a} \times \overline{b + bx} = \frac{z}{a} \times b + \frac{z}{a} \times bx$.

Therefore

Therefore the binomial quantity $\frac{z}{a} \times r - r^t$ will be = the compound quantity $\frac{z}{a} \times b + \frac{z}{a} \times bx -$ the series $b^t + t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c =$ the compound quantity

$$\left\{ \begin{array}{l} \frac{z}{a} \times b + \frac{z}{a} \times bx \\ - b^t - t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c. \end{array} \right\}$$

But the said binomial quantity is = $\frac{z}{a} - 1$.

Therefore the compound quantity

$$\left\{ \begin{array}{l} \frac{z}{a} \times b + \frac{z}{a} \times bx \\ - b^t - t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c \end{array} \right\}$$

will also be = $\frac{z}{a} - 1$.

To simplify the equation, put $e = \frac{z}{a}$, and $f = \frac{z}{a} - 1$.

And we shall then have the compound quantity

$$\left\{ \begin{array}{l} eb + eb \times x \\ - b^t - t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c \end{array} \right\} = f.$$

Add $t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2$ to both sides of this equation.

And we shall then have $eb - b^t + eb \times x - \&c = f + t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2$.

And subtract $eb \times x$ from both sides; and we shall then have $eb - b^t - \&c = f + t \times b^t \times x - eb \times x + t \times \frac{t-1}{2} \times b^t \times x^2$.

Now, because b is supposed to be less than r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, it follows that $\frac{z}{a} \times b - b^t$ will be greater

greater than $\frac{z}{a} \times r - r^t$, or (it's equal,) the absolute term $\frac{z}{a} - 1$; that is, $e \times b - b^t$ in the last equation will be greater than f . Therefore f may be subtracted from $e \times b - b^t$, and consequently from the other side of the equation. Let it be so subtracted. And we shall then have $eb - b^t - f - \&c = t \times b^t \times x - eb \times x + t \times \frac{t-1}{2} \times b^t \times x^2$, or (ranging the terms of the equation in the more usual manner, with the absolute term on the right-hand side,) $t \times b^t \times x - ebx + t \times \frac{t-1}{2} \times b^t \times x^2 = eb - b^t - f - \&c$.

Now in this equation $t \times b^t \times x - ebx + t \times \frac{t-1}{2} \times b^t \times x^2 = eb - b^t - f - \&c$, the co-efficient $t \times b^t$ must always be greater than the co-efficient eb . For b , though it is less than $M + d$ and also than r , will yet be greater than M , which is the lower limit of the magnitude of r ; that is, b will be

greater (than $\left[\frac{P}{m}\right]^{\frac{1}{m-1}}$, or than $\left[\frac{P}{t}\right]^{\frac{1}{t-1}}$, or) than $\left[\frac{z}{at}\right]^{\frac{1}{t-1}}$, by art. 12,

page 370. Therefore b^{t-1} will be greater than $\left[\frac{z}{at}\right]^{\frac{1}{t-1}}$, or than $\left[\frac{z}{at}\right]^{\frac{t-1}{t-1}}$, or than $\frac{z}{at}$. Therefore $t \times b^{t-1}$ must be greater than $t \times \frac{z}{at}$, or

than $\frac{z}{a}$, or than e , and $t \times b^{t-1} \times b$ must be greater than $e \times b$, or $t \times b^t$ must be greater than eb . Q. E. D.

Since therefore $t \times b^t$ is greater than eb , the quantity $t \times b^t x$ will be greater than the quantity $eb \times x$, and the equation $t \times b^t \times x - ebx + t \times \frac{t-1}{2} \times b^t \times x^2 = eb - b^t - f - \&c$ will become $t \times b^t - eb \times x + t \times \frac{t-1}{2} \times b^t \times x^2 = eb - b^t - f - \&c$, and (multiplying all the terms into the fraction $\frac{2}{t \times t-1 \times b^t}$, or the reciprocal of the co-efficient of x^2 , in

order to free the said square of x from it's co-efficient,) we shall have $\frac{2tb^t - 2eb}{t^2b^t - tb^t} \times x + x^2 = \frac{2eb - 2b^t - 2f}{t^2b^t - tb^t}$; and, putting G for $\frac{tb^t - eb}{t^2b^t - tb^t}$, or half

the

the co-efficient of x , and H for $\frac{2eb - 2b^t - 2f}{r^2b^t - b^t}$, or the absolute term of the equation, we shall have $2G \times x + x^2 = H$. Therefore $GG + 2Gx + xx$ will be $= GG + H$, and $G + x$ will be $= \sqrt{GG + H}$, and x will be $= \sqrt{GG + H} - G$. Therefore w (which is equal to $b \times x$) will be $=$ to $b \times \sqrt{GG + H} - G$, or to $b \times \sqrt{GG + H} - b \times G$; and r , or $b + w$, will be $= b + b \times \sqrt{GG + H} - b \times G$. Q. E. I.

The Resolution of the Numeral Equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, by this Method, or by means of a Quadratick Equation.

Art. 25. I will now proceed to try this last method of finding a second near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by applying it to the resolution of the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, in which $\frac{z}{a}$ is $= 967.648,136,7$, and $\frac{z}{a} - 1$ is $= 966.648,136,7$. And in doing so I will chuse the number 1.058,195,6 for b , or the first near value of r , by means of which we propose to obtain a much more exact value of it by resolving a quadratick equation in the manner just now described.

This number was obtained above in art. 18 by supposing the greater root of this equation to be nearly equal to an arithmetical mean between $M + d$ and M , or 1.077,594,2 and 1.038,797,1; and it was shewn in the same article to be something less than the true value of r , or the greater root of the said equation. We must therefore proceed according to the directions of the last, or 24th, article.

Here then t is $= 70$, and b is $= 1.058,195,6$, and e , or $\frac{z}{a}$, is $= 967.648,136,7$, and f , or $\frac{z}{a} - 1$, is $= 966.648,136,7$; and b^t is $= 1.058,195,6^{70} = 52.434,000,0$, as is shewn above in art. 18; and $eb - b^t$, or $967.648,136,7 \times b - b^{70}$, or $967.648,136,7 \times 1.058,195,6 - 1.058,195,6^{70}$, is $(= 1023.961,000,6 - 52.434,000,0) = 971.527,000,6$. Therefore $eb - b^t - f$ will be $(= 971.527,000,6 - 966.648,136,7) = 4.878,863,9$, and $2eb - 2b^t - 2f$ will be $(= 2 \times 4.878,863,9) = 9.757,727,8$. And $t \times b^t$ will

will be $(= 70 \times 52.434,000,0) = 3670.380,000,0$, and $t^2 \times b^t$ will be $(= 70^2 \times 52.434,000,0) = 4900 \times 52.434,000,0 = 256,926.600,000,0$, and $t^2 \times b^t - t \times b^t$ will be $(= 256,926.600,000,0 - 3670.380,000,0) = 253,256.220,000,0$; and consequently $\frac{2eb - 2b^t - 2f}{t^2b^t - tb^t}$ will be $(= \frac{0.757,727,8}{253,256.220,000,0}) = 0.000,038,5$; that is, H will be $= 0.000,038,5$.

Further, since $t \times b^t$ is $= 3670.380,000,0$, and eb is $(= 967.648,136,7 \times 1.058,195,6) = 1023.961,000,6$, we shall have $G (= \frac{t \times b^t - eb}{t^2b^t - tb^t} = \frac{3670.380,000,0 - 1023.961,000,6}{253,256.220,000,0} = \frac{2646.418,999,4}{253,256.220,000,0}) = 0.010,449,5$; and the quadratick equation $2Gx + x^2 = H$ will become $2 \times 0.010,449,5 \times x + x^2 = 0.000,038,5$, of which the root x will be $= \sqrt{GG + H} - G$.

Now, since G is $= 0.010,449,5$, we shall have $GG = 0.010,449,5^2 = 0.000,109,192,050,25$, and $GG + H (= 0.000,109,192,050,25 + 0.000,038,5) = 0.000,147,692,050,25$, and $\sqrt{GG + H} (= \sqrt{0.000,147,692,050,25}) = 0.012,152,8$, and $\sqrt{GG + H} - G (= 0.012,152,8 - 0.010,449,5) = 0.001,703,3$. Therefore x will be $= 0.001,703,3$, and consequently w , or $b \times x$, will be $(= b \times 0.001,703,3 = 1.058,195,6 \times 0.001,703,3) = 0.001,802,4$. Therefore r , or $b + w$, will be $(= 1.058,195,6 + 0.001,802,4) = 1.059,998,0$; that is, the greater root of the proposed equation $967.648,136,7 \times r - r^2 = 966.648,136,7$ will be very nearly $= 1.059,998,0$. Q. E. I.

This value of r is wonderfully near it's true value, which is 1.06. And therefore we may conclude that this last method of finding the value of r by resolving the quadratick equation $2Gx + xx = H$ is a very exact one.

A Third Method of resolving the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ to a great Degree of Exactness by means of a Series invented by Sir Isaac Newton for reverting an Infinite Series.

Art. 26. This equation may also be resolved to a great degree of exactness by means of a certain series invented by Sir Isaac Newton for reverting a given infinite

infinite series of quantities involving the successive powers of a quantity less than 1, and of which the powers consequently form a decreasing progression. This series is as follows.

If y be a quantity less than 1, so that it's powers $y, y^2, y^3, y^4, y^5, y^6, y^7$, &c form a decreasing progression of terms, and a, b, c, d, e, f, g , &c are numerical co-efficients of the several successive powers of y , to wit, $y, y^2, y^3, y^4, y^5, y^6, y^7$, &c; and, if the series $ay + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$ is $= z$, and z is also less than 1, so that it's several successive powers $z, z^2, z^3, z^4, z^5, z^6, z^7$, &c form likewise a decreasing progression of terms; then will y be equal to an infinite series of quantities, of which the first five terms

$$\text{will be } \frac{z}{a} - \frac{b}{a^2} \times z^2 + \frac{2bb - ac}{a^3} \times z^3 - \frac{5b^3 - 5abc + aad}{a^4} \times z^4 + \frac{30a^2c^2 - 21ab^2c + 6a^2bd + 14b^4 - a^3e}{a^5} \times z^5.$$

This is the first of two very useful Theorems given us by Sir Isaac Newton for the reversion of infinite serieses, and published in the *Commercium Epistolicum* of Mr. John Collins and other learned Mathematicians of the last century, which was printed in the year 1712 by order of the Royal Society of London, in consequence of the dispute between Sir Isaac Newton and Mr. Leibnitz, of Hanover, concerning the first invention of the Method of Fluxions. These Theorems make a part of Sir Isaac Newton's (then Mr. Newton's,) Second Letter to Mr. Oldenburgh, the Secretary of the Royal Society, with directions to communicate it to Mr. Leibnitz, dated October 24, 1676, which is full of curious and useful mathematical discoveries. But they are given without a demonstration or investigation. It seems probable, however, from some expressions in Sir Isaac Newton's Letter, that he investigated them by his second method of reverting infinite serieses, which he has described in that Letter to Mr. Oldenburgh, and which is founded on the plainest and clearest principles of Arithmetick, and the common operations of Addition, Subtraction, Multiplication, and Division. And accordingly I have, in my Discourse on the Reversion of Infinite Serieses published in the Third Volume of this Collection of Tracts called *Scriptores Logarithmici*, investigated both these Theorems at great length by this second and most easy method of reverting infinite serieses contained in this celebrated Letter. The investigation of the first of these Theorems, or that which I have above set down, is contained in art. 313, 314, 315, &c - - 319, of that Discourse on the Reversion of Serieses, pages 695, 696, 697, &c - - - 700, of the said Third Volume of the *Scriptores Logarithmici*; to which I refer the reader.

Art. 27. If a , or the co-efficient of y in the first term, ay , of the series $ay + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$, which is to be reverted, is 1, (as is very often the case,) the notation of the foregoing theorem will be much shorter and simpler than it is in the foregoing article, and the said theorem will

then be as follows; to wit, that, if the quantity y is less than 1, and consequently its successive powers $y, y^2, y^3, y^4, y^5, y^6, y^7$, &c form a decreasing progression of terms; and the quantity z is also less than 1, and consequently its successive powers $z, zz, z^3, z^4, z^5, z^6, z^7$, &c form likewise a decreasing progression of terms; and, if the series $y + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$ is equal to the quantity z , the quantity y will be equal to an infinite series of terms of which the five first terms will be $z - b \times zz + \frac{2bb - c}{2bb - c} \times z^3 - (5b^3 - 5bc + d) \times z^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{3c^2 - 21bbc + 6bd + 14b^4 - e} \times z^5$.

Now this last series may be applied to the resolution of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ to a great degree of exactness, when we have already obtained a pretty near value of r , or the greater root of the said equation, (which may be done by means of its two limits M and $M + d$ in the manner above-described,) by proceeding as follows.

Art. 28. To avoid confusion in the notation, it will now be expedient to use the letter p , instead of the letter b , to denote the first near value of r , or the greater root of the proposed equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, that we have already discovered by means of its two limits M and $M + d$. And I will suppose that p is somewhat less than the true value of r , or the said greater root, as we found 1.058,195,6 to be in the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, which has been already resolved. And I will put w (as before,) for the unknown small excess of r , or the said greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, above p , the known near value of it. And, to simplify the notation, I will substitute P instead of $\frac{z}{a}$, the co-efficient of r , and Q instead of the absolute term $\frac{z}{a} - 1$; whereby the said equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ will be converted into the equation $P \times r - r^t = Q$.

Further, let y be $= \frac{w}{p}$.

And then we shall have $r (= p + w = p \times \sqrt[1 + \frac{w}{p}]{} = p \times \sqrt[1 + y]{} = p + py$, and $P \times r (= P \times \sqrt[p + py]{} = P \times p + P \times py$.

And r^t will be $= \overline{p + w}^t = p \times \overline{1 + \frac{w}{p}}^t = p^t \times \overline{1 + \frac{w}{p}}^t = p^t \times \overline{1 + y}^t =$, by the binomial theorem, to $p^t \times$ the series $1 + t \times y + t \times$
 $\times$

$$\times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 \\ + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c.$$

Therefore the binomial quantity $Pr - r^t$ will be = the compound quantity $Pp + Ppy - p^t \times$ the series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c.$

But the binomial quantity $Pr - r^t$ is = Q .

Therefore the compound quantity $Pp + Pp \times y - p^t \times$ the series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c$ will also be = Q .

Therefore (dividing both sides by p^t), we shall have $\frac{Pp}{p^t} + \frac{Pp}{p^t} \times y -$ the series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c = \frac{Q}{p^t}$, and (adding the said series to both sides,) we shall have $\frac{Pp}{p^t} + \frac{Pp}{p^t} \times y = \frac{Q}{p^t} +$ the series $1 + t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c$, and (subtracting 1 from both sides,) $\frac{Pp}{p^t} + \frac{Pp}{p^t} \times y - 1 = \frac{Q}{p^t} +$ the series $t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c,$

+ &c, and (subtracting $\frac{Q}{p^t}$ from both sides,) $\frac{Pp}{p^t} + \frac{Pp}{p^t} \times y \rightarrow 1 - \frac{Q}{p^t} =$ the series $t \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c,$ and (subtracting $\frac{Pp}{p^t} \times y$ from both sides,) $\frac{Pp}{p^t} + \frac{Q}{p^t} - 1 =$ the series $t \times y \rightarrow \frac{Pp}{p^t} \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c.$

Art. 29. Now, since t is greater than $\frac{Pp}{p^t}$, and consequently $t \times p^t$ is greater than Pp , it follows that $t \times y - \frac{Pp}{p^t} \times y$ (which is $= \frac{t \times p^t}{p^t} \times y - \frac{Pp}{p^t} \times y$) will be $= \frac{t \times p^t - Pp}{p^t} \times y$; and consequently the last equation will become $\frac{Pp}{p^t} - \frac{Q}{p^t} - 1 =$ the series $\frac{t \times p^t - Pp}{p^t} \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c,$ or $\frac{Pp - Q - p^t}{p^t} =$ the series $\frac{p^t - Pp}{p^t} \times y + t \times \frac{t-1}{2} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times y^5 + \&c;$ and therefore (multiplying all the terms into p^t), we shall have $Pp - Q - p^t =$ the series $(p^t - Pp) \times y + t \times \frac{t-1}{2} \times p^t \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times p^t \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times p^t \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times p^t \times y^5 + \&c$

$\times \frac{t-3}{4} \times \frac{t-4}{5} \times p^t \times y^5 + \&c$, and (dividing all the terms by $tp^t - Pp$,
 the co-efficient of y), we shall have $\frac{Pp - p^t - Q}{tp^t - Pp} =$ the series $y + t \times \frac{t-1}{2}$
 $\times \frac{p^t}{tp^t - Pp} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3}$
 $\times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{tp^t - Pp} \times y^5$
 $+ \&c$, or (ranging the terms of the equation in the usual order, with the
 known quantity, or absolute term, on the right hand,) the series $y + t \times \frac{t-1}{2}$
 $\times \frac{p^t}{tp^t - Pp} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp} \times y^3 + t \times \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times$
 $\frac{p^t}{tp^t - Pp} \times y^5 + \&c = \frac{Pp - p^t - Q}{tp^t - Pp}$; which is an equation of the same
 form with the equation $y + by^2 + cy^3 + dy^4 + ey^5 + fy^6 + gy^7 + \&c$
 $= z$, which is the subject of Sir Isaac Newton's first theorem above-men-
 tioned.

Now let q be put for the absolute term $\frac{Pp - p^t - Q}{tp^t - Pp}$ of this last equation
 $y + t \times \frac{t-1}{2} \times \frac{p^t}{tp^t - Pp} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp} \times y^3 + t \times$
 $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times$
 $\times \frac{p^t}{tp^t - Pp} \times y^5 + \&c = \frac{Pp - p^t - Q}{tp^t - Pp}$, and let b be put $= t \times \frac{t-1}{2} \times$
 $\frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^2 ; and c be put $= t \times \frac{t-1}{2} \times \frac{t-2}{3}$
 $\times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^3 ; and d be put $= t \times \frac{t-1}{2} \times$
 $\times$

$\times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^4 ; and e be put
 $= t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient
 of y^5 . And the said equation will then be converted into the equation $y + by^2 + cy^3 + dy^4 + ey^5 + \&c = q$. And consequently, by the above-mentioned
 first theorem of Sir Isaac Newton, we shall have $y = q - b \times q^2 + \frac{2bb - c}{a} \times q^3 - \frac{5b^3 - 5bc + d}{a^2} \times q^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{a^3} \times q^5 - \&c$.

And, when y is found by computing some of the terms of this series, we shall have $w = p \times y$, and $r = p + w = p + p \times y$; that is, the second near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, will be $= p + p \times y$. Q. E. I.

will now proceed to try the exactness of this method of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by applying it to the resolution of the same numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, which has been above resolved by the two preceding methods of resolution explained in the foregoing articles.

The Resolution of the Numeral Equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, by the foregoing Method of Resolution founded on Sir Isaac Newton's second Method of reverting an Infinite Series.

Art. 30. Let p , or the first near value of r , or the greater root of the proposed equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, be 1.058,195,6, which is the value that has been already obtained in art. 18 by supposing the said greater root to be equal to an arithmetical mean between $M + d$ and M . This value has been shewn, in the same art. 18, to be somewhat less than the true value of the said greater root. Therefore, if w be put for the difference between this greater root and 1.058,195,6, we shall have $r = 1.058,195,6 + w$, or $p + w$, as in the two foregoing articles, 28 and 29.

And,

And, since t is $= 70$, we shall have $p^t = \overline{1.058,195,0}^{70} =$ (by art. 18) $52.434,000,0$, and $t \times p^t (= 70 \times 52.434,000,0) = 3670.380,000,0$.

And, because P is $= 967.648,136,7$, we shall have $Pp (= 957.648,136,7 \times 1.058,195,6) = 1023.961,000,6$, and consequently $t \times p^t - Pp (= 3670.380,000,0 - 1023.961,000,6) = 2646.418,999,4$. Therefore

$$\frac{p^t}{t \times p^t - Pp} \text{ will be } = \frac{52.434,000,0}{2646.418,999,4} = 0.019,813,1.$$

Further, since Q is $= 966.648,136,7$, we shall have $Pp - p^t - Q (= 1023.961,000,6 - 52.434,000,0 - 966.648,136,7 = 1023.961,000,6 - 1019.082,136,7) = 4.878,863,9$, and $\frac{Pp - p^t - Q}{t \times p^t - Pp} (= \frac{4.878,863,9}{2646.418,999,4}) = 0.001,843,5$; that is, q will be $= 0.001,843,5$.

$$\text{And, since } b \text{ is } = t \times \frac{t-1}{2} \times \frac{p^t}{t \times p^t - Pp},$$

$$\text{and } c \text{ is } = t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{t \times p^t - Pp} = b \times \frac{t-2}{3},$$

$$\text{and } d \text{ is } = t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{t \times p^t - Pp} = c \times \frac{t-3}{4},$$

$$\text{and } e \text{ is } = t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{t \times p^t - Pp} = d \times \frac{t-4}{5},$$

$$\text{we shall have } b (= 70 \times \frac{70-1}{2} \times 0.019,813,1 = 70 \times \frac{69}{2} \times 0.019,813,1 = 35 \times 69 \times 0.019,813,1 = 2415 \times 0.019,813,1) = 47.848,636,5,$$

$$\text{and } c (= b \times \frac{t-2}{3} = b \times \frac{70-2}{3} = b \times \frac{68}{3} = 47.848,636,5 \times \frac{68}{3} = \frac{3253.707,282,0}{3}) = 1084.569,094,0;$$

$$\text{and } d (= c \times \frac{t-3}{4} = c \times \frac{70-3}{4} = c \times \frac{67}{4} = 1084.569,094,0 \times \frac{67}{4} = \frac{72,666.129,298,0}{4}) = 18,166.532,324,5;$$

$$\text{and } e (= d \times \frac{t-4}{5} = d \times \frac{70-4}{5} = d \times \frac{66}{5} = 18,166.532,324,5 \times \frac{66}{5} = \frac{1,194,991.133,417,0}{5}) = 238,998.226,683,4.$$

Therefore

Therefore the final equation $y + by^3 + cy^3 + dy^4 + ey^5 + \&c = q$ obtained in art. 29 will, in this case, become $y + 47.848,636,5 \times y^3 + 1084.569,094,0 \times y^3 + 18,166.532,324,5 \times y^4 + 238,998.226,683,4 \times y^5 + \&c = 0.001,843,5$.

Further, since q is $= 0.001,843,5$, we shall have $q^2 (= 0.001,843,5^2) = 0.000,003,398,492,25$, and $q^3 (= 0.000,003,398,492 \times 0.001,843,5) = 0.000,000,006,265,120,002$, and $q^4 (= 0.000,000,006,265,120,002 \times 0.001,843,5) = 0.000,000,000,011,549,748$.

Therefore y (which is $= q - b \times q^2 + \overline{2bb - c} \times q^3 - \overline{5b^3 - 5bc + d} \times q^4 + \&c$.) will be $= 0.001,843,5 - b \times 0.000,003,398,492,25 + \overline{2bb - c} \times 0.000,000,006,265,120,002 - \overline{5b^3 - 5bc + d} \times 0.000,000,000,011,549,748 + \&c$.

Now, since b is $= 47.848,636,5$, we shall have $bb (= 47.848,636,5^2) = 2289.492,014,709$, and $b^3 (= bb \times b = 2289.492,014,709 \times 47.848,636,5) = 109,549.071,181,463$, and $2bb (= 2 \times 2289.492,014,709) = 4578.984,029,418$, and $5b^3 (= 5 \times 109,549.071,181,463) = 547,745.355,907,315$.

And, since c is $= 1084.569,094,0$, and d is $= 18,166.532,324,5$, we shall have $2bb - c (= 4578.984,029,418 - 1084.569,094,0) = 3494.414,935,418$, and $bc (= 47.848,636,5 \times 1084.569,094,0) = 51,895.152,337,940$, and $5bc (= 5 \times 51,895.152,337,940) = 259,475.761,689,700$. Therefore $5b^3 - 5bc$ will be $(= 547,745.355,907,315 - 259,475.761,689,700) = 288,269.594,217,615$, and $5b^3 - 5bc + d$ will be $(= 288,269.594,217,615 + 18,166.532,324,5) = 306,436.126,541,615$.

Therefore y will be $= 0.001,843,5 - 47.848,636,5 \times 0.000,003,398,492 + 3494.414,935,418 \times 0.000,000,006,265 - 306,436.126,541,615 \times 0.000,000,000,011 + \&c = 0.001,843,5 - 0.000,162,589 + 0.000,021,892 - 0.000,003,370 + \&c = 0.001,865,392 - 0.000,165,959 + \&c = 0.001,699,433 + \&c$, or (neglecting the two last figures,) $0.001,699,4$.

Therefore w , or $p \times y$, or $1.058,195,6 \times y$, will be $(= 1.058,195,6 \times 0.001,699,4) = 0.001,798,2$, and consequently $1.058,195,6 + w$ will be $= 1.058,195,6 + 0.001,798,2 = 1.059,993,8$; that is, the second near value of r , or the greater root of the proposed equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, obtained by means of the four first terms of the series $q - b \times q^3 + \overline{2bb - c} \times q^3 - \overline{5b^3 - 5bc + d} \times q^4 + \overline{3c^2 - 21bbc + 6bdd + 14b^4 - e} \times q^5 - \&c$ given us by Sir Isaac Newton for the reversion of an infinite series, will be $= 1.059,993,8$. Q. E. I.

This value of r is extremely near it's true value 1.06 . For the interest of a hundred pounds for a year, computed according to this value of r , would amount to $(100 \times 0.059,993,8\% =) 5.999,38\%$, or more than $5\text{ l. } 19\text{ s. } 11\frac{1}{2}\text{ d.}$, instead

instead of being exactly six pounds. This is a very great degree of exactness, and therefore proves that this third method of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by means of the said series, is very just and accurate. And it is also founded on clear and satisfactory principles. But yet, as it requires us to go through such a number of troublesome arithmetical operations before we arrive at the value of w , or the difference of the first near value of r from it's true value, it appears to me to be much less convenient than the first method of finding a second near value of r , explained above in art. 19, which proceeds, first, by the differential method of approximation, and afterwards, (if still greater exactness is required,) by a process of Mr. Raphson's method. I also prefer this first method of resolving equations of this kind to the second method of resolving them mentioned above in art. 23, (which proceeds by the resolution of a quadratick equation,) and am inclined to think it, upon the whole, the most judicious and convenient method that can be taken for this purpose.

Art. 31. As that first method of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ seems to me to be preferable to the other two, and to be, upon the whole, the best method, of all those I have any knowledge of, that can be taken for that purpose, I will now endeavour to recapitulate in a short and summary manner the several grounds and reasonings by which we obtained in that method the three successive near values of r , or the greater root of the said equation, in art. 7, 8, &c - - 12 of this discourse.

A Recapitulation of the several Processes of the first of the foregoing three Methods of resolving the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, by which we obtained three successive near Values of r , or the greater Root of the said Equation, in Art. 7, 8, &c, - - 12.

In the first place then we observed, that this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is an equation of that form which admits of two roots; the term r^t , or the higher power of the unknown quantity r , being subtracted from $\frac{z}{a} \times r$, which involves it's simple power r .

Secondly, we observed, that the lesser root of this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is 1, and consequently that the root of it which is necessary to the solution of the Problem from which this equation is derived (in which Problem it is supposed that r must always be greater than 1,) is the other, or greater, root.

Thirdly, we observed, that in every binomial equation, (or equation containing two different powers of the unknown quantity,) of the more general form $Px - x^m = Q$ (of which form the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is a particular case,) the lesser root must be always less than $\sqrt[m-1]{\frac{P}{m}}$, or than $\sqrt[m-1]{\frac{P}{m}}$, and that the greater root must always be greater than $\sqrt[m-1]{\frac{P}{m}}$, but less than $P^{\frac{1}{m-1}}$. This observation is true, whenever there are two roots to the equation. But, when the absolute term Q is of the greatest possible magnitude, (which is the magnitude of the binomial quantity $Px - x^m$ when x is equal to $\sqrt[m-1]{\frac{P}{m}}$,) the equation will have only one root, which will be $\sqrt[m-1]{\frac{P}{m}}$.

Fourthly, it is shewn that the greatest possible magnitude of the binomial quantity $Px - x^m$, and consequently of, it's equal, the absolute term Q , is that value of the said binomial quantity which it has when x is equal to $\sqrt[m-1]{\frac{P}{m}}$; which middle value of x may, for the sake of brevity, be conveniently denoted by the letter M . And it is also shewn that that greatest magnitude of the binomial quantity $Px - x^m$ will therefore be equal to $\frac{m}{m-1} \times \sqrt[m-1]{\frac{P}{m}}$, or to $\frac{m}{m-1} \times M^m$.

Fifthly, it is shewn that, if d be put for the excess of $\sqrt[m-1]{\frac{P}{m}}$, or M , or that middle value of x , above the lesser root of the equation $Px - x^m = Q$ so

so that the said lesser root shall be $= M - d$, the greater root of the equation will be always less than $M + d$, or than $\sqrt[m]{\frac{P}{m}} + d$. But their difference will often be but small, so that $M + d$ will be a pretty good first near value of the said greater root, or a good ground-work for a further approach to it's true value by Mr. Raphson's, or some other, method of approximation.

And further, it is observed, that we can always discover by a trial whether the difference between $M + d$ and the said greater root of the equation is great or small. For, if we substitute $M + d$, instead of x , in the binomial quantity $Px - x^m$, and the value of the said binomial quantity resulting from such substitution is but a little less than Q , or the absolute term of the equation $Px - x^m = Q$, we may conclude that $M + d$ will be but a little greater than the true value of the said greater root, and therefore will be fit to be employed as a first near value of the said greater root and made the ground-work of a further approach to it's true value; but, if the value of the said binomial quantity resulting from such substitution is much less than the absolute term Q , we may conclude that the said quantity $M + d$ is considerably greater than the true value of the said greater root; and in this case it will be expedient to take for our first near value of the said greater root a quantity somewhat less than $M + d$. And, as the said greater root is always greater than M as well as less than $M + d$, it will sometimes be found convenient in these cases to take an arithmetical mean between $M + d$ and M , to wit, the quantity $M + \frac{d}{2}$, for the first near value of the said greater root, and the ground-work of a further approach to it's true value by Mr. Raphson's, or some other, method of approximation.

Sixthly, by transferring the foregoing conclusions from the general equation $Px - x^m = Q$ to the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ (which comes under the said general equation,) and by substituting $\frac{z}{a}$ for P , and $\frac{z}{a} - 1$ for Q , and r for x , and t for m , we found that $\frac{\frac{z}{a}}{t}$, or $\frac{z}{at}$, would be $= \frac{P}{m}$, and

that $\sqrt[t]{\frac{z}{at}}$ would be $= \sqrt[m]{\frac{P}{m}}$, or M .

Seventhly, we observed, that, since the lesser root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ is $= 1$, the quantity d , or the excess of M above such lesser root, will be $= M - 1$ ($= \sqrt[m]{\frac{P}{m}}^{\frac{1}{m-1}} - 1$) $= \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$; and consequently $M + d$ will be ($= M + \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1 = \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} + \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$) $= 2 \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$. So that $2 \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$ will, probably, be a fit quantity to be taken for a first near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$. This, however, must be tried by substituting this quantity $2 \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$ instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, agreeably to the fifth branch of the present recapitulation: and, if the value of the said binomial quantity resulting from this substitution is found to be but a little less than $\frac{z}{a} - 1$, or the absolute term of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, we may conclude that this quantity $2 \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$ will be proper to be made use of as a first near value of r , or the greater root of the said equation; but, if the value of the said binomial quantity resulting from such substitution is considerably less than $\frac{z}{a} - 1$, or the absolute term of the said equation, we must chuse some quantity less than $2 \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - 1$ for our first near value of r , or the greater root of the said equation; and in this case it may sometimes be expedient to chuse $M + \frac{d}{2}$, or the arithmetical mean between $M + d$ and M , (which arithmetical mean will be $= \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} + \frac{1}{2} \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - \frac{1}{2}$, or $\frac{3}{2} \times \sqrt[t]{\frac{z}{at}}^{\frac{1}{t-1}} - \frac{1}{2}$), for

our

our first near value of r , or the said greater root, or the ground work of a further approach to the true value of the said greater root by Mr. Raphson's, or some other, method of approximation.

Eighthly, let $2 \times \left[\frac{z}{at} \right]^{\frac{1}{t-1}} - 1$, or $\frac{3}{2} \times \left[\frac{z}{at} \right]^{\frac{1}{t-1}} - \frac{1}{2}$, or whatever other quantity we shall have chosen for our first near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, be called y ; and, to obtain a second near value of r , or the said greater root, which shall be much nearer to it's true value than the said former root, we may proceed as follows.

Substitute y , instead of r , in the binomial quantity $\frac{z}{a} \times r - r^t$, and call the value of the said binomial quantity resulting from such substitution D.

Then subtract from y (if y is greater than the true value of r , or the said greater root,) a 200th part of it's value, and substitute the remainder, or $y - \frac{y}{200}$, instead of r , in the same binomial quantity $\frac{z}{a} \times r - r^t$, and call the value of the said binomial quantity resulting from such substitution E. And this quantity E will be greater than D.

But, if y is less than the true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, it will be better (though not necessary,) to proceed in a different manner, and to add to y a 200th part of it, so as to produce the quantity $y + \frac{y}{200}$, and then to substitute $y + \frac{y}{200}$, instead of r , in the binomial quantity $\frac{z}{a} \times r - r^t$, and to call the result of such substitution E. And in this case E will be less than D.

But, not to make these directions too intricate by providing for this variety of cases, I shall suppose y to be greater than the true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and $y - \frac{y}{200}$ (and not $y + \frac{y}{200}$;) to be substituted instead of r in the binomial quantity $\frac{z}{a} \times r - r^t$, and the result of the said substitution to be called E; in which case E will be greater than D.

We must then subtract D from E, and likewise from $\frac{z}{a} - 1$, or the absolute term of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and must make the following

lowing proportion; As $E - D$ is to $\frac{z}{a} - 1 - D$, so, very nearly, will $y - \sqrt{y - \frac{y}{200}}$, (or $y - y + \frac{y}{200}$), or $\frac{y}{200}$, be to $y - r$; and we shall thence

have $y - r$, very nearly, $= \frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, and consequently $y =$, very

nearly, $r +$ the fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, and r , very nearly, $= y -$ the

fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$. Therefore $y =$ the fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$

will be a second near value of r , much nearer to it's true value than y was.

Q. E. I.

Ninthly and lastly, make this second near value of y , to wit, $y =$ the

fraction $\frac{\frac{y}{200} \times \frac{z}{a} - 1 - D}{E - D}$, the ground-work of another approach to the

true value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$,

by a process of Mr. Raphson's method of approximation. And we shall thereby obtain a third near value of r , or the said greater root, that will be accurate enough for all purposes.

Q. E. I.

A SCHOLIUM.

Art. 32. I have now gone through the whole of what I meant to offer to the reader's consideration on the resolution of the general equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, which is derived from a Problem relating to the amount, z , of an annuity a , that has been due, but never paid, for a given number, t , of years, when the quantity, a , of the annuity, the number, t , of the years during which it has been due, and the amount z , that is become due in a single payment at the end of the said t years, are all given or known, and r , the rate of interest corresponding to the said given amount, z , is to be derived from

from the knowledge of those other three quantities. This equation Dr. Halley has directed his readers to resolve by computing the expression $1 + \sqrt[2]{bb + 2ay}$

$-b$, in which y is $= \frac{z}{at} \sqrt[2]{t-1} - 1$, and b is $= \frac{6}{t+1}$: and this expression

is found, upon trial, to give us a very accurate near value of r , or the greater root of the said equation, which is the root wanted for the solution of the aforesaid Problem. But he has not given us any investigation of this expression; which makes the use of it, to those persons who have not found, or seen, an investigation of it, an unscientific and unsatisfactory mode of obtaining the quantity r that they are in search of. And now, after the said expression has been investigated by the learned and skilful Mr. William Morgan, of the Equitable-Assurance Office, in the manner set forth above in the first of the foregoing Notes on Dr. Halley's Discourse, it is found to depend on a very long and intricate Algebraical computation, which is grounded upon Mr. De Moivre's Theorem for raising a multinomial quantity to any given power, and on a supposition that the said Theorem is true in the case of fractional powers, or of the roots, and powers of the roots, of a multinomial quantity, as well as in the more simple case of it's integral and affirmative powers; though it has never, I believe, been demonstrated, either by Mr. De Moivre himself or any other Mathematician, in any but that first and most simple case. It is, indeed, highly probable that this Theorem (as well as Sir Isaac Newton's binomial theorem,) is true in all those cases as well as in the more simple case of integral and affirmative powers. But till the truth of it in those cases has been demonstrated, it is, in the eyes of a lover of truth and accuracy in these sciences, an obscure and uncertain Proposition. And, even if it had been demonstrated by Mr. De Moivre, or Mr. Stirling, or Mr. Mac Laurin, or Mr. Thomas Simpson of Woolwich, or some other profound Algebraist, the application of it to the investigation of this expression of Dr. Halley is attended with such a number of intricate and troublesome substitutions of one number, or Algebraick quantity, for another (such as indexes of powers, and co-efficients of the terms of serieses, and other complicated Algebraick quantities,) that the reader feels himself in continual apprehension of making some false step in his computation, and consequently of falling into some error in his conclusion. At least I must confess that such were my feelings, both when I first read and examined Mr. Morgan's investigation of Dr. Halley's said expression, and when I afterwards considered it with still greater care and attention in order to draw up the full and expanded view of it which is set forth in the first of the foregoing Notes on Dr. Halley's Discourse. These circumstances, attending the said expression of Dr. Halley, lessen both our pleasure in perusing the investigation of this expression, and our confidence in the expression itself when we have obtained it, untill we have tried it in several numeral equations, or examples, and have found it always to succeed: and then our confidence in the

the truth of it is of the *experimental* rather than of the *scientific* kind, and is paid to the success of the trials rather than to the evidence of the investigation. And for these reasons I should be rather inclined to prefer either of the three methods of resolving the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, that have been given above in this Appendix, and more especially the first of the said three methods, (of the particulars of which I have given a recapitulation in the preceeding art. 31,) to the resolution of it by the expression $1 + \sqrt[3]{bb + 2by}^{\frac{1}{3}} - b$, which Dr. Halley has given us for this purpose: acknowledging, however, that it is good to have several different methods of doing the same thing, and that Dr. Halley's aforesaid expression of the value of r , or the greater root of the said equation, is very useful in practice and comes very near the true value of the said greater root.

End of the Consideration of the first of the three Problems in Dr. Halley's Discourse on Compound Interest which have given occasion to the foregoing Notes,

*and the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$,
deduced from it.*

Art. 33. We come next to consider the second of the three Problems in Dr. Halley's Discourse which have given occasion to the foregoing Notes, and the equation deduced from it. This Problem relates to the finding of the rate of interest of money allowed in a bargain made for the purchase of an annuity for a number of years that are still to come, when the quantity of the annuity purchased, the number of years during which it is to continue, and the price that is to be paid for it by the purchaser, are all given, or known; and it occurs above in pages 226, 227, and 228 of the present Volume, in these words.

PROBLEM IV.

“The annuity (a ,) present value (z ,) and time (t ,) being given, to find (r ,) the rate of interest.

SOLUTION.

“This Problem being more difficult than it appears at first sight, and requiring the resolution of this equation, $\frac{a}{z} = \frac{z+a}{z} \times r^t - r^{t+1}$, [or, in the more usual

usual way of ranging the terms, with the known quantity, or absolute term, on the right-hand,) $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$,] to which it is reduced, there must be applied some method of approaching the root r , which is by no means evident. And that approximation, as the number of years and [the] rate [of interest] are greater, or less, cannot properly be obtained by one general rule; but rather by two, according as the value of the reversion is greater, or less.

“If the number of years be great, (as, suppose, 40, or upwards,) and, especially, if the rate of interest be high, $1 + \frac{a}{z}$ will be nearly the rate; or, more accurately, $\frac{z+a}{z} - \left[\frac{z}{z+a}\right]^t \times \frac{a}{z}$. Call it [this last quantity $\frac{z+a}{z} - \left[\frac{z}{z+a}\right]^t \times \frac{a}{z}$] r . And $\frac{a}{r^t \times r - 1}$ will be exceeding near the value of the reversion; which [quantity $\frac{a}{r^t \times r - 1}$] let be z . Then $1 + \frac{a}{z+x}$ shall approach the true rate sufficiently. But, if greater exactness be required, by repeating this process it will be obtained. Hence this rule. From the logarithm of a , &c,” as in page 227, to the bottom of the page.

“If the number of years be small, the aforesaid rule will avail little. In this case it will be requisite to approach the rate thus. Let $\frac{t+1}{2}$ be the index of a root of $\frac{at}{z}$; from which root subtract 1, and the remainder call y . And let $\frac{6}{t-1}$ be called b . I say, that $b - \sqrt{bb - 2by}$ is sufficiently near to $r-1$, and will be still nearer to the truth as the number of years is smaller; and that the error will be always in excess. Hence the rule. Divide the logarithm of $\frac{at}{z}$ by $\frac{t+1}{2}$, &c,” as in page 228.

These are the words of Dr. Halley. But the Problem may be stated more fully in the following manner.

PROBLEM IV. *Concerning the present Value of an Annuity that is to continue for a given Number of Years.*

If an annuity of a pounds a year, that is to continue for t years, is sold for the sum of z pounds; and all these quantities, a , t , and z , are given, or known: it is required, from these three known quantities, to determine the

value of r , or the rate of interest that corresponds to the price z that has been paid for the said annuity.

Art. 34. Dr. Halley (as we have seen,) divides this Problem into two cases, according as t , or the number of years during which the annuity is to continue, is greater, or not greater, than 39 years: and he gives us different expressions for the solution of these two different cases of the Problem. When t is greater than 39 years, he gives us, 1st, the expression $1 + \frac{a}{z}$ for a tolerably near value of r , or the rate sought; and, 2ndly, the expression $\left[\frac{z+a}{z} - \frac{z}{z+a} \right]^t \times \frac{a}{z}$ for a second and nearer value of it; and, lastly, the expression $1 + \frac{a}{z+x}$ for a third and still nearer value of it. These three expressions have been examined and investigated above in the Second of the foregoing Notes on Dr. Halley's Discourse: and I have nothing to add to what is there said concerning them; the principles from which the said expressions are derived being simple, clear, and certain. But the expression $b - \overline{bb - 2by}^{\frac{1}{2}}$, which Dr. Halley gives us for the solution of the second case of the Problem, in which t , or the number of years during which the annuity is to continue, is less than 40 years, is derived from Mr. De Moivre's Multinomial Theorem, and from a supposition that the said Theorem is true in the case of powers that are both fractional and negative, which makes the investigation of it (which is given above in the Third of the foregoing Notes on Dr. Halley's Discourse,) exceedingly intricate, obscure, and subtle. In this case therefore I should be inclined to make use of some other method of determining the value of r , or the rate of interest, that should be derived from some clearer principles than those by which Dr. Halley's said expression $b - \overline{bb - 2by}^{\frac{1}{2}}$ has been investigated by Mr. Morgan in the Third of the foregoing Notes on his Discourse.

Art. 35. One such method of finding the value of r in this second case of the foregoing Problem, instead of resorting to the expression $b - \overline{bb - 2by}^{\frac{1}{2}}$ given us by Dr. Halley, has been already described above in the latter part of the said Third Note on Dr. Halley's Discourse, art. 15, 16, &c; and consists of two parts, to wit, 1st, of a designation of $r - 1$ by the letter v , and a substitution of the binomial quantity $1 + v$ instead of r in the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$; and, 2ndly, of the computation of the three or four first terms of a certain series of decreasing quantities, which is equal to v , or $r - 1$, and which is obtained by means of Sir Isaac Newton's second method of reverting an infinite series. This second method of reverting an infinite series is much simpler and clearer than Sir Isaac's first method of doing the same thing, which is founded on the assumption of a series with unknown numeral co-efficients (denoted usually by the capital

pital letters, A, B, C, D, E, F, G, H, &c.) of the several powers of the quantity involved in it's terms, which unknown co-efficients are afterwards successively determined, together with the signs + and - which are to be prefixed to them, by resolving the same number of separate, simple equations of a very difficult and complicated nature. But the said second method of reverting an infinite series requires no such assumption of another series with unknown co-efficients to be afterwards successively determined by the resolution of those difficult simple equations, but is founded on the common Arithmetical operations of Addition, Subtraction, Multiplication, and Division. Both these methods of reverting infinite serieses are explained, and illustrated by examples, in my Discourse on that subject in the Third Volume of this Collection of Tracts called *Scriptores Logarithmici*; and the description and illustration of the second of these methods begins in page 668, and continues to page 725; and the subject is then resumed in page 762, and continued to page 785, or the end of the Volume. And it is there shewn, in page 711, that, if v and b are two quantities less than 1, so that their powers $v, v^2, v^3, v^4, v^5, v^6, v^7, \&c.$ and $b, b^2, b^3, b^4, b^5, b^6, b^7, \&c.$ will form decreasing progressions of terms; and, more especially, if v and b are much smaller than 1, so that those two progressions will decrease very swiftly; and, if $b, c, d, e, f, \&c.$ are numeral co-efficients of $v^2, v^3, v^4, v^5, v^6, \&c.$ and the series $v - bv^2 + cv^3 - dv^4 + ev^5 - fv^6 + \&c.$ is $= b$;—I say, it is there shewn that, upon these suppositions, v , or the root of the equation $v - bv^2 + cv^3 - dv^4 + ev^5 - fv^6 + \&c. = b$, will be equal to the series $b + b \times b^2 + \frac{2bb - c}{1} \times b^3 + \frac{5b^3 - 5bc + d}{1} \times b^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{1} \times b^5 + \&c.$ And I have shewn above in this present Volume, in Note III, art. 15, 16, 17, &c, how this reverted series $b + b \times b^2 + \frac{2bb - c}{1} \times b^3 + \frac{5b^3 - 5bc + d}{1} \times b^4 + \frac{3c^2 - 21bbc + 6bd + 14b^4 - e}{1} \times b^5 + \&c.$ may be applied to the finding of v , or $r - 1$, and consequently to the determination of $1 + v$, or r . These things I shall therefore not repeat in this place, as I have nothing to add to what is delivered on this subject in that latter part of the said Third Note. But I will here mention another method of determining the value of r in the foregoing Problem concerning the present value of an annuity for a term of years, without having recourse to Dr. Halley's expression $1 + b - \frac{bb - 2by}{1}^{\frac{1}{2}}$, which is derived from Mr. De Moivre's Multinomial Theorem in the case of fractional and negative powers. And this method will be very different from that just now mentioned, (which is grounded on the reversion of serieses,) and will bear a great resemblance to the first of the three methods by which we found a near value of r , or the greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, (resulting from the former Problem of Dr. Halley concerning the amount of an annuity that had been unpaid during a given number of years past,) in the preceeding part of this Appendix.

Art. 36. Now, in order to shew how the value of r in the present Problem may be determined by this other method, which I now propose to describe, it will be necessary, in the first place, to shew that the said Problem may be reduced to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, as Dr. Halley affirms, and consequently that the resolution of that equation will be equivalent to the solution of the said Problem. Now this may be shewn in the following manner.

The Reduction of the foregoing Problem to the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

It has been shewn in Dr. Halley's aforesaid Discourse on Compound Interest, in pages 224 and 225 of the present Volume, that z , or the price of an annuity of a pounds a year for a term of t years, when the rate of interest of money is r , is $= \frac{a}{r-1} - \frac{a}{r^t \times r-1}$. Therefore (multiplying both sides into $r-1$), we

shall have $rz - z = a - \frac{a}{r^t}$; and (multiplying both sides into r^t , we shall have

$r^{t+1} \times z - r^t \times z = r^t \times a - a = \overbrace{r^t - 1} \times a$; and (dividing both sides by $r^t - 1$), we shall have $\frac{r^{t+1} \times z - r^t \times z}{r^t - 1} = a$; and (dividing both sides

by z), we shall have $\frac{r^{t+1} - r^t}{r^t - 1} = \frac{a}{z}$. Therefore (multiplying both sides

into $r^t - 1$), we shall have $r^{t+1} - r^t = \frac{ar^t}{z} - \frac{a}{z}$, and (adding $\frac{a}{z}$ to

both sides,) $r^{t+1} - r^t + \frac{a}{z} = \frac{ar^t}{z}$; and (adding r^t to both sides,) r^{t+1}

$+ \frac{a}{z} = \frac{ar^t}{z} + r^t = \frac{ar^t}{z} + \frac{z \times r^t}{z} = \frac{z+a}{z} \times r^t$; and, lastly, subtracting r^{t+1}

from both sides,) $\frac{a}{z} = \frac{z+a}{z} \times r^t - r^{t+1}$, or (in the more common way of ranging the terms, with the known quantity, or absolute term, on the right hand,) $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. Q. E. D.

Of the Resolution of the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Art. 37. Having thus shewn that the foregoing Problem may be reduced to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, in which r is the only unknown quantity,) we must now proceed to shew how this equation may be resolved.

Now, in order to the resolution of this equation, we may observe in the first place, that one of it's roots will be $= 1$. For, if we suppose r to be $= 1$, we shall have $r^t (= 1^t) = 1$, and $r^{t+1} (= 1^{t+1})$ also $= 1$. And consequently $\frac{z+a}{z} \times r^t - r^{t+1}$ will be $(= \frac{z+a}{z} \times 1 - 1 = \frac{z+a}{z} - 1 = \frac{z+a}{z} - \frac{z}{z} = \frac{z}{z} + \frac{a}{z} - \frac{z}{z}) = \frac{a}{z}$, or the absolute term of the equation.

Therefore 1 is a root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. Q. E. D.

Art. 38. But, though 1 is one of the roots of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, yet it is not the root that will solve the foregoing Problem :

because in that Problem r denotes 1 pound together with it's interest for a year, and therefore must always be greater than 1 pound, or 1. Therefore the root of the said equation which will solve the foregoing Problem, must be it's other and greater root. We must therefore now endeavour to find the value of the other and greater root of this equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Art. 39. The observation made in art. 37, "that 1 is equal to one of the roots of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$," is not without some degree of utility in our present inquiry, notwithstanding it is not the root that will solve the foregoing Problem. For it will often (though not always) assist us in finding another quantity, greater than itself, that will approach pretty nearly to the other, or greater, root of the said equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, of which we are in search. But that we may be the better able to apply it to this purpose, it will be convenient, first, to make a few remarks on the number of roots in a binomial equation of the general form $Px^m - x^{m+1} = Q$, and on the limits of their magnitudes; which may afterwards be transferred to the roots of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which comes under that more general form.

Of the Number of Roots in the Binomial Equation $Px^m - x^{m+1} = Q$, and the Limits of their Magnitudes.

Art. 40. Now, if P and Q are known quantities, and m any whole number whatsoever, and the binomial quantity $Px^m - x^{m+1}$ is equal to Q , it is evident, in the first place, that x^{m+1} will be less than Px^m , and consequently that $\frac{x^{m+1}}{x^m}$, or x , will be less than P .

Secondly, if x be supposed to increase continually from 0 *ad infinitum*, it is evident that, while x is less than 1, x^m will be less than x , and x^{m+1} will be less than x^m ; and that, when x is equal to 1, x^m will be equal to 1, and x^{m+1} will also be equal to 1 and to x and to x^m ; and that, when x is greater than 1, x^m will be greater than 1, and greater than x , and x^{m+1} will be greater than 1, and than x , and also than x^m ; and, lastly, that, when x becomes greater than P , x^{m+1} will become greater than Px^m , and the difference between Px^m and x^{m+1} will be no longer $Px^m - x^{m+1}$, but $x^{m+1} - Px^m$; and, as x increases further from P *ad infinitum*, the binomial quantity $x^{m+1} - Px^m$ will increase continually from 0 *ad infinitum*, so as to become equal, in some instant of time during its increase, to any quantity, how small, or how great, soever.

Thirdly, if the greatest magnitude which x ever attains in the equation $Px^m - x^{m+1} = Q$, namely, the quantity P , be divided into some very great number of small and equal parts, (as, for example, into ten thousand million of such parts,) and each of these small and equal parts be denoted by $\dot{x}$, or the letter x with a point placed over it, it is evident that, while x , in its increase from 0 to its greatest magnitude P , increases from any one of its successive values to its next value, or receives the small increment $\dot{x}$, or, in other words, increases from x to $x + \dot{x}$, the term Px^m will increase at the same time from $P \times x^m$ to $P \times \overbrace{x + \dot{x}}^m$, (or to $P \times x^m + m \times x^{m-1} \dot{x} + \&c.$) or to Px^m

+

+ $mPx^{m-1}\dot{x}$ + &c, or will receive the small increment $mPx^{m-1}\dot{x}$ + &c, and x^{m+1} will increase at the same time from x^{m+1} to $\overline{x+x}^{m+1}$, or to $x^{m+1} + \overline{m+1} \times x^m \times \dot{x}$ + &c, or will receive the small increment $\overline{m+1} \times x^m \times \dot{x}$ + &c.

Fourthly, while $\overline{m+1} \times x^m \times \dot{x}$, or the increment of x^{m+1} , continues less than $mPx^{m-1}\dot{x}$, or the contemporary increment of Px^m , the binomial quantity $Px^m - x^{m+1}$ (which is the excess of Px^m above x^{m+1} , and therefore will be increased in consequence of the increment of Px^m , but diminished in consequence of the increment of the second term x^{m+1} , which is subtracted from Px^m), will continually increase: but, when $\overline{m+1} \times x^m \times \dot{x}$, or the increment of x^{m+1} , becomes greater than $mPx^{m-1}\dot{x}$, or the contemporary increment of Px^m , the said binomial quantity will continually decrease, till it becomes equal to 0, when x^{m+1} is equal to Px^m , or x is equal to P . But the increment $\overline{m+1} \times x^m \times \dot{x}$ will be less than the increment $mPx^{m-1}\dot{x}$ so long as $\overline{m+1} \times x^m$ is less than mPx^{m-1} , or as x^m is less than $\frac{m}{m+1} \times Px^{m-1}$, or as $\frac{x^m}{x^{m-1}}$, or x , is less than $\frac{m}{m+1} \times P$; and it will be greater than the increment $mPx^{m-1}\dot{x}$ when x is greater than $\frac{m}{m+1} \times P$. Therefore, while x is increasing from 0 to $\frac{m}{m+1} \times P$, the binomial quantity $Px^m - x^{m+1}$ will continually increase; and, while x increases further from $\frac{m}{m+1} \times P$ to P , or $\frac{m+1}{m+1} \times P$, (which is it's greatest possible magnitude in the equation $Px^m - x^{m+1} = Q$.) the said binomial quantity will decrease from it's greatest magnitude to 0. And consequently the greatest magnitude of the said binomial quantity $Px^m - x^{m+1}$ will be that which it has when x is equal to $\frac{m}{m+1} \times P$.

But, when x is $= \frac{m}{m+1} \times P$, the term Px^m will be $= P \times \left(\frac{mP}{m+1}\right)^m$, and x^{m+1} will be $= \left(\frac{mP}{m+1}\right)^m \times \frac{mP}{m+1}$; and consequently the binomial quantity Px^m

Of the Number of Roots in the Binomial Equation $Px^m - x^{m+1} = Q$, and the Limits of their Magnitudes.

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Thirdly, if the greatest magnitude which x ever attains in the equation $Px^m - x^{m+1} = Q$, namely, the quantity P , be divided into some very great number of small and equal parts, (as, for example, into ten thousand million of such parts,) and each of these small and equal parts be denoted by $\dot{x}$, or the letter x with a point placed over it, it is evident that, while x , in its increase from 0 to its greatest magnitude P , increases from any one of its successive values to its next value, or receives the small increment $\dot{x}$, or, in other words, increases from x to $x + \dot{x}$, the term Px^m will increase at the same time from $P \times x^m$ to $P \times \overbrace{x + \dot{x}}^m$, (or to $P \times x^m + m \times x^{m-1} \dot{x} + \&c.$) or to Px^m +

+ $mPx^{m-1}\dot{x}$ + &c, or will receive the small increment $mPx^{m-1}\dot{x}$ + &c, and x^{m+1} will increase at the same time from x^{m+1} to $\overline{x+x}^{m+1}$, or to $x^{m+1} + \overline{m+1} \times x^m \times \dot{x}$ + &c, or will receive the small increment $\overline{m+1} \times x^m \times \dot{x}$ + &c.

Fourthly, while $\overline{m+1} \times x^m \times \dot{x}$, or the increment of x^{m+1} , continues less than $mPx^{m-1}\dot{x}$, or the contemporary increment of Px^m , the binomial quantity $Px^m - x^{m+1}$ (which is the excess of Px^m above x^{m+1} , and therefore will be increased in consequence of the increment of Px^m , but diminished in consequence of the increment of the second term x^{m+1} , which is subtracted from Px^m ;) will continually increase: but, when $\overline{m+1} \times x^m \times \dot{x}$, or the increment of x^{m+1} , becomes greater than $mPx^{m-1}\dot{x}$, or the contemporary increment of Px^m , the said binomial quantity will continually decrease, till it becomes equal to 0, when x^{m+1} is equal to Px^m , or x is equal to P . But the increment $\overline{m+1} \times x^m \times \dot{x}$ will be less than the increment $mPx^{m-1}\dot{x}$ so long as $\overline{m+1} \times x^m$ is less than mPx^{m-1} , or as x^m is less than $\frac{m}{m+1} \times Px^{m-1}$, or as $\frac{x^m}{x^{m-1}}$, or x , is less than $\frac{m}{m+1} \times P$; and it will be greater than the increment $mPx^{m-1}\dot{x}$ when x is greater than $\frac{m}{m+1} \times P$. Therefore, while x is increasing from 0 to $\frac{m}{m+1} \times P$, the binomial quantity $Px^m - x^{m+1}$ will continually increase; and, while x increases further from $\frac{m}{m+1} \times P$ to P , or $\frac{m+1}{m+1} \times P$, (which is its greatest possible magnitude in the equation $Px^m - x^{m+1} = Q$;) the said binomial quantity will decrease from its greatest magnitude to 0. And consequently the greatest magnitude of the said binomial quantity $Px^m - x^{m+1}$ will be that which it has when x is equal to $\frac{m}{m+1} \times P$.

But, when x is $= \frac{m}{m+1} \times P$, the term Px^m will be $= P \times \left(\frac{mP}{m+1}\right)^m$, and x^{m+1} will be $= \left(\frac{mP}{m+1}\right)^m \times \frac{mP}{m+1}$; and consequently the binomial quantity Px^m

$Px^m - x^{m+1}$ will be $(= P \times \left[\frac{mP}{m+1}\right]^m - \left[\frac{mP}{m+1}\right] \times \left[\frac{mP}{m+1}\right]^m = P - \frac{mP}{m+1} \times \left[\frac{mP}{m+1}\right]^m = \frac{mP + P - mP}{m+1} \times \left[\frac{mP}{m+1}\right]^m = \frac{P}{m+1} \times \left[\frac{mP}{m+1}\right]^m$. Therefore the greatest possible magnitude of the binomial quantity $Px^m - x^{m+1}$ will be $\frac{P}{m+1} \times \left[\frac{mP}{m+1}\right]^m$; and consequently the greatest possible magnitude of the absolute term Q of the binomial equation $Px^m - x^{m+1} = Q$ will be $\frac{P}{m+1} \times \left[\frac{mP}{m+1}\right]^m$.

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Art. 41. And hence it follows that, if Q , or the absolute term of the binomial equation $Px^m - x^{m+1} = Q$, is equal to $\frac{P}{m+1} \times \left[\frac{mP}{m+1}\right]^m$, the said equation will have but one root, which will be $\frac{mP}{m+1}$, or $\frac{m}{m+1} \times P$; but, if the absolute term Q is less than $\frac{P}{m+1} \times \left[\frac{mP}{m+1}\right]^m$, the said equation will have two roots, of which the lesser will be less than $\frac{m}{m+1} \times P$, and the greater will be greater than $\frac{m}{m+1} \times P$, but less than P , or than $\frac{m+1}{m+1} \times P$.

Art. 42. These limits of the greater root of the equation $Px^m - x^{m+1} = Q$ are so near to each other, (their difference being only $\frac{1}{m+1}$ th part of P ,) that, if we take an arithmetical mean between them, the said arithmetical mean (which will be $\frac{m + \frac{1}{2}}{m+1} \times P$, or $\frac{2m+1}{2m+2} \times P$,) will be a very good approximation to the true value of the said greater root, which lies between the said limits $\frac{m}{m+1} \times P$ and $\frac{m+1}{m+1} \times P$. And thus we may easily obtain a good first near value of the greater root of the equation $Px^m - x^{m+1} = Q$, even without a previous knowledge of the lesser root of this equation.

Let the said arithmetical mean, or quantity $\frac{2m+1}{2m+2} \times P$, be denoted by the letter b , and be substituted instead of x in the binomial quantity $Px^m - x^{m+1}$. And, if the binomial quantity $Pb^m - b^{m+1}$, resulting from such substitution, is

is greater than Q , or the absolute term of the equation $Px^m - x^{m+1} = Q$, it will follow that b is less than the greater root of the said equation; and, if the said binomial quantity is less than Q , it will follow that b will be greater than the said greater root; and, further, if the said binomial quantity is nearly equal to Q , it will follow that the said quantity b will be very nearly equal to the said greater root. By this substitution, therefore, of b for x in the binomial quantity $Px^m - x^{m+1}$, we shall be enabled to judge whether b will be near enough to the true value of x , or the greater root of the equation $Px^m - x^{m+1} = Q$, to be taken for a first near value of it, or made the ground-work, or basis, of a further approach to it's true value by the differential, or by Mr. Raphson's, or any other method of approximation: and, if we do not think it near enough to the truth to be so employed, we may increase it, or diminish it, by some small quantity, (such as a 100th, or a 200th, part of itself,) as may be thought necessary, in order to make it approach nearer to the true value of the said greater root. And then it will be expedient to make use of such nearer value of x , or the said greater root, as a first near value of it, or as a ground-work of a further approach to the true value of the said greater root by the differential method of approximation. And, if the second near value thereby obtained is still judged to be not sufficiently exact, it will be proper to investigate a third near value of it by a process of Mr. Raphson's method of approximation. And thus we may obtain a very exact value of the greater root of the equation $Px^m - x^{m+1} = Q$ without a previous knowledge of the lesser root of the said equation.

Art. 43. But, if we already know the value of the lesser root of the equation $Px^m - x^{m+1} = Q$, we may thereby obtain a pretty good approximation towards the value of the greater root of the said equation by proceeding as follows.

Let $\frac{m}{m+1} \times P$, (or the value of x at the instant of time at which the binomial quantity $Px^m - x^{m+1}$, during the increase of x from 0 to P , attains it's greatest possible magnitude,) be denoted by the letter M ; and let d be the difference by which M exceeds the lesser root of the equation $Px^m - x^{m+1} = Q$, which is supposed to be known. Then it will be found that, if we add half the quantity d to the said middle quantity M , the sum $M + \frac{d}{2}$ will often be pretty nearly equal to the greater root of the said equation. It will therefore be adviseable to suppose this greater root to be equal to $M + \frac{d}{2}$, and to substitute $M + \frac{d}{2}$ instead of x in the terms of the binomial quantity

$Px^m - x^{m+1}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or less, than Q , or the absolute term of the equation $Px^m - x^{m+1} = Q$, and consequently whether the said quantity $M + \frac{d}{2}$ will be less, or greater, than the true value of x , or the greater root of the said equation. And, if it shall appear that the result of the said substitution is pretty nearly equal to Q , or the absolute term of the said equation, (as, for example, if it differs from Q by less than a 20th part of Q .) we may conclude that the quantity $M + \frac{d}{2}$ is a pretty good first near value of x , or the greater root of the said equation, and is fit to be employed as such, or to be made the ground-work of a further approach to the true value of the said greater root either by the differential method, or Mr. Raphson's method, or some other method, of approximation.

I have found, by the examples of equations of this form which I have undertaken to resolve, that $M + \frac{d}{2}$ is much fitter than $M + d$ to be adopted in this conjectural manner for a first near value of x , or the greater root of the equation.

Art. 44. When we have thus made the substitutions of both the quantities $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ instead of x in the binomial quantity $Px^m - x^{m+1}$, and found each of the values of the said binomial quantity resulting from these substitutions to be pretty nearly equal to the absolute term Q , we may afterwards either make use of the quantity which produces the result which comes nearest to Q , (whether the said quantity be $\frac{2m+1}{2m+2} \times P$ or $M + \frac{d}{2}$), as a first near value of x , or the greater root of the said equation, or as a ground-work for a further approach to the true value of the said greater root, without taking any further notice of the other quantity, or we may make use of both the said quantities $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ at once in the investigation of a second near value of x , or the greater root of the said equation $Px^m - x^{m+1} = Q$, by the differential method of approximation. For, if b be put for $\frac{2m+1}{2m+2} \times P$, and c be put for $M + \frac{d}{2}$, and B be put $= Pb^m - b^{m+1}$, and C be put $= Pc^m - c^{m+1}$, we may make the following proportion, according to the directions of the differential method of approximation; to wit, As the difference of

of the results B and C is to the difference of B and Q, so, very nearly, will the difference of b and c (which correspond to B and C,) be to the difference of b and the true value of x , or the greater root of the equation $Px^m - x^{m+1} = Q$; and by this proportion we shall obtain a second near value of x , or the said greater root, which will be much nearer to it's true value than either b or c , or $\frac{2m+1}{2m+2} \times P$ or $M + \frac{d}{2}$, was. And, by thus making use of both b and c for this purpose, we may save ourselves the trouble of making a third substitution of a quantity nearly equal to b or c , (such as $b + \frac{b}{200}$, or $b - \frac{b}{200}$, or $c + \frac{c}{200}$, or $c - \frac{c}{200}$,) instead of x in the binomial quantity $Px^m - x^{m+1}$, which would be a necessary operation if we made use of only one of the two quantities b and c , or $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, in the application of the differential method of approximation.

If it should be thought expedient to make use of only one of these two quantities b and c , or $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, as a first near value of x , or the greater root of the equation $Px^m - x^{m+1} = Q$, or as a ground-work for a further approach to the true value of the said greater root, without taking the pains to substitute both these quantities successively instead of x in the binomial quantity $Px^m - x^{m+1}$, and comparing the results of such substitutions, I believe it will be found that the former quantity b , or $\frac{2m+1}{2m+2} \times P$, will, for the most part, deserve to be employed in this manner in preference to the latter quantity c , or $M + \frac{d}{2}$. But I should rather advise the employment of both these quantities b and c , or $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, together with B and C, the results of their substitutions in the binomial quantity $Px^m - x^{m+1}$, as the grounds of a further approach to the true value of the greater root sought by the differential method of approximation, in the manner that has been just now described.

The Application of the Conclusions obtained in the five preceding Articles to the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which is a particular Case of the more general Equation $Px^m - x^{m+1} = Q$.

Art. 45. I now return to the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, of which we know the lesser root to be $= 1$, and will apply the conclusions obtained in the five last articles concerning the roots of the more general binomial equation $Px^m - x^{m+1} = Q$, to the roots of the said equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, of which we wish to find the greater root.

Now in this equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ the letter r answers to the letter x in the foregoing equation $Px^m - x^{m+1} = Q$, and the index t of the unknown quantity r answers to the index m of the unknown quantity x in the other equation, and $t + 1$ answers to $m + 1$, and $\frac{z+a}{z}$ (the co-efficient of r^t) answers to P (the co-efficient of x^m), and the absolute term $\frac{a}{z}$ answers to Q , the absolute term of the other equation.

Therefore M , or $\frac{m}{m+1} \times P$, will be $= \frac{t}{t+1} \times \frac{z+a}{z}$; and consequently, since the greater root of the equation $Px^m - x^{m+1} = Q$ was shewn to be greater than M , or $\frac{m}{m+1} \times P$, but less than P , we may conclude that the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will be greater than $\frac{t}{t+1} \times \frac{z+a}{z}$, but less than $\frac{z+a}{z}$, or than $\frac{t+1}{t+1} \times \frac{z+a}{z}$.

Further, $\frac{2m+1}{2m+2} \times P$ will be $= \frac{2t+1}{2t+2} \times \frac{z+a}{z}$. And therefore, since $\frac{2m+1}{2m+2} \times P$ has been shewn to be a near value of the greater root of the former equation $Px^m - x^{m+1} = Q$, it follows that $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ will be a near value of the greater root of the present equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Further,

Further, since we know that the lesser root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ is $= 1$, it follows that the excess of M , (or $\frac{m}{m+1} \times P$), or $\frac{t}{t+1} \times \frac{z+a}{z}$, above the said lesser root will be $= \frac{t}{t+1} \times \frac{z+a}{z} - 1$; that is, d will be $= \frac{t}{t+1} \times \frac{z+a}{z} - 1$. Therefore $\frac{d}{2}$ will be $= \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, and $M + \frac{d}{2}$ will be $= \frac{t}{t+1} \times \frac{z+a}{z} + \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ ($= \frac{2t}{2t+2} \times \frac{z+a}{z} + \frac{t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$) $= \frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$. And therefore, since $M + \frac{d}{2}$ is a near value of the greater root of the former equation $Px^m - x^{m+1} = Q$, we may conclude that $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ will be a near value of the greater root of the present equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$.

Art. 46. Having thus obtained the two quantities $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$ for tolerably near values of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, let the former of these quantities, to wit, $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$, be put $= b$, and the latter of them, to wit, $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, be put $= c$; and let b be substituted instead of r in the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, and let the value of the said binomial quantity resulting from such substitution be called B ; and let c be, in like manner, substituted instead of r in the said binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$, and let the value of the said binomial quantity resulting from such substitution be called C .

Then, if the difference of the result B and $\frac{a}{z}$ (the absolute term of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$), is less than the difference of the result C and $\frac{a}{z}$, we may conclude that b will be nearer to the true value of r , or the greater root sought, than c is; and, if the difference of B and $\frac{a}{z}$ is greater than

than the difference of C and $\frac{a}{z}$, we may conclude that c will be nearer to the true value of r , or the said greater root, than b is. And thus we shall be able to determine with certainty, by means of these two substitutions, which of the two quantities b and c , or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, approaches nearest to the true value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which we are seeking.

Art. 47. Having thus discovered which of the two quantities b and c , or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, approaches nearest to the true value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, we must make choice of it in preference to the other as a first near value of the said greater root, or as a ground-work of a further approach to the true value of the said greater root, either by the differential, or Mr. Raphson's, or some other, method of approximation. And, if we chuse to make use of the differential method of approximation on this occasion, it will, for the most part, be convenient to make use of the other, or less exact, of the said two quantities b and c , or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, as well as of the more exact of them, in order to save ourselves the trouble of making a third substitution of a new quantity a little greater, or less, than the more exact of the said two quantities, (as, for example, of the quantity $b + \frac{b}{200}$ or $b - \frac{b}{200}$, or the quantity $c + \frac{c}{200}$ or $c - \frac{c}{200}$,) instead of r in the binomial quantity $\frac{z+a}{z} \times r^t - r^{t+1}$. And, if we thus make use of both the quantities b and c , and of the results B and C that correspond to them, we must proceed to make the following proportion; to wit, As the difference of the results B and C is to the difference of the result B and the absolute term $\frac{a}{z}$, so, very nearly, will the difference of the two quantities b and c be to the difference of the quantity b and the true value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, which we are seeking. And, by means of this proportion, we shall obtain a second near value of such greater root, that will be much nearer to its true

true value than either of the two former quantities b and c , or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$. Q. E. I.

And if this second near value of r , or the greater root of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, is not thought to be sufficiently exact, we must have recourse to Mr. Raphson's method of approximation to find a third near value of it from the second near value of it. And the said third near value, that will be thereby obtained, will be as near the truth as ever need be desired.

Q. E. I.

Art. 48. I will now proceed to apply this method of resolving the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ to the resolution of the numeral equation 1.090,909,0 $\times r^{21} - r^{22} = 0.090,909,0$, which comes under the general form $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and which Dr. Halley has resolved by means of the expression $1 + b - (bb - 2by)^{\frac{1}{2}}$, which he has given us for that purpose.

This equation results from the following Problem.

A P R O B L E M.

An annuity of 20*l.* *per annum*, which is to continue for 21 years, is sold for the sum of 220*l.* It is required to find the rate of interest allowed to the purchaser.

S O L U T I O N.

Here a is = 20*l.*, and t is = 21 years, and z is = 220*l.*

Therefore $z + a$ will be (= 220*l.* + 20*l.*) = 240*l.*, and $\frac{z+a}{z}$ will be ($= \frac{240}{220}$) = 1.090,909,0 &c, and $\frac{a}{z}$ will be ($= \frac{20}{220}$) = 0.090,909,0 &c; and consequently the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will, in this case, become 1.090,909,0 $\times r^{21} - r^{22} = 0.090,909,0$, &c; which numeral equation

equation must therefore be resolved, in order to find the value of r , or the greater root of the said equation, which denotes the rate of interest sought.

Now the resolution of this numeral equation in the manner just now recommended, (to wit, by, first, finding the two quantities $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, or b and c , and substituting them instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, and calling the results of the said substitutions B and C; and then deriving from b , c , B, C, and $\frac{a}{z}$, by means of the differential method of approximation, a second near value of r , or the greater root of the said equation, that shall be nearer to its true value than either b or c was; and, lastly, by finding a third near value of the said greater root by Mr. Raphson's method of approximation,) may be performed as follows.

The Resolution of the Equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, &c in the Manner explained and recommended in the foregoing Articles.

Art. 49. In this equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ &c, the index t is $= 21$, and the absolute term $\frac{a}{z}$ is $= 0.090,909,0$ &c, and the co-efficient $\frac{z+a}{z}$ is $= 1.090,909,0$ &c. Therefore $\frac{2t+1}{2t+2}$ will be $(= \frac{2 \times 21 + 1}{2 \times 21 + 2} = \frac{42 + 1}{42 + 2}) = \frac{43}{44}$, and consequently $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ will be $(= \frac{43}{44} \times \frac{z+a}{z} = \frac{43}{44} \times 1.090,909,0 = \frac{43 \times 1.090,909,0}{44} = \frac{46.909,087,0}{44} = 1.066,115,6$. Therefore $1.066,115,6$ will be a near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$.

And $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, will be $(= \frac{3 \times 21}{2 \times 21 + 2} \times 1.090,909,0 - \frac{1}{2} = \frac{63}{44} \times 1.090,909,0 - \frac{1}{2} = \frac{67 \times 1.090,909,0}{44} - \frac{1}{2} = \frac{68.727,267,0}{44} - \frac{1}{2} = 1.561,983,3 - \frac{1}{2} = 1.561,983,3 - 0.500,000,0) = 1.061,983,3$. Therefore $1.061,983,3$ will also be a near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$.

The

The former of these two near values of r , to wit, 1.066,115,6, must now be substituted instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$.

Now, if r is = 1.066,115,6, the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ will be = $1.090,909,0 \times \overline{1.066,115,6}^{21} - \overline{1.066,115,6}^{22}$. We must therefore find the values of $\overline{1.066,115,6}^{21}$ and $\overline{1.066,115,6}^{22}$.

Now the logarithm of 1.066,115,6 is = 0.027,804,2. Therefore the logarithm of $\overline{1.066,115,6}^{21}$ will be (= $21 \times 0.027,804,2$) = 0.583,888,2; which is the logarithm of the number 3.836,084,9. Therefore $\overline{1.066,115,6}^{21}$ will be = 3.836,084,9.

Therefore $\overline{1.066,115,6}^{22}$ will be (= $\overline{1.066,115,6}^{21} \times 1.066,115,6$) = $3.836,084,9 \times 1.066,115,6$ = 4.089,709,954,814,44; and $1.090,909,0 \times \overline{1.066,115,6}^{21}$ will be (= $1.090,909,0 \times 3.836,084,9$) = 4.184,819,542,174,10; and consequently the binomial quantity $1.090,909,0 \times \overline{1.066,115,6}^{21} - \overline{1.066,115,6}^{22}$ will be (= $4.184,819,542,174,10 - 4.089,709,954,814,44$) = 0.095,109,587,359,66.

This result is a little greater than 0.090,909,0 &c, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$. And consequently the number 1.066,115,6 (from the substitution of which instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ it arises,) must be somewhat less than the true value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, &c. But the difference is not great; and therefore the number 1.066,115,6 (obtained by means of the expression $\frac{2m+1}{2m+2} \times P$, or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$), may be considered as a pretty good first near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ &c.

The Investigation of a second near Value of r , or the greater Root of the Equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, by means of the Differential Method of Approximation.

Art. 50. Since the number 1.066,115,6 appears to be less than the true value of r , or the greater root sought, and since the other near value of r , to wit, the number 1.061,983,3 (which was obtained by means of the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$), is considerably less than 1.066,115,6, it follows that the said other near value of r will be considerably more distant

from it's true value than 1.066,115,6 is; and therefore it will not be so fit to be substituted instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, in order to lay a foundation for a process of the differential method of approximation towards the true value of r , as some quantity a little greater than 1.066,115,6, and which therefore (unless it be so great that it's excess above 1.066,115,6 shall be greater than the excess of 1.066,115,6 above 1.061,983,3,) must be nearer than 1.061,983,3 to the true value of r . Now the excess of 1.066,115,6 above 1.061,983,3 is = 0.004,132,3. Therefore, if we add to 1.066,115,6 any number not greater than 0.004,132,3, the number thence arising must be less distant than 1.061,983,3 from the true value of r . We will therefore add to 1.066,115,6 one third part of the number 0.004,132,3, that is, the number 0.001,377,4; and the sum thence arising, to wit, 1.067,493,0, will necessarily be much nearer than 1.061,983,3 to the true value of r . We will, therefore, substitute this number, 1.067,493,0, instead of r , in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, in order to lay a foundation for a process of the differential method of approximation towards the said true value.

Now, if r is supposed to be = 1.067,493,0, the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ will be = $1.090,909,0 \times \overline{1.067,493,0}^{21} - \overline{1.067,493,0}^{22}$. We must therefore find the values of $\overline{1.067,493,0}^{21}$ and $\overline{1.067,493,0}^{22}$; which may be done as follows.

The logarithm of 1.067,493,0 is = 0.028,365,0. Therefore the logarithm of $\overline{1.067,493,0}^{21}$ will be (= $21 \times 0.028,365,0$) = 0.595,665,0; which is the logarithm of the number 3.941,531,8. Therefore $\overline{1.067,493,0}^{21}$ is = 3.941,531,8. Therefore $\overline{1.067,493,0}^{22}$ will be (= $\overline{1.067,493,0}^{21} \times 1.067,493,0$) = $3.941,531,8 \times 1.067,493,0$ = 4.207,557,605,777,40; and $1.090,909,0 \times \overline{1.067,493,0}^{21}$ will be (= $1.090,909,0 \times 3.941,531,8$) = 4.299,852,514,406,20; and consequently the binomial quantity $1.090,909,0 \times \overline{1.067,493,0}^{21} - \overline{1.067,493,0}^{22}$ will be (= $4.299,852,514,406,20 - 4.207,557,605,777,40$) = 0.092,294,908,628,80.

This result is somewhat greater than 0.090,909,0, &c, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ &c. And therefore the number 1.067,493,0 (by the substitution of which instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ that result has been obtained,) will be less than the greater root of that equation.

We have now three different values of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, that are contiguous to each other, or differ but little from each other, to wit, 1st, the true value of the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, and, 2ndly, the number 1.067,493,0,

493,0, which is less than the said greater root, and, 3dly, the number 1.066,115,6, which is less than 1.067,493,0; and we have likewise the three values of the said binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ that correspond to the said three values of r , or result from the substitution of the said values of r in it's terms; to wit, 1st, the number 0.090,909,0, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, which corresponds to the greater root of the said equation; and, 2ndly, the number 0.092,294,9, &c, which corresponds to 1.067,493,0, or the middle value of r ; and, 3dly, the number 0.095,109,5, &c, which corresponds to 1.066,115,6, or the third and least value of r . We may therefore (according to the directions of the differential method of approximation,) make the following proportion; to wit, As 0.095,109,5 — 0.092,294,9, (or the difference of the second and third values of the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$,) is to 0.092,294,9 — 0.090,909,0 (or the difference of the first and second of these values,) so, very nearly, will 1.067,493,0 — 1.066,115,6 (or the difference of the second and third values of r ,) be to $r - 1.067,493,0$, or the difference of the first and second of those values; that is, As 0.002,814,6 is to 0.001,385,9, so, very nearly, will 0.001,377,4 be to $r - 1.067,493,0$. And hence it follows that $r - 1.067,493,0$ will be, very nearly, $(= \frac{0.001,385,9 \times 0.001,377,4}{0.002,814,6} = \frac{0.000,001,908,938,66}{0.002,814,6}) = 0.000,678,2$. Therefore r will be, very nearly, $(= 0.000,678,2 + 1.067,493,0) = 1.068,171,2$; that is, the second near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, obtained by this process of the differential method of approximation, will be 1.068,171,2.

Q. E. I.

This value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, is a very exact one, considering that it is the result of only one process of the differential method of approximation. For it is true in the first five figures 1.0681; the true value of r in this equation being 1.068,14, as is declared by Dr. Halley, and has been shewn above in this present Volume in the Third Note upon Dr. Halley's Discourse on Compound Interest, in page 312.

Art. 51. This number 1.068,171,2, thus obtained for a second near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ by the differential method of approximation, is a little greater than the true value of the said greater root. For, if it be substituted instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, the value of the said quantity resulting from such substitution will be somewhat less than 0.090,909,0, or the absolute term of the said equation. This substitution may be made as follows.

If r be supposed to be $= 1.068,171,2$, the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$ will be $= 1.090,909,0 \times \overline{1.068,171,2}^{21} - \overline{1.068,171,2}^{22}$. We must therefore

therefore find the values of $\overline{1.068,171,2}^{21}$ and $\overline{1.068,171,2}^{22}$; which may be done as follows.

The logarithm of 1.068,171,2 is = 0.028,640,8. Therefore the logarithm of $\overline{1.068,171,2}^{21}$ will be ($= 21 \times 0.028,640,8$) = 0.601,456,8; which is the logarithm of the number 3.994,448,4. Therefore $\overline{1.068,171,2}^{21}$ will be = 3.994,448,6; and consequently $\overline{1.068,171,2}^{22}$ will be ($= \overline{1.068,171,2}^{21} \times 1.068,171,2$) = 3.994,448,6 $\times$ 1.068,171,2 = 4.266,754,954,400,32, and $1.090,909,0 \times \overline{1.068,171,2}^{21}$ will be ($= 1.090,909,0 \times 3.994,448,6$) = 4.357,579,927,777,40. Therefore the binomial quantity $1.090,909,0 \times \overline{1.068,171,2}^{21} - \overline{1.068,171,2}^{22}$ will be ($= 4.357,579,927,777,40 - 4.266,754,954,400,32$) = 0.090,824,973,377,08; which is less than 0.090,909,0, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$; and consequently the number 1.068,171,2 must be greater than the greater root of that equation.

Q. E. D.

The Investigation of a third near Value of r, or the greater Root of the Equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, by a Process of Mr. Raphson's Method of Approximation.

Art. 52. I will now make use of the said number 1.068,171,2, or the second near value of r above obtained, to find a still more exact value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ by a process of Mr. Raphson's method of approximation.

Let the excess of 1.068,171,2 above the true value of r , or the greater root of the said equation, be called w .

Then will r be = $1.068,171,2 - w$; and consequently r^{21} will be ($= \overline{1.068,171,2 - w}^{21}$) =, by the binomial theorem, $\overline{1.068,171,2}^{21} - 21 \times \overline{1.068,171,2}^{20} \times w + \&c = \overline{1.068,171,2}^{21} - 21 \times \frac{\overline{1.068,171,2}^{21}}{1.068,171,2} \times w + \&c = 3.994,448,6 - 21 \times \frac{3.994,448,6}{1.068,171,2} \times w + \&c = 3.994,448,6 - 21 \times 3.739,520,9 \times w + \&c = 3.994,448,6 - 78.529,938,9 \times w + \&c$; and $1.090,909,0 \times r^{21}$ will be ($= 1.090,909,0 \times 3.994,448,6 - 1.090,909,0 \times 78.529,938,9 \times w + \&c$) = 4.357,579,927,777,40 - 85.669,017,115,460,10 $\times w + \&c$.

And r^{22} will be ($= \overline{1.068,171,2 - w}^{22}$) =, by the binomial theorem, to $\overline{1.068,171,2}^{22} - 22 \times \overline{1.068,171,2}^{21} \times w + \&c = 4.266,754,954,400,32$

$$- 22 \times 3.994,448,6 \times w + \&c) = 4.266,754,954,400,32 - 87.877,869,2 \times w + \&c.$$

Therefore the binomial quantity $1.090,909,0 \times r^{22} - r^{22}$ will be equal to the compound quantity

$$\begin{aligned} & \left\{ \begin{aligned} & 4.357,579,927,777,40 - 85.669,017,115,460,10 \times w + \&c. \\ & - 4.266,754,954,400,32 + 87.877,869,2 \times w - \&c \end{aligned} \right. \\ & = 0.090,824,973,377,08 + 2.208,852,084,539,90 \times w - \&c. \end{aligned}$$

But the binomial quantity $1.090,909,0 \times r^{22} - r^{22}$ is $= 0.090,909,0$.

Therefore the compound quantity $0.090,824,9 \&c + 2.208,852,0 \&c \times w$ will also be $= 0.090,909,0$; and consequently $2.208,852,0 \times w$ will be $= 0.090,909,0 - 0.090,824,9 = 0.000,084,1$, and w will be $= \frac{0.000,084,100,000,000}{2.208,852,0} = 0.000,038,0$. Therefore r , or $1.068,171,2 - w$, will be $(= 1.068,171,2 - 0.000,038,0) = 1.068,133,2$; that is, the third near value of r , or the greater root of the equation $1.090,909,0 \times r^{22} - r^{22} = 0.090,909,0$ obtained by this process of Mr. Raphson's method of approximation, will be $1.068,133,2$. Q. E. I.

The difference of this number, $1.068,133,2$, from $1.068,14$, or $1.068,140,0$, (which is the true value of the greater root of the equation $1.090,909,0 \times r^{22} - r^{22} = 0.090,909,0$), is only $0.000,006,8$, or less than the 158,564th part of $1.068,140,0$, or $1.068,14$, the true value of the said greater root. This is as great a degree of exactness as need ever be desired.

Art. 53. The foregoing method of resolving the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ may be employed with success when t , or the number of years during which the annuity is to continue, is a great number, as well as when it is a small number; whereas Dr. Halley's expression $1 + b - \overline{bb - 2by}^{\frac{1}{2}}$ is recommended by him as useful only when t is not greater than 39 years. And, when t is greater than 39 years, Dr. Halley has given us other methods, (and very good ones, founded upon clear and simple principles,) of resolving the said equation. But by the foregoing method of resolving it, the greater the number t is, the nearer will it's first near value $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ (which is an arithmetical mean quantity between $\frac{t}{t+1} \times \frac{z+a}{z}$, and $\frac{z+a}{z}$, or $\frac{t+1}{t+1} \times \frac{z+a}{z}$; the two limits of the greater root of the equation,) approach to the true value of r , or the said greater root: So that this method of resolution may justly be considered

sidered as equally well adapted to all the cases of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$. But that this may appear the more clearly, I will now proceed to give another example of this method of resolving an equation of this form, in which t , or the number of years during which the annuity is to continue, shall be 40 years.

Another Example of the Resolution of the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ by the Method described in the foregoing Articles, when t , or the Number of Years during which the Annuity is to continue, is 40 Years.

Art. 54. It has been shewn above, in the Third of the foregoing Notes upon Dr. Halley's Discourse on Compound Interest, art. 12, page 316, that, if r , or the rate of interest of money, be supposed to be $= 1.06$, and an annuity of 20*l.* a year for a term of 40 years be sold to a purchaser for its true value, corresponding to the said rate of interest 1.06, the said value, or fair price of the said annuity, will be $= 300.926,042,2*l.*$, or 300*l.* 18*s.* 6*d.* And hence it follows, & *converso*, that, if the sum of 300.926,042,2*l.*, or 300*l.* 18*s.* 6*d.* has been paid for an annuity of 20*l.* a year, that is to continue for 40 years, the rate of interest that has been allowed to the purchaser of such annuity must have been 1.06, or 6 per cent.

Art. 55. Now let it be supposed that such an annuity has been sold for the said price of 300.926,042,2*l.*, or 300*l.* 18*s.* 6*d.*, but that it is not known what rate of interest was made the ground of the said valuation of it, or intended to be allowed to the purchaser on that occasion: and let it be required to determine, what that rate of interest must have been, by reasonings founded on our knowledge of the other circumstances belonging to it, to wit, the quantity of the annuity, the time during which it is to continue, and the price that has been paid for it.

If a be put for the quantity of the annuity, or the number of pounds to be annually paid to the annuitant, and t be put for the time of its duration, or the number of years during which it is to continue, and z for the price that has been paid for the said annuity, or the number of pounds and parts of a pound contained in the said price of it, and r for the rate of interest allowed the purchaser in the purchase of it, (which rate of interest is supposed to be unknown, and is required to be found from the knowledge of the other three quantities,) the solution of the said Problem may be reduced to the resolution of the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, as has been shewn above in art. 36; and the

greater

greater value of r in this equation, or it's greater root, will be the rate of interest sought.

Art. 56. Now, since a is, in this case, $= 20\%$, and z is $= 300.926,042,2\%$, and t is $= 40$ years, we shall have $\frac{a}{z} = \frac{20\%}{300.926,042,2\%} = 0.066,461,5$, and $\frac{z+a}{z} (= \frac{z}{z} + \frac{a}{z} = 1 + \frac{a}{z} = 1 + 0.066,461,5) = 1.066,461,5$, and $r^t = r^{40}$, and $r^{t+1} = r^{41}$; and consequently the general equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ will, in this case, become $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and the greater value of r in this equation, or the greater root of this equation, will be the rate of interest sought. We must therefore now endeavour to resolve the numeral equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$ by the method described in the foregoing articles.

The Investigation of a first near Value of r , or the greater Root of the Equation
 $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$.

Art. 57. Now, since t is $= 40$, and $\frac{z+a}{z}$ is $= 1.066,461,5$, we shall have $\frac{2t+1}{2t+2} \times \frac{z+a}{z} (= \frac{2 \times 40 + 1}{2 \times 40 + 2} \times 1.066,461,5 = \frac{80+1}{80+2} \times 1.066,461,5 = \frac{81}{82} \times 1.066,461,5 = \frac{81 \times 1.066,461,5}{82} = \frac{86,383,381,5}{82}) = 1.053,455,8$. Therefore, (by art. 42 and 45,) the number $1.053,455,8$ will be a first near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$.

Further, $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, will be $(= \frac{3 \times 40}{2 \times 40 + 2} \times 1.066,461,5 - \frac{1}{2} = \frac{120}{80+2} \times 1.066,461,5 - \frac{1}{2} = \frac{120}{82} \times 1.066,461,5 - \frac{1}{2} = \frac{60}{41} \times 1.066,461,5 - \frac{1}{2} = \frac{60 \times 1.066,461,5}{41} - \frac{1}{2} = \frac{63,987,690,0}{41} - \frac{1}{2} = 1.560,675,3 - \frac{1}{2} = 1.560,675,3 - 0.500,000,0) = 1.060,675,3$.

Therefore (by art. 43,) the number $1.060,675,3$ will also be another near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$.

Art. 58. These two near values of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, have been obtained with very little trouble; and the latter of them, to wit, the number 1.060,675,3 (which was derived from the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$), is exceedingly near the true value of the said greater root, which we know to be 1.06. But this will not always be the case; and I am inclined to think that, in most equations of this form $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, the former near value of r , that is derived from the expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$, will be nearer the truth than the latter near value of it that is derived from the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$.

Art. 59. The number 1.060,675,3 is so little greater than the true value of r , or the greater root sought, which is $= 1.06$, that the equation may be considered as already sufficiently resolved; or, if a greater degree of exactness should be required, we ought to adopt this latter number 1.060,675,3 in preference to the former number 1.053,455,8, as the ground-work of a further approach to it's true value by the differential method, or by Mr. Raphson's method, or by some other method, of approximation. But this near approach of the number 1.060,675,3 to the true value of r would not be yet apparent, if it were not for our previous knowledge that 1.06 is the true value of r , or the said greater root. And I think it will be best on the present occasion not to make use of this previous knowledge, but to proceed in the resolution of this equation in the manner which would be proper to be adopted if we were ignorant what the said true value was, any further than it had been discovered by the computation of the two expressions $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$ and $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, by which we obtained the two numbers 1.053,455,8 and 1.060,675,3. And in this case the next step to be taken would have been to substitute the number 1.053,455,8 (obtained by means of the expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$), instead of z in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$. This may be done in the manner following.

Art. 60. If r is supposed to be $= 1.053,455,8$, the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ will be $= 1.066,461,5 \times 1.053,455,8^{40} - 1.053,455,8^{41}$. We must therefore find the values of $1.053,455,8^{40}$ and $1.053,455,8^{41}$; which may be done as follows.

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The logarithm of 1.053,455,8 is = 0.022,616,2. Therefore the logarithm of $\overline{1.053,455,8}^{40}$ will be = $40 \times 0.022,616,2 = 0.904,648,0$; which is the logarithm of the number 8.028,751,8. Therefore $\overline{1.053,455,8}^{40}$ will be = 8.028,751,8. Therefore $\overline{1.053,455,8}^{41}$ will be $(= \overline{1.053,455,8}^{40} \times 1.053,455,8 = 8.028,751,8 \times 1.053,455,8) = 8.456,934,150,470,44$; and $1.066,461,5 \times \overline{1.053,455,8}^{40}$ will be $(= 1.066,461,5 \times 8.028,751,8) = 8.562,354,687,755,70$. Therefore the binomial quantity $1.066,461,5 \times \overline{1.053,455,8}^{40} - \overline{1.053,455,8}^{41}$ will be $(= 8.562,354,687,755,70 - 8.456,934,150,470,44) = 0.105,420,537,285,26$.

This number is greater than 0.066,461,5, or the absolute term of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and consequently the number 1.053,455,8 (from the substitution of which instead of r in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ it arises,) must be less than the true value of r , or the greater root of that equation.

We will now proceed to substitute the other near value of r , (obtained by means of the expression $\frac{3t}{2t+2} \times \frac{z+a}{a} - \frac{1}{2}$), to wit, the number 1.060,675,3, instead of r in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$; which may be done as follows.

If r is supposed to be = 1.060,675,3, the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ will be = $1.066,461,5 \times \overline{1.060,675,3}^{40} - \overline{1.060,675,3}^{41}$. We must therefore find the values of $\overline{1.060,675,3}^{40}$ and $\overline{1.060,675,3}^{41}$; which may be done as follows.

The logarithm of 1.060,675,3 is = 0.025,582,4. Therefore the logarithm of $\overline{1.060,675,3}^{40}$ will be $(= 40 \times 0.025,582,4) = 1.023,296,0$; which is the logarithm of the number 10.551,058,2. Therefore $\overline{1.060,675,3}^{40}$ will be = 10.551,058,2. Therefore $\overline{1.060,675,3}^{41}$ will be $(= \overline{1.060,675,3}^{40} \times 1.060,675,3 = 10.551,058,2 \times 1.060,675,3) = 11.191,246,821,602,46$, and $1.066,461,5 \times \overline{1.060,675,3}^{40}$ will be $(= 1.066,461,5 \times 10.551,058,2) = 11.252,297,354,559,30$. Therefore the binomial quantity $1.066,461,5 \times \overline{1.060,675,3}^{40} - \overline{1.060,675,3}^{41}$ will be $(= 11.252,297,354,559,30 - 11.191,246,821,602,46) = 0.061,050,532,956,84$.

This number is less than 0.066,461,5, or the absolute term of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and consequently the number

1.060,675,3 (by the substitution of which instead of r in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ it arises,) must be greater than the true value of r , or the greater root of that equation.

Art. 61. We are now possessed of three different values of the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$, to wit, the numbers 0.105,420,5 &c, 0.066,461,5, and 0.061,050,5 &c, which correspond to three different values of r , that differ but little from each other, to wit, 1st, the number 1.053,455,8, and, 2ndly, the greater value of r in the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, and, 3dly, the number 1.060,675,3. We may therefore make the following proportion; As 0.105,420,5 &c $-$ 0.061,050,5 &c, (or the difference of the first and third of the first set of numbers,) is to 0.066,461,5 &c $-$ 0.061,050,5 &c, (the difference of the second and third of the first set of numbers,) so, very nearly, will 1.060,675,3 $-$ 1.053,455,8 (the difference of the first and third of the second set of numbers,) be to 1.060,675,3 $-$ r , or the difference of the second and third of the second set of numbers; that is, as 0.044,370,0 is to 0.005,411,0, so, very nearly, will 0.007,219,5 be to 1.060,675,3 $-$ r ; whence we shall have $1.060,675,3 - r$, nearly, ($= \frac{0.005,411,0 \times 0.007,219,5}{0.044,370,0} = \frac{0.000,039,064,714,50}{0.044,370,0}$) $= 0.000,880,4$. Therefore 1.060,675,3 will be $= 0.000,880,4 + r$, and r will be ($= 1.060,675,3 - 0.000,880,4$) $= 1.059,794,9$; that is, 1.059,794,9 will be a second near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$. Q. E. I.

Art. 62. This number 1.059,794,9 will be somewhat less than the true value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; because, if it is substituted instead of r in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$, the value of the said binomial quantity resulting from such substitution will be greater than 0.066,461,5, or the absolute term of the said equation. This will appear by making the said substitution, which may be done as follows.

If r is supposed to be $= 1.059,794,9$, the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ will be $= 1.066,461,5 \times \overline{1.059,794,9}^{40} - \overline{1.059,794,9}^{41}$. We must therefore find the values of $\overline{1.059,794,9}^{40}$ and $\overline{1.059,794,9}^{41}$; which may be done as follows.

The logarithm of 1.059,794,9 is $= 0.025,221,8$. Therefore the logarithm of $\overline{1.059,794,9}^{40}$ will be ($= 40 \times 0.025,221,8$) $= 1.008,872,0$; which is the logarithm of the number 10.206,385,8. Therefore $\overline{1.059,794,9}^{40}$ is $= 10.206,385,8$. Therefore $\overline{1.059,794,9}^{41}$ will be ($= \overline{1.059,794,9}^{40} \times 1.059,794,9$) $= 10.206,385,8 \times 1.059,794,9 = 10.816,675,618,272,42$; and

and $1.066,461,5 \times 1.059,794,9^{40}$ will be $(= 1.066,461,5 \times 10.206,385,8)$
 $= 10.884,717,509,846,70$. Therefore the binomial quantity $1.066,461,5 \times$
 $1.059,794,9^{40} - 1.059,794,9^{41}$ will be $(= 10.884,717,509,846,70 -$
 $10.816,675,618,272,42) = 0.068,041,891,574,28$; which is greater than
 $0.066,461,5$, or the absolute term of the equation $1.066,461,5 \times r^{40} - r^{41} =$
 $0.066,461,5$. And consequently the number $1.059,794,9$ (from the substitu-
tion of which instead of r in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$
the said number $0.068,041,891,574,28$ arises,) must be less than the true value
of r , or the greater root of that equation. Q. E. D.

Art. 63. To find a still nearer value of r , let w be put for the excess of the
true value of r above the number $1.059,794,9$, and let the value of w be in-
vestigated by Mr. Raphson's method of approximation.

Then will r be $= 1.059,794,9 + w$. And consequently r^{40} will be $=$
 $1.059,794,9 + w^{40}$ ($=$, by the binomial theorem, to $1.059,794,9^{40} + 40$
 $\times 1.059,794,9^{39} \times w + \&c = 1.059,794,9^{40} + 40 \times \frac{1.059,794,9^{40}}{1.059,794,9} \times w$
 $+ \&c = 10.206,385,8 + 40 \times \frac{10.206,385,8}{1.059,794,9} \times w + \&c = 10.206,385,8$
 $+ 40 \times 9.630,529,2 \times w + \&c) = 10.206,385,8 + 385.221,168,0 \times w$
 $+ \&c$; and $1.066,461,5 \times r^{40}$ will be $(= 1.066,461,5 \times 10.206,385,8 +$
 $1.066,461,5 \times 385.221,168,0 \times w + \&c) = 10.884,717,509,846,70 +$
 $410.823,544,657,032,00 \times w + \&c$.

And r^{41} will be $= 1.059,794,9 + w^{41}$ ($=$, by the binomial theorem, to
 $1.059,794,9^{41} + 41 \times 1.059,794,9^{40} \times w + \&c = 10.816,675,618,272,42$
 $+ 41 \times 10.206,385,8 \times w + \&c) = 10.816,675,618,272,42 + 418.461,$
 $817,8 \times w + \&c$.

Therefore the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ will be $=$ to the
compound quantity

$$\left\{ \begin{array}{l} 10.884,717,509,846,70 + 410.823,544,657,032,00 \times w + \&c \\ - 10.816,675,618,272,42 - 418.461,817,8 \times w - \&c \end{array} \right\}$$

$$= 0.068,041,891,574,28 - 7.638,273,142,968,00 \times w - \&c.$$

But the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$ is $= 0.066,461,5$.

Therefore the compound quantity $0.068,041,8 \&c - 7.638,273,1 \&c \times w$
 $- \&c$ will also be $= 0.066,461,5$. And consequently $0.068,041,8$ will be
 $= 0.066,461,5 + 7.638,273,1 \&c \times w$, and $0.068,041,8 - 0.066,461,5$
will be $= 7.638,273,1 \&c \times w$, or $0.001,580,3$ will be $= 7.638,273,1 \&c \times w$,

3 K 2

and

and w will be $= \frac{0.001,580,3}{7.638,273,1} = 0.000,242,2$. Therefore r , or $1.059,794,9 + w$, will be $(= 1.059,794,9 + 0.000,242,2) = 1.060,037,1$; that is, the third near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, obtained by this process of Mr. Raphson's method of approximation, will be $1.060,037,1$. Q. E. I.

This third near value of r is very near it's true value, which is 1.06 . For the interest of a hundred pounds, reckoned according to this value of r , would be $(= 100 \times 0.060,037,1\%$, or $6.003,71\%$, or 6% and $0.003,71$ of a pound, or 6% and $240 \times 0.003,71$ of a penny, or 6% and $0.890,40$ of a penny, or) 6 pounds and less than one penny, instead of being exactly 6 pounds.

End of the Consideration of the second of the three Problems in Dr. Halley's Discourse on Compound Interest which have given occasion to the foregoing Notes,

$$\text{and of the Equation } \frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z},$$

which is deduced from it.

Art. 64. We come now to consider the third and last of the three Problems in Dr. Halley's Discourse on Compound Interest which have given occasion to the foregoing Notes, and the equation which is deduced from it, and on the resolution of which the Solution of the said Problem depends. This Problem relates to the finding of the rate of Interest of Money allowed in a bargain made for the prolongation of a given annuity that has been already granted for a given number of years, for an additional number of years, when the quantity of the annuity purchased, the number of years during which it is to continue by virtue of the first grant, the number of years during which it is to be prolonged beyond the first term by virtue of the new bargain, and the price that is to be paid for such prolongation of it by the purchaser in consequence of such new bargain, are all given or known; and it occurs in page 229 of the present Volume in these words.

“Lastly, by way of corollary to the former, let it be required to find the rate of interest allowed the purchaser, when he pays a sum z for an annuity a , wherein he has already a term t , to have it prolonged for a certain time $= x$.”

“Example. An annuity of 20% per annum that is already granted for a term of years of which 21 years are still to come, may, for the sum of 40 pounds paid down, be prolonged for 10 years more, or to 31 years. What is the rate of Interest required?”

S O;

SOLUTION.

“Put $T = 2t + x + 1$, and $\frac{1}{2}T$ shall be the index of $\frac{ax}{z}$. Let $\sqrt[\frac{ax}{z}]{T}$ be
 “ $= 1 + y$, and $\frac{6T+6}{xx}$ be $= b$. I say, $r - 1$ is very near to $b -$
 “ $\sqrt[bb - 2by]{\frac{1}{2}}$.”

These are the words of Dr. Halley. But the Problem may be stated more fully in the following manner.

A fuller Statement of the said Problem of Dr. Halley.

Art. 65. If an annuity of a certain number of pounds *per annum* denoted by the letter a , that is to continue for a certain number of years, of which t years are unexpired, has been already granted; and the annuitant is desirous of prolonging the annuity for an additional number of years denoted by the letter x , and agrees to pay to the grantor of the annuity a certain sum of money denoted by the letter z , (or containing z pounds and parts of a pound) for such prolongation of the annuity; and r be put for the rate of Interest of Money corresponding to the said price z that is to be paid for the said prolongation of the annuity for the additional term of x years; and the four quantities a , t , x , and z are all given, or known; but the last quantity r , or the said rate of Interest, is unknown: It is required, from the four known quantities a , t , x , and z , to determine the unknown quantity r , or the rate of Interest allowed to the annuitant in the purchase of the said prolongation of his annuity.

The Reduction of the said Problem to an Equation.

Art. 66. This Problem may be reduced to the following *trinomial* equation, (or equation involving three different powers of the unknown quantity r), to wit, $\frac{a}{z} \times r^x + r^{t+x} - r^{t+x+1} = \frac{a}{z}$, by the following train of reasoning.

The value of an annuity of 1*l.* *per annum* for t years is known to be $= \frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$. Therefore the value of an annuity of 1*l.* *per annum* for $t+x$ years will be $= \frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$. Therefore the excess of the value of an annuity of 1*l.* *per annum* for $t+x$ years above the value of the like annuity for only t years will be equal to the excess of the sum of $\frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$ above the sum of $\frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$, that is, to the sum of $\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$. But the excess of the value of an annuity of 1*l.* a year for $t+x$ years above the value of an annuity of 1*l.* a year for only t years is equal to the value of an annuity of 1*l.* a year to commence at the end of t years, or to be added to an annuity of 1*l.* a year already granted for a term of t years. Therefore the value of such an additional annuity of 1*l.* a year for x years, to be added to a like annuity of 1*l.* a year for t years already granted, will be $= \frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$. And consequently the value of an annuity of a pounds a year for x years, to be added to a like annuity of a pounds a year for t years already granted, will be a times the former value, or will be $= a \times \left(\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1} \right)$
 $= \frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$; that is, according to Dr. Halley's notation, z (or the price paid for the prolongation of the annuity of a pounds for the additional term of x years, beyond the term of t years for which it had been originally granted,) will be $= \frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$. Therefore (multiplying all the terms into $r-1$,) we shall have $z \times r-1 = \frac{a}{r^t} - \frac{a}{r^{t+x}}$, or $rz - z = \frac{a}{r^t} - \frac{a}{r^{t+x}}$, and (multiplying all the terms into r^t ,) we shall have $r^{t+1} \times z - r^t \times z = a - \frac{a \times r^t}{r^{t+x}} (= a - \frac{ar^t}{r^t \times r^x}) = a - \frac{a}{r^x}$, and (multiplying all the terms into r^x ,) we shall have $r^{x+t+1} \times z - r^{x+t} \times z = a \times r^x - a$. Therefore, if we add a to both sides, we shall have $r^{x+t+1} \times z - r^{x+t} \times z + a = a \times r^x$, and (adding $r^{x+t} \times z$ to both sides,) $r^{x+t+1} \times z + a =$

$= a \times r^x + r^{x+t} \times z$, and (subtracting $r^{x+t+1} \times z$ from both sides,) $a = a \times r^x + r^{x+t} \times z - r^{x+t+1} \times z$; and, lastly, dividing all the terms by z , we shall have $\frac{a}{z} = \frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, or (in the more usual way of ranging the terms of an equation, with the absolute term, or known quantity, on the right hand,) $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$. Q. E. D.

In order therefore to solve the foregoing Problem of Dr. Halley concerning the rate of interest allowed in a bargain for the prolongation of an annuity for a term of years, it will be necessary for us to resolve this trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$.

Art. 67. This equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ is of such a form as to admit of having two different roots, or, in the language of modern Algebraists, two different real and affirmative roots; there being one change of the signs $+$ and $-$ amongst the terms on the left-hand side of the equation, and the highest power of the unknown quantity r , to wit, r^{x+t+1} , having the sign $-$ prefixed to it, or being subtracted from the sum of the other two terms $\frac{a}{z} \times r^x$ and r^{x+t} . And of the two roots of this equation the lesser will be $= 1$. For, if we suppose r to be $= 1$, we shall have $r^x (= 1^x) = 1$, and $r^{x+t} (= 1^{x+t})$ also $= 1$, and $r^{x+t+1} (= 1^{x+t+1})$ also $= 1$, because all the powers of 1 are equal to 1 . And therefore $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$ will be $(= \frac{a}{z} \times 1 + 1 - 1 = \frac{a}{z} \times 1) = \frac{a}{z}$, which is the absolute term of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$. And consequently 1 is a root of the said equation. Q. E. D.

But, though 1 is a root of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, it is not the root that will solve Dr. Halley's aforesaid Problem; because in that Problem r is supposed to be 1 pound together with its interest for a year, and therefore must be greater than 1 pound, or 1 . It is therefore the other and greater root of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ that will solve Dr. Halley's Problem, and that we must consequently endeavour

endeavour to find in order to effect that solution. But in our investigation of that other and greater root of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, the knowledge that 1 is the lesser root of it may be made useful, by enabling us to find a quantity greater than 1, and greater likewise than the value of r when the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$ is of it's greatest possible magnitude, for a first near value of the greater root, that will approach pretty nearly to it's true value, and may therefore be employed with good success as the basis, or ground-work, of a further approach to the said true value by the differential method, or Mr. Raphson's method, or some other method, of approximation. But, that we may apply this knowledge of the lesser root of this equation to the discovery of a near value of it's greater root, it will be convenient, first, to make a few remarks on the number of roots in a trinomial equation of the general form $Px^m + x^{m+n} + x^{m+n+1} = Q$, and on the limits of their magnitudes; which may afterwards be transferred to the roots of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, which comes under that more general form.

Of the Number of Roots in the Trinomial Equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, and the Limits of their Magnitudes.

Art. 68. If P and Q in this equation are known quantities, and m and n are any whole numbers whatsoever, it is evident, in the first place, that, since x^{m+n+1} is always less than $Px^m + x^{m+n}$, the fraction $\frac{x^{m+n+1}}{x^m}$ must always be less than the fraction $\frac{Px^m + x^{m+n}}{x^m}$. But the fraction $\frac{x^{m+n+1}}{x^m}$ is $(= \frac{x^m \times x^{n+1}}{x^m}) = x^{n+1}$, and the fraction $\frac{Px^m + x^{m+n}}{x^m}$ is $(= \frac{Px^m}{x^m} + \frac{x^{m+n}}{x^m} = \frac{Px^m}{x^m} + \frac{x^m \times x^n}{x^m}) = P + x^n$. Therefore x^{n+1} must always be less than $P + x^n$; and consequently $x^{n+1} - x^n$ must always be less than P .

Secondly,

Secondly, if x be supposed to increase from being equal to 1 (at which time x^n is = 1, and x^{n+1} is also = 1, and consequently $x^{n+1} - x^n$ is = 1 - 1 = 0,) to any greater magnitude, the increment of x^{n+1} will be greater than the contemporary increment of x^n , and consequently the binomial quantity $x^{n+1} - x^n$, or the excess of x^{n+1} above x^n , will continually increase; and therefore the said binomial quantity $x^{n+1} - x^n$ will be greatest when x is greatest; and, *vice versa*, the quantity x will be greatest when the said binomial quantity $x^{n+1} - x^n$ is greatest. But the greatest possible magnitude of the binomial quantity $x^{n+1} - x^n$, when x is a root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, is P ; as has been just now shewn. Therefore the greatest possible magnitude of the quantity x in the said trinomial equation will be the magnitude which it has when the binomial quantity $x^{n+1} - x^n$ is equal to P , or will be the root of the binomial equation $x^{n+1} - x^n = P$.

Therefore, in order to find the greatest possible magnitude of the quantity x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, it will be necessary to resolve the binomial equation $x^{n+1} - x^n = P$, (which is an equation that admits of only one root,) or to find the value of it's root x . Let this root be called A .

Thirdly, let us suppose the quantity x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ to increase gradually from 0 till it becomes equal to it's greatest possible magnitude, which will be A , or the root of the binomial equation $x^{n+1} - x^n = P$; and let us inquire how the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will increase, or decrease, at the same time.

Now it is evident that, when x is less than 1, x^m will be less than 1, and than x , and x^{m+n} will be less than x^m , and x^{m+n+1} will be less than x^{m+n} ; and that, when x is = 1, x^m will be also = 1, and x^{m+n} will also be = 1, and x^{m+n+1} will also be = 1; and, lastly, that, when x is become greater than 1, x^m will be greater than 1 and than x , and x^{m+n} will be greater than x^m , and x^{m+n+1} will be greater than x^{m+n} .

Further, when x is $= 0$, x^m , and consequently Px^m , will be $= 0$ likewise; and so will x^{m+n} and x^{m+n+1} ; and consequently the whole trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will be $= 0$ likewise. And, when x is become equal to A , or it's greatest magnitude, or to the root of the binomial equation $x^{n+1} - x^n = P$, the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will be a second time equal to 0 . It follows therefore that the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ must, during the increase of x from 0 to it's greatest magnitude A , have first increased from 0 to some certain finite quantity, and then have decreased from the said finite quantity to 0 . And therefore, if Q , or the absolute term of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, is exactly equal to the said finite quantity to which the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ becomes equal at the instant it has attained it's greatest magnitude and begins to decrease, the said trinomial equation will have but one root; which will be the value of x at the said instant of the said trinomial quantity's attaining it's greatest magnitude, and which (being an intermediate, or middle, value of x between 0 , it's least magnitude, or it's lower limit, and A , it's greatest magnitude, or it's highest limit,) may be conveniently denoted by the letter M ; but, if the said absolute term Q is less than the said finite quantity, or greatest magnitude of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, the said trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ will have two roots, to wit, one that will be less than the said middle quantity M , and another that will be greater than the said middle quantity M , but less than A , or the root of the binomial equation $x^{n+1} - x^n = P$.

Art. 69. We will next endeavour to determine the value of M , or the said middle value of x , which makes the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ be of it's greatest possible magnitude.

Let us suppose A , or the greatest possible magnitude of x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1} = Q$, to be divided into some very great number of small and equal parts, as, for instance, into ten thousand million of such

such parts, and that each of these equal parts is denoted by $\dot{x}$, or the letter x with a point placed over it. And let us suppose x to increase gradually from 0 to A (which is it's greatest possible magnitude,) by the continual addition of one of these equal parts, denoted by $\dot{x}$, in so many successive small and equal portions of time, to it's former value.

Then, if x be put for the value of x at the beginning of any one of these small and equal portions of time during it's increase from 0 to A , it is evident that $x + \dot{x}$ will be it's value at the end of the said small portion of time. And in the same small portion of time the quantity x^m will increase from x^m to $\overline{x + \dot{x}}^m$, or (by the binomial theorem,) to $x^m + m \times x^{m-1} \times \dot{x} + \&c$; and the quantity x^{m+n} will increase from x^{m+n} to $\overline{x + \dot{x}}^{m+n}$, or (by the binomial theorem,) to $x^{m+n} + \overline{m+n} \times x^{m+n-1} \times \dot{x} + \&c$; and the quantity x^{m+n+1} will increase from x^{m+n+1} to $\overline{x + \dot{x}}^{m+n+1}$, or, (by the binomial theorem,) to $x^{m+n+1} + \overline{m+n+1} \times x^{m+n} \times \dot{x} + \&c$; and Px^m will increase from $P \times x^m$ (to $P \times \overline{x + \dot{x}}^m$, or to $P \times x^m + m \times x^{m-1} \times \dot{x} + \&c$, or) to $P \times x^m + mP \times x^{m-1} \times \dot{x} + \&c$. Therefore, while x receives the increment $\dot{x}$, the binomial quantity $Px^m + x^{m+n}$ will receive the increment $mP \times x^{m-1} \times \dot{x} + \&c + \overline{m+n} \times x^{m+n-1} \times \dot{x} + \&c$, or (neglecting the terms included under the two marks of $\&c$, on account of their extreme smallness,) the increment $mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$; and the single quantity x^{m+n+1} will receive the increment $\overline{m+n+1} \times x^{m+n} \times \dot{x}$.

Now, while x increases from 0 to it's greatest magnitude A , the quantity x^{m+n} will at first be very much less than x^{m+n-1} , or $\frac{x^{m+n}}{x}$, and will continue to be less than x^{m+n-1} so long as x is less than 1; and then it will be equal to x^{m+n-1} ; and, when x is greater than 1, x^{m+n} will be greater than x^{m+n-1} . And hence it follows that, while x is increasing from 0 to 1, and for some time after, the quantity $\overline{m+n+1} \times x^{m+n} \times \dot{x}$ (which is the increment of x^{m+n+1} , or the third term of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$), will be less than $mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$, or the contemporary

increment of the binomial quantity $Px^m + x^{m+n}$, or the sum of the first and second terms of the said trinomial quantity; and then, at a certain instant of time during the increase of x from 0 to A , and, when x is arrived at a certain magnitude greater than 1, the increment $\overline{m+n+1} \times x^{m+n} \times \dot{x}$ will be equal to the increment $mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$; and afterwards, when x is increasing further, from that certain magnitude that is greater than 1, to A , it's greatest magnitude, the increment $\overline{m+n+1} \times x^{m+n} \times \dot{x}$ will be greater than the increment $mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$.

But, while the increment of the third term x^{m+n+1} of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, (which is marked with the sign $-$, or is subtracted from the sum of the other two terms Px^m and x^{m+n} of the said trinomial quantity,) is less than the contemporary increment of the sum of the said other two terms, it is evident that the said trinomial quantity will increase: and, when the former increment is become greater than the latter, the said trinomial quantity will decrease. And consequently the said trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will have attained it's greatest magnitude at the instant of time at which the increment $\overline{m+n+1} \times x^{m+n} \times \dot{x}$ is equal to the increment $mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$.

But, when $\overline{m+n+1} \times x^{m+n} \times \dot{x}$ is $= mP \times x^{m-1} \times \dot{x} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$, we shall have $\overline{m+n+1} \times x^{m+n} = mP \times x^{m-1} + \overline{m+n} \times x^{m+n-1}$, and (multiplying all the terms into x), $\overline{m+n+1} \times x^{m+n+1} = mP \times x^m + \overline{m+n} \times x^{m+n}$, and (dividing all the terms by x^m), $\overline{m+n+1} \times x^{n+1} = mP + \overline{m+n} \times x^n$, and (subtracting $\overline{m+n} \times x^n$ from both sides,) $\overline{m+n+1} \times x^{n+1} - \overline{m+n} \times x^n = mP$, and, lastly, (dividing all the terms by $m+n+1$), $x^{n+1} - \frac{\overline{m+n}}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$.

Therefore the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ will have attained it's greatest magnitude when the binomial quantity $x^{n+1} - \frac{\overline{m+n}}{m+n+1} \times x^n$ is equal

equal to $\frac{m}{m+n+1} \times P$, or when x is equal to the root of the binomial equation $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$; that is, M , or the above-mentioned middle value of x , which will make the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ be of the greatest magnitude, will be the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$. Q. E. I.

Art. 70. It appears therefore that, if we resolve the binomial equation $x^{n+1} - x^n = P$, and call it's root A , and if we also resolve the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, and call it's root M , the said quantities A and M will be the limits of the magnitudes of the two roots of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$; or that, if Q , the absolute term of the said equation, be less than the trinomial quantity $P \times M^m + M^{m+n} - M^{m+n+1}$, (or that value of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, which results from the substitution of M in it's terms instead of x), the said trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ will have two roots; of which the lesser will be less than M , (or the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$), and the greater will be greater than M , or the root of the said binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, but less than A , or the root of the former binomial equation $x^{n+1} - x^n = P$; but, if the absolute term Q be exactly equal to the said quantity $P \times M^m + M^{m+n} - M^{m+n+1}$, the said trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ will have only one root, to wit, M , or the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$.

In order therefore to discover the values of the quantities A and M , (which are the limits of the magnitudes of the two roots of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$),

$-x^{m+n+1} = Q$, it will be necessary to resolve the two binomial equations $x^{n+1} - x^n = P$ and $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$. We will therefore now endeavour to resolve the equation $x^{n+1} - x^n = P$.

The Resolution of the Binomial Equation $x^{n+1} - x^n = P$.

Art. 71. The value of x in this equation is always greater than 1, and may be of any magnitude greater than 1, how great soever, if the absolute term P be of a magnitude proportionally great. For, if x be supposed to increase, from being equal to 1, *ad infinitum*, the binomial quantity $x^{n+1} - x^n$ will at the same time increase continually from 0 *ad infinitum*, and therefore will become equal, at one time or other of its increase, to any quantity, how small, or how great, soever. But, if P is less than 1, (as is generally the case in trinomial equations of this form, $Px^m + x^{m+n} - x^{m+n+1} = Q$, that result from the foregoing Problem concerning the rate of interest allowed to a purchaser of the prolongation of a given annuity for an additional term of years,) the value of x will be very little greater than 1, and much less than 2. For, if x was = 2, and n (which is supposed to be a whole number,) was the smallest possible whole number, or 2, we should have $x^n (= 2^2) = 4$, and $x^{n+1} (= 2^{2+1} = 2^3) = 8$; and consequently $x^{n+1} - x^n$ would be $(= 8 - 4) = 4$. And therefore, if x was = 2, and n was 10, or 20, or some greater number, the quantity $x^{n+1} - x^n$ would be much greater than 4. Therefore, if P is less than 1, and n is any whole number whatsoever, (even the small number 2; and, *a fortiori*, if n is 10 or 20, or any greater number,) the value of x in the equation $x^{n+1} - x^n = P$ must be much less than 2, though greater than 1, and will probably be nearly equal to 1.2, or 1.3. And therefore, if we put v for the excess of x above 1, and substitute $1 + v$ instead of x in the equation $x^{n+1} - x^n = P$, (whereby the said equation will be transformed into another equation of which the small quantity v will be the root,) the powers of v in the said transformed equation will decrease with considerable swiftness, and therefore the said equation may be resolved to a considerable degree of exactness by a single process of Mr. Raphson's, or some other, method of approximation. The substitution of $1 + v$ instead of x in the equation $x^{n+1} - x^n = P$ may be made as follows.

Since

Since x is $= 1 + v$, we shall have $x^n = \overline{1 + v}^n =$, by the binomial theorem, to the series $1 + n \times v + n \times \frac{n-1}{2} \times v^2 + \&c$, and $x^{n+1} = \overline{1 + v}^{n+1} =$, by the binomial theorem, to the series $1 + \overline{n+1} \times v + \overline{n+1} \times \frac{n}{2} \times v^2 + \&c$, and consequently $x^{n+1} - x^n =$ the series

$$1 + \overline{n+1} \times v + \overline{n+1} \times \frac{n}{2} \times v^2 + \&c - \text{the series}$$

$$1 + n \times v + n \times \frac{n-1}{2} \times v^2 + \&c = \text{the series}$$

$$* \quad v + n \times v^2 + \&c.$$

But $x^{n+1} - x^n$ is $= P$.

Therefore the series $v + n \times v^2 + \&c$ will also be $= P$.

Therefore $\frac{v}{n} + v^2 + \&c$ will be $= \frac{P}{n}$, and $v^2 + \frac{v}{n} + \frac{1}{4nn} + \&c$ will be $= \frac{P}{n} + \frac{1}{4nn} (= \frac{4nP}{4nn} + \frac{1}{4nn}) = \frac{4nP + 1}{4nn}$. Therefore $v + \frac{1}{2n} + \&c$ will be $= \frac{\sqrt{4nP + 1}}{2n}$, and v will be $= \frac{\sqrt{4nP + 1} - 1}{2n} - \&c$. And consequently x , or $1 + v$, will be $= 1 + \frac{\sqrt{4nP + 1} - 1}{2n} - \&c$, or will be somewhat less than $1 + \frac{\sqrt{4nP + 1} - 1}{2n}$; that is, the root of the binomial equation $x^{n+1} - x^n = P$ will be nearly equal to, but somewhat less than, $1 + \frac{\sqrt{4nP + 1} - 1}{2n}$.

Q. E. I.

This way of obtaining the value of x in the binomial equation $x^{n+1} - x^n = P$ by the resolution of the quadratick equation $v + n \times v^2 + \&c = P$, or $\frac{v}{n} + v^2 + \&c = \frac{P}{n}$, will be attended with very little difficulty of computation; more especially, if we do not carry the extraction of the square-root of $4nP + 1$ to more than four places of figures, which will be enough for our present purpose, which is to obtain the value of the root x , or of the quantity A , (which is the greatest possible magnitude of x in the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, or the higher limit of it's greater root,) to a sufficient degree of exactness to enable us to derive from it a probable first near value of the greater root of the said trinomial equation.

Art. 72.

Art. 72. The other binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, (the root of which is equal to M , or the middle value of x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, which, by being substituted instead of x in it's terms, will make the said quantity be of the greatest possible magnitude, may be resolved with nearly the same ease as the former equation $x^{n+1} - x^n = P$, and with rather more exactness, by substituting $1 + v$ instead of x in it's terms, and resolving the transformed equation, thence resulting, in the same manner as we before resolved the transformed equation derived from the equation $x^{n+1} - x^n = P$ in the foregoing article, that is, by retaining only the terms that involve v and vv in the said transformed equation, and resolving it as a quadratick equation. For in this equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ the absolute term $\frac{m}{m+n+1} \times P$ is considerably less than P , or the absolute term of the former equation $x^{n+1} - x^n = P$, and consequently the root x will also be less than in the former equation, and therefore v , or the excess of x , or $1 + v$, above 1 , will be less than it was in the former transformed equation, and it's powers will consequently decrease with greater swiftness. This transformation of the said binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, and resolution of the transformed equation derived from it, may be performed in general terms, as follows.

The Resolution of the Binomial Equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$.

Art. 73. Let f be put $= \frac{m+n}{m+n+1}$, the co-efficient of x^n , and g be put $=$ the absolute term $\frac{m}{m+n+1} \times P$. And we shall then have $x^{n+1} - fx^n = g$.

Now let x be $= 1 + v$.

Then

Then we shall have $x^n = \overline{1+v}^n =$, by the binomial theorem, to the series $1 + n \times v + n \times \frac{n-1}{2} \times v^2 + \&c$, and $x^{n+1} = \overline{1+v}^{n+1} =$, by the binomial theorem, to the series $1 + \overline{n+1} \times v + \frac{n}{2} \times v^2 + \&c$, and $f \times x^n = f \times$ the series $1 + n \times v + n \times \frac{n-1}{2} \times v^2 + \&c =$ the series $f + fnv + \frac{fnn - fn}{2} \times v^2 + \&c$.

Therefore the binomial quantity $x^{n+1} - f \times x^n$ will be =

$$\left\{ \begin{array}{l} \text{the series } 1 + \overline{n+1} \times v + \frac{n}{2} \times v^2 + \&c \\ - \text{the series } f + fn \times v + \frac{fnn - fn}{2} \times v^2 + \&c \end{array} \right\}$$

$$= \text{the series } 1 - f + \overline{n+1 - fn} \times v + \frac{n - fnn + n + fn}{2} \times v^2 + \&c.$$

But the binomial quantity $x^{n+1} - f \times x^n$ is = g .

Therefore the series $1 - f + \overline{n+1 - fn} \times v + \frac{n - fnn + n + fn}{2} \times v^2 + \&c$ will also be = g .

Therefore (adding f to both sides, we shall have $1 + \overline{n+1 - fn} \times v + \frac{n - fnn + n + fn}{2} \times v^2 + \&c = g + f$, and (subtracting 1 from both sides,) we shall have $\overline{n+1 - fn} \times v + \frac{n - fnn + n + fn}{2} \times v^2 + \&c = g + f - 1$, and (multiplying all the terms by 2,) we shall have $\overline{2n+2 - 2fn} \times v + \overline{nn - fnn + n + fn} \times v^2 + \&c = 2g + 2f - 2$, and (dividing all the terms by $nn - fnn + n + fn$) we shall have $\frac{\overline{2n+2 - 2fn} \times v}{nn - fnn + n + fn} + vv + \&c = \frac{2g + 2f - 2}{nn - fnn + n + fn}$. Therefore (adding to both sides the square of the fraction $\frac{n+1 - fn}{nn - fnn + n + fn}$) we shall have $vv + 2 \times \frac{n+1 - fn}{nn - fnn + n + fn} \times v + \frac{n+1 - fn}{nn - fnn + n + fn}^2 = \frac{2g + 2f - 2}{nn - fnn + n + fn} + \frac{n+1 - fn}{nn - fnn + n + fn}^2$.

To simplify this notation, put $b = n + 1 - fn$, and $i = 2g + 2f - 2$, and $k = nn - fnn + n + fn$.

And we shall then have $vv + \frac{2b}{k} \times v + \frac{b^2}{k^2} + \&c = \frac{i}{k} + \frac{b^2}{k^2} = \frac{ik + b^2}{k^2}$. Therefore (extracting the square-roots of both sides,) we shall have $v + \frac{b}{k} + \&c = \frac{\sqrt{ik + b^2}}{k}$, and (subtracting $\frac{b}{k} + \&c$ from both sides,) $v = \frac{\sqrt{ik + b^2}}{k} - \frac{b}{k} - \&c = \frac{\sqrt{ik + b^2} - b}{k} - \&c$. Therefore $1 + v$, or x , will be $= 1 + \frac{\sqrt{ik + b^2} - b}{k} - \&c$; that is, the root of the binomial equation $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1}$ will be $= 1 + \frac{\sqrt{ik + b^2} - b}{k} - \&c$, or will be nearly equal to, but somewhat less than, $1 + \frac{\sqrt{ik + b^2} - b}{k}$. Q. E. I.

Art. 74. The general algebraïck expression of this root is so complicated that it would probably be more easy and convenient in practice to neglect it, and to proceed to the resolution of any particular numeral equation of the form

$x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$, that we might have occasion to resolve, by applying to it all the foregoing reasonings and operations above set forth in art. 73, instead of computing the values of b and i and k , in order to find the value of the general expression $1 + \frac{\sqrt{ik + b^2} - b}{k}$, just now obtained.

But this must be left to the judgement and discretion of the calculator in every particular example.

Art. 75. When we have thus obtained the values of the roots of the two binomial equations $x^{n+1} - x^n = P$ and $x^{n+1} - \frac{m+n}{m+n+1} \times x^n = \frac{m}{m+n+1} \times P$, (which roots have been shewn to be equal to the quantities A and M , or to the limits of the magnitude of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$;) we may derive from them a conjectural value of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, that will probably be pretty near the truth, by computing the value of the trinomial quantity $P \times M^m + M^{m+n} - M^{m+n+1}$, or the greatest possible magnitude of the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, and comparing it with the absolute term Q , or the value of the said trinomial quantity in the equation $Px^m + x^{m+n} - x^{m+n+1} = Q$. And in this manner we may obtain

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a pretty good first near value of the greater root of the said trinomial equation, independently of the lesser root of the said equation, or without any knowledge of the value of the said lesser root.

Art. 76. But, if we already know the magnitude of the lesser root of the said equation, we may in such case obtain another conjectural value of it's greater root by means of it's said lesser root. For, if d be put for the excess of M , (or the middle value of x which, being substituted instead of x in the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$, makes that quantity to be of the greatest magnitude possible,) and we add $\frac{d}{2}$ to M , it will, I believe, often be found that the sum thence arising, to wit, $M + \frac{d}{2}$, will be pretty nearly equal to the greater root of the said equation. And thus we shall in these cases have two ready methods of obtaining a tolerably good first near value of the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$.

Art. 77. And, when we have, by either of the two foregoing methods, obtained a pretty good first near value of x , or the greater root of the trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, we may derive from it a second near value of the said root, that shall be much nearer to the truth than the said first near value, by a process of the Differential method of approximation: and, if such second near value of the said greater root should not be thought sufficiently exact, we may easily find a third near value of it, that shall be much more exact than the said second near value of it, by a process of Mr. Raphson's method of approximation.

The Application of the Conclusions obtained in the ten preceeding Articles to the

Equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, which is a particular Case of the more general Equation $Px^m + x^{m+n} - x^{m+n+1} = Q$.

Art. 78. I now return to the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, (of which we know the lesser root to be $= 1$.) and will apply the conclusions obtained in the ten last articles, concerning the roots of the more general trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, to the roots of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, of which we wish to find the greater root.

Now in this equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, the letter r (which denotes the unknown quantity,) answers to the letter x , which denotes the unknown quantity in the foregoing equation $Px^m + x^{m+n} - x^{m+n+1} = Q$; and the index, x , of the unknown quantity r in the first term, r^x , of the present equation, answers to the index, m , of the unknown quantity x in the first term, Px^m , of the former equation; and the index, $x+t$, of the unknown quantity r in the second term, r^{x+t} , of the present equation, answers to the index, $m+n$, of the unknown quantity x in the second term, x^{m+n} , of the former equation; and the index, $x+t+1$, of the unknown quantity r in the third term, r^{x+t+1} , of the present equation, answers to the index, $m+n+1$, of the unknown quantity x in the third term, x^{m+n+1} , of the former equation; and $\frac{a}{z}$, the co-efficient of r^x in the first term, $\frac{a}{z} \times r^x$, of the present equation, answers to P , the co-efficient of x^m in the first term, Px^m , of the former equation; and the absolute term, $\frac{a}{z}$, of the present equation, answers to Q , the absolute term of the former equation, as well as to P , the co-efficient of x^m in the said former equation, because the absolute term of the present equation is equal to the co-efficient of r^x in it's first term.

Art. 79. Since the index t in the present equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ answers to the index n in the former equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, and the co-efficient $\frac{a}{z}$ answers to the co-efficient P , the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ will answer to the binomial equation $x^{n+1} - x^n = P$.

And, since the index x in the present equation answers to the index m in the former equation, the binomial equation $r^{x+t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$ will answer to the binomial equation $x^{m+n+1} - \frac{m+n}{m+n+1} \times x^m = \frac{m}{m+n+1} \times P$. And consequently A , or the greatest possible magnitude of r in the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, will be the value of

of r in the binomial equation $r^{t+1} - r^t = \frac{a}{z}$; and M (or the middle value of r in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, which, if it is substituted instead of r in the said trinomial quantity, will make it be of the greatest magnitude possible,) will be the value of r in the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1}$, or the root of the said equation.

And therefore, in order to obtain the values of A and M , (which are the higher and lower limits of the magnitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$;) we must resolve the two binomial equations $r^{t+1} - r^t = \frac{a}{z}$ and $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$; and the root of the former of these binomial equations will be $= A$, or the higher limit of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$; and the root of the latter of the said binomial equations will be $= M$, or the lower limit of the greater root of the said trinomial equation.

And, since t , in the present trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, answers to n in the former trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$;—and $\frac{a}{z}$, the co-efficient of r^x in the present equation, answers to P , the co-efficient of x^m in the former equation;—it follows that the expression $1 + \frac{\sqrt{4nP+1}-1}{2n}$ (found above in art. 71 for the root of the binomial equation $x^{n+1} - x^n = P$;) will be $= 1 + \frac{\sqrt{4t \times \frac{a}{z} + 1} - 1}{2t}$. Therefore

$1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$ will be the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, or will be $= A$, or the higher limit of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$. Q. E. I.

As for the other binomial equation, the root of which is $= M$, to wit, the equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$, it will be most convenient to

to resolve it in each particular example, or numeral equation, by substituting $1 + v$ in it's terms instead of r , and resolving the transformed equation that will result from such substitution, as if it were a quadratick equation. The value of $1 + v$, or x , obtained by this way of proceeding, will be a pretty good near value of M , (or the middle value of r in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, which, by being substituted instead of r in the said trinomial quantity, will make it be of the greatest possible magnitude,) which is the lower limit of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$.

Art. 80. When A and M , the two limits of the magnitude of the greater root of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, are thus obtained; we must substitute M instead of r in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, and compare the quantity $\frac{a}{z} \times M^x + M^{x+t} - M^{x+t+1}$, resulting from this substitution, with $\frac{a}{z}$, the absolute term of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$; and from this comparison we shall be able to find a conjectural value of the greater root of this equation, which will not differ greatly from it's true value, but will be sufficiently near it to form a very good ground-work of a further approach to it's true value by the Differential method of approximation, by which we shall obtain a second near value of the said greater root that will be much more exact than the said first near value of it. And, if this second near value, obtained by a process of the Differential method of approximation, is not thought to be sufficiently exact; we may obtain a third and more exact value by making the said second near value the ground work of a further approach to the true value of the greater root sought by a process of Mr. Raphson's method of approximation.

Art. 81. And, when we have obtained the value of M , or the lower limit of the greater root of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, in the manner above-described, we may employ it in obtaining another first near value of the said greater root by means of the lesser root of the said equation, which we already know to be ≈ 1 . For, if we put d for the excess of the said quantity M above 1 , and add $\frac{d}{2}$ to M , the sum thence arising, to wit, $M + \frac{d}{2}$, will often, I believe, be found to be pretty nearly equal to the said greater root, and

and near enough to be used as a first near value of it, or to be made the ground-work of a further approach to the true value of such greater root by the Differential method of approximation. And thus, when we have found the value of M , we shall have two ready methods of obtaining a tolerably good first near value of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$.

Art. 82. And, if we doubt which of these two first near values of the greater root of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ comes nearest to its true value, that doubt may be removed by substituting them both successively, instead of r , in the terms of the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, and observing which of the two results of these substitutions will differ least from $\frac{a}{z}$, the absolute term of the said equation. For that value of r which produces the result that differs least from the absolute term $\frac{a}{z}$ may be concluded to differ least from the true value of the greater root of the said equation. And we may observe further, that our trouble in making both these substitutions in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$ will not be thrown away; because both these substitutions may be usefully employed in performing a process of the Differential method of approximation, by which we shall obtain a second near value of the said greater root, that will be much more exact than either of the two first values that have been so substituted instead of r in the said trinomial quantity. And, if this second near value of r , or the said greater root, is not thought to be sufficiently exact, we may use it as the ground-work of a further approach to the true value of the said greater root by a process of Mr. Raphson's method of approximation, by which we shall obtain a third near value of it that will be as exact as need be desired.

An Example of the Resolution of an Equation of the foregoing Form, $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, by the Method described in the foregoing Articles.

Art. 83. I will now proceed to apply the method of resolving the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, described in the foregoing articles,

ticles, to the resolution of the numeral equation of this form that results from the Problem proposed by Dr. Halley, which is set forth above in art. 64; in which a , or the annuity granted, is 20 pounds a year; and t , or the number of years of the first term for which it was granted, that are unexpired, or still to come, is 21 years; and x , or the number of years for which the annuity is to be prolonged beyond the said 21 years, is 10 years; and z , or the sum of money paid down as the price of such prolongation, is 40 pounds; and r denotes the rate of Interest of Money corresponding to the price z , or 40 pounds, and which is required to be found.

Now, since a is = 20*l.*, and t is = 21 years, and x is = 10 years, and z is = 40*l.*, we shall have $\frac{a}{z}$ ($= \frac{20}{40} = \frac{1}{2}$) = 0.500,000,0, and $x + t$ ($= 10 + 21$) = 31, and $x + t + 1$ ($= 31 + 1$) = 32. Therefore the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ will in this case be $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. We must therefore endeavour to resolve this equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and discover the value of it's greater root, by the method described in the foregoing articles.

Of the Magnitudes of A and M, the higher and lower Limits of the Magnitude of the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Art. 84. Since x is = 10, and t is = 21, and $\frac{a}{z}$ is = 0.500,000,0, the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ (obtained in art. 79,) will, in this case, be $r^{32} - r^{31} = 0.500,000,0$, and the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$ will be $r^{32} - \frac{31}{32} \times r^{31} = \frac{10}{32} \times 0.500,000,0$ ($= \frac{5.000,000,0}{32}$) = 0.156,250,0, or $r^{32} - 0.968,750,0 \times r^{31} = 0.156,250,0$. Therefore A, or the higher limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ will be the root of the binomial equation $r^{32} - r^{31} = 0.500,000,0$; and the quantity M, or the lesser limit of the greater root of the said trinomial equation, will be the root of the binomial equation $r^{32} - 0.968,750,0 \times r^{31} = 0.156,250,0$. In order therefore to discover the said limits A and M, we must resolve the said binomial equations $r^{32} - r^{31} = 0.500,000,0$ and $r^{32} - 0.968,750,0 = 0.156,250,0$.

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The Investigation of the Magnitude of A, or the higher Limit of the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Art. 85. It has been shewn in art. 79, that the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ is nearly = the expression $1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$, which, when t is = 21, and $\frac{a}{z}$ is = 0.500,000,0, or $\frac{1}{2}$, becomes = $1 + \frac{\sqrt{4 \times 21 \times \frac{1}{2} + 1} - 1}{2 \times 21}$ ($= 1 + \frac{\sqrt{2 \times 21 + 1} - 1}{42} = 1 + \frac{\sqrt{42 + 1} - 1}{42} = 1 + \frac{\sqrt{43} - 1}{42} = 1 + \frac{6.5574 - 1}{42} = 1 + \frac{5.5574}{42} = 1 + 0.1323$) = 1.1323.

But in the numeral equation $r^{22} - r^{21} = 0.500,000,0$, (which comes under the general equation $r^{t+1} - r^t = \frac{a}{z}$), t is = 21; and $t + 1$ is = 22, and $\frac{a}{z}$ is = 0.500,000,0.

Therefore the root of the numeral equation $r^{22} - r^{21} = 0.500,000,0$ will be = 1.1323. Therefore A, or the higher limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ (which has been shewn to be equal to the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$), will be nearly = 1.1323. Q. E. I.

The Investigation of the Magnitude of M, or the lower Limit of the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Art. 86. We must next endeavour to find the value of M, or the lower limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Now this limit is the root of the other binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, which may be found in the following manner.

Let v be put for the excess of the root of the said binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$ above 1, or r be put = $1 + v$. And let $1 + v$ be substituted instead of r in the said binomial equation.

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Then, since r is $= 1 + v$, we shall have $r^{22} = \overline{1 + v}^{22}$ ($=$, by the binomial theorem, to the series $1 + 22 \times v + 22 \times \frac{22-1}{2} \times v^2 + \&c =$ the series $1 + 22v + 22 \times \frac{21}{2} \times v^2 + \&c =$ the series $1 + 22v + 11 \times 21 \times v^2 + \&c$) $=$ the series $1 + 22v + 231 \times vv + \&c$; and we shall have $r^{21} = \overline{1 + v}^{21}$ ($=$, by the binomial theorem, to the series $1 + 21 \times v + 21 \times \frac{21-1}{2} \times v^2 + \&c =$ the series $1 + 21v + 21 \times \frac{20}{2} \times v^2 + \&c =$ the series $1 + 21v + 21 \times 10 \times v^2 + \&c$) $=$ the series $1 + 21v + 210vv + \&c$, and consequently $0.968,750,0 \times r^{21}$ ($= 0.968,750,0 \times$ the series $1 + 21v + 210vv + \&c =$ the series $0.968,750,0 \times 1 + 0.968,750,0 \times 21v + 0.968,750,0 \times 210vv + \&c$) $=$ the series $0.968,750,0 + 20.343,750,0 \times v + 203.437,500,0 \times vv + \&c$.

Therefore the binomial quantity $r^{22} - 0.968,750,0 \times r^{21}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1.000,000,0 + 22.000,000,0 \times v + 231.000,000,0 \times vv + \&c \\ - 0.968,750,0 - 20.343,750,0 \times v - 203.437,500,0 \times vv - \&c \end{array} \right\} \\ = 0.031,250,0 + 1.656,250,0 \times v + 27.562,500,0 \times vv + \&c.$$

But the binomial quantity $r^{22} - 0.968,750,0 \times r^{21}$ is $= 0.156,250,0$.

Therefore the compound quantity $0.031,250,0 + 1.656,250,0 \times v + 27.562,500,0 \times vv + \&c$ will also be $= 0.156,250,0$.

Therefore $1.656,250,0 \times v + 27.562,500,0 \times vv + \&c$ will be ($= 0.156,250,0 - 0.031,250,0$) $= 0.125,000,0$, and $\frac{1.656,250,0}{27.562,500,0} \times v + vv + \&c$ will be $= \frac{0.125,000,0}{27.562,500,0}$, that is, $0.004,535,1$, or (neglecting the three last figures of $0.004,535,1$), $0.004,535,1$, or (neglecting the three last figures of $0.004,535,1$), $0.004,535,1$. Therefore (adding $\frac{0.0600}{2}$, or 0.0300 , to both sides,) we shall have $vv + 0.0600 \times v + 0.0300^2 + \&c = 0.004,535,1 + 0.0009 = 0.005,435,1$; and (extracting the square-roots of both sides,) we shall have $v + 0.0300 + \&c (= \sqrt{0.005,435,1}) = 0.0737$, and (subtracting $0.0300 + \&c$ from both sides,) $v (= 0.0737 - 0.0300 - \&c) = 0.0437 - \&c$; that is, v will be nearly equal to, but somewhat less than, 0.0437 . Therefore r , or $1 + v$, will be nearly equal to, but somewhat less than, $1 + 0.0437$, or 1.0437 . Therefore M , or the lower limit of the magnitude of the greater root of the trinomial equation $0.500,000,0 \times r^{20} + r^{21} - r^{22} = 0.500,000,0$, will be nearly equal to, but somewhat less than, 1.0437 .

Q. E. D.

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The Substitution of 1.0437, or the Value of M, instead of r, in the trinomial Quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, in order to obtain the greatest possible Magnitude of the said trinomial Quantity.

Art. 87. Having thus discovered that the two quantities A and M, or the two limits of the magnitude of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, are nearly equal to 1.1323 and 1.0437, we will now substitute the latter quantity M, or 1.0437, instead of r, in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, whereby we shall obtain the greatest possible magnitude which the said trinomial quantity can ever attain; and, when we have made this substitution, we will compare the value of the said trinomial quantity resulting from such substitution with 0.500,000,0, or the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, in order that we may be enabled from that comparison to form a probable conjecture concerning the greater value of r in the said equation. This substitution of 1.0437 instead of r in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ may be made in the following manner.

If r is supposed to be = 1.0437, we shall have $r^{10} = \overline{1.0437}^{10}$, and $r^{31} = \overline{1.0437}^{31}$, and $r^{32} = \overline{1.0437}^{32}$; and consequently $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be = $0.500,000,0 \times \overline{1.0437}^{10} + \overline{1.0437}^{31} - \overline{1.0437}^{32}$. We must therefore find the values of $\overline{1.0437}^{10}$, $\overline{1.0437}^{31}$, and $\overline{1.0437}^{32}$.

The logarithm of 1.0437 is = 0.018,575,7. Therefore the logarithm of $\overline{1.0437}^{10}$ will be (= $10 \times 0.018,575,7$) = 0.185,757,0; which is the logarithm of the number 1.533,758,6. Therefore $\overline{1.0437}^{10}$ will be = 1.533,758,6.

And the logarithm of $\overline{1.0437}^{31}$ will be (= $31 \times 0.018,575,7$) = 0.575,846,7; which is the logarithm of the number 3.765,708,7. Therefore $\overline{1.0437}^{31}$ will be = 3.765,708,7; and consequently $\overline{1.0437}^{32}$ will be (= $\overline{1.0437}^{31} \times 1.0437$) = $3.765,708,7 \times 1.0437$ = 3.930,270,1.

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.0437}^{10} + \overline{1.0437}^{31} - \overline{1.0437}^{32}$ will be (= $0.500,000,0 \times 1.533,758,6 + 3.765,708,7 - 3.930,270,1$) = $0.766,879,3 + 3.765,708,7 - 3.930,270,1$ = 4.532,588,0 - 3.930,270,1 = 0.602,317,9. Therefore 0.602,317,9 is the value of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ when r is = M, or 1.0437, and consequently is the greatest possible magnitude of the said trinomial quantity.

Q. E. I.

The Investigation of a first near Value of the greater Root of the Equation
 $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, by means of A and M,
 or 1.1323 and 1.0437, the two Limits of the Magnitude of the said greater
 Root.

Art. 88. Now, while r increases from 1.0437, or M, to 1.1323, or A, (which is it's greatest possible magnitude while r^{32} continues to be less than the binomial quantity $0.500,000,0 \times r^{10} + r^{31}$;) the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will decrease from 0.602,317,9 to 0, so that the increment of r that is contemporary with the said decrement, 0.602,317,9, of the said trinomial quantity, will be $(= 1.1323 - 1.0437) = 0.0886$.

Further, while the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ decreases from 0.602,317,9 to 0.500,000,0, (or the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$;) r will increase from M, or 1.0437, or the value corresponding to 0.602,317,9 (the greatest possible value of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$;) to the value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore, if we suppose the increment of r from M, or 1.0437, to the said greater root of the said trinomial equation, to bear the same, or nearly the same, proportion to it's whole increment from M, or 1.0437, to 1.1323, or to the increment 0.0886, as the decrement of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ from 0.602,317,9 to the absolute term 0.500,000,0, to wit, the decrement $(0.602,317,9 - 0.500,000,0)$, or 0.102,317,9, (which is contemporary with the increment of r from M, or 1.0437, to the value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$;) bears to the whole decrement, 0.602,317,9, of the said trinomial quantity from 0.602,317,9 to 0, we shall have $r - 1.0437$, nearly, $= \frac{0.0886 \times 0.102,317,9}{0.602,317,9} (= \frac{0.009,065,365,94}{0.602,317,9}) = 0.0150$, and consequently

r , nearly, $= 0.0150 + 1.0437 = 1.0587$; that is, 1.0587 will be, probably, not very distant from the value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. And thus, by having computed the values of A and M, or the higher and lower limits of the magnitude of the said greater root of the said equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and found the said limits to be equal to 1.1323 and 1.0437, we have been enabled to form a probable conjecture concerning the value of the said greater root, and to conclude that it is nearly equal to 1.0587, independently of the lesser root, or without making use of our previous knowledge that the said lesser root is $= 1$.

And

And this number 1.0587, thus found for the first near value of the greater root of the said trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, is tolerably near to the true value of the said greater root, for a first, or conjectural, value of it. For the true value of the said greater root has been found above, in page 356 of the present Volume, to be $= 1.063,235.9$; which exceeds 1.0587 by the small quantity 0.004,535.9, which is less than the 234th part of the said true value 1.063,235.9. This is a very considerable degree of exactness for a first near value of r , and such as will make the said first near value 1.0587 be an excellent basis, or ground-work, for a further approach to the true value of r , or the said greater root, either by the Differential method, or Mr. Raphson's method, or any other method, of approximation.

The Investigation of another near Value of r , or the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, by means of M , or 1.0437, it's lower Limit, and of the lesser Root of the said Equation, which is known to be $= 1$.

Art. 89. But, if we make use of our previous knowledge of the magnitude of the lesser root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ in order to obtain a probable first near value of the greater root of the said equation, by putting d for the excess of M , or 1.0437, above the said lesser root, (which we know to be $= 1$), and supposing the greater root to be nearly equal to $M + \frac{d}{2}$, we shall thereby obtain another near value of the said greater root, which will be still nearer to it's true value than 1.0587. For, since M is $= 1.0437$, and the lesser root of the equation is $= 1$, we shall have $d (= 1.0437 - 1) = 0.0437$, and $\frac{d}{2} (= \frac{0.0437}{2}) = 0.0218$, and consequently $M + \frac{d}{2} (= 1.0437 + 0.218) = 1.0665$; which is greater than 1.063,235.9, or the true value of the said greater root, by only the small quantity 0.002,264.1, which is less than the 469th part of 1.063,235.9, or the true value of the said greater root. This difference from the true value of the said greater root is but half the former difference, 0.004,535.9, of the number 1.0587 from the said true value. So that this number 1.0665, (obtained by means of the expression $M + \frac{d}{2}$, and derived from our previous knowledge of the magnitude of the lesser root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$) is a better first near value of the greater root of the said equation

equation than the former number 1.0587, which was obtained from the two quantities A and M, or 1.1323 and 1.0437, which were the limits of the magnitude of the said greater root. But both these numbers 1.0587 and 1.0655 are near enough to the true value of the said greater root to be fit to be taken for the first near values of it, or employed as the basis, or ground-work, of a further approach to the said true value either by the Differential method, or Mr. Raphson's method, or some other method, of approximation.

The Investigation of a second near Value of r, or the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, by means of the Differential Method of Approximation.

Art. 90. We will now make use of both these numbers 1.0587 and 1.0665 as the means of a further approach to the true value of the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, by a process of the Differential method of approximation; and for that purpose we will substitute, 1st, 1.0587, and afterwards, 1.0665, instead of r in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, and compare the values of the said trinomial quantity resulting from those substitutions with 0.500,000,0, or the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, in the manner prescribed by the said Differential method.

Now, if we suppose r to be $= 1.0587$, we shall have $r^{10} = \overline{1.0587}^{10}$, and $r^{31} = \overline{1.0587}^{31}$, and $r^{32} = \overline{1.0587}^{32}$, and $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0 \times \overline{1.0587}^{10} + \overline{1.0587}^{31} - \overline{1.0587}^{32}$. We must therefore find the values of $\overline{1.0587}^{10}$, $\overline{1.0587}^{31}$, and $\overline{1.0587}^{32}$; which may be done as follows.

The logarithm of 1.0587 is $= 0.024,772,9$. Therefore the logarithm of $\overline{1.0587}^{10}$ will be $= 10 \times 0.024,772,9 = 0.247,729,0$; which is the logarithm of the number 1.769,004,8. Therefore $\overline{1.0587}^{10}$ will be $= 1.769,004,8$. Therefore $0.500,000,0 \times \overline{1.0587}^{10}$ will be $(= 0.500,000,0 \times 1.769,004,8 = \frac{1}{2} \times 1.769,004,8) = 0.884,502,4$.

And the logarithm of $\overline{1.0587}^{31}$ will be $(= 31 \times 0.024,772,9) = 0.767,959,9$, which is the logarithm of the number 5.860,840,5. Therefore $\overline{1.0587}^{31}$ will be $= 5.860,840,5$. Therefore $\overline{1.0587}^{32}$ will be $(= \overline{1.0587}^{31} \times 1.0587 = 5.860,840,5 \times 1.0587) = 6.204,871,8$.

Therefore

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.0587}^{10} + \overline{1.0587}^{31} - \overline{1.0587}^{32}$ will be $(= 0.884,502,4 + 5.860,840,5 - 6.204,871,8 = 6.745,342,9 - 6.204,871,8) = 0.540,471,1$; which is greater than $0.500,000,0$, or the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore 1.0587 must be less than the greater root of that equation.

We must next proceed to substitute 1.0665 instead of r in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$.

Now, if we suppose r to be $= 1.0665$, we shall have $r^{10} = \overline{1.0665}^{10}$, and $r^{31} = \overline{1.0665}^{31}$, and $r^{32} = \overline{1.0665}^{32}$; and consequently the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be $= 0.500,000,0 \times \overline{1.0665}^{10} + \overline{1.0665}^{31} - \overline{1.0665}^{32}$. We must therefore find the values of $\overline{1.0665}^{10}$, $\overline{1.0665}^{31}$, and $\overline{1.0665}^{32}$; which may be done as follows.

The logarithm of 1.0665 is $= 0.027,960,9$. Therefore the logarithm of $\overline{1.0665}^{10}$ will be $(= 10 \times 0.027,960,9) = 0.279,609,0$; which is the logarithm of $1.903,746,0$. Therefore $\overline{1.0665}^{10}$ will be $= 1.903,746,0$, and consequently $0.500,000,0 \times \overline{1.0665}^{10}$ will be $(= 0.500,000,0 \times 1.903,746,0 = \frac{1}{2} \times 1.903,746,0) = 0.951,873,0$.

And the logarithm of $\overline{1.0665}^{31}$ will be $(= 31 \times 0.027,960,9) = 0.866,787,9$; which is the logarithm of the number $7.358,476,2$. Therefore $\overline{1.0665}^{31}$ will be $= 7.358,476,2$. Therefore $\overline{1.0665}^{32}$ will be $(= \overline{1.0665}^{31} \times 1.0665 = 7.358,476,2 \times 1.0665) = 7.847,814,8$.

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.0665}^{10} + \overline{1.0665}^{31} - \overline{1.0665}^{32}$ will be $(= 0.951,873,0 + 7.358,476,2 - 7.847,814,8 = 8.310,349,2 - 7.847,814,8) = 0.462,534,4$; which is less than $0.500,000,0$, or the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore 1.0665 must be greater than the greater root of that equation.

We have now three different values of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, that are contiguous, or nearly equal, to each other, to wit, the three quantities $0.540,471,1$, $0.500,000,0$, and $0.462,534,4$, of which the first, or greatest, arises from the substitution of the quantity 1.0587 instead of r in the said trinomial quantity, and the second, or middle, value is the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and the third, or least, arises from the substitution of the quantity 1.0665 instead of r in the said trinomial quantity. We may therefore now

make the following proportion, in order to discover a second near value of r , or the greater root of the said equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ that shall be nearer to the true value of the said greater root than either of the two numbers 1.0587 and 1.0665, which have been already found for it; to wit, As $0.540,471,1 - 0.462,534,4$, or the difference of the first and third of the first set of quantities, is to $0.540,471,1 - 0.500,000,0$, or the difference of the first and second of the first set of quantities, so, very nearly, will $1.0665 - 1.0587$, or the difference of the first and third of the second set of quantities, be to $r - 1.0587$, or the difference of the first and second of the second set of quantities; that is, as $0.077,936,7$ is to $0.040,471,1$, so, very nearly, will 0.0078 be to $r - 1.0587$. Therefore $r - 1.0587$ will be, nearly, $= \frac{0.040,471,1 \times 0.0078}{0.077,936,7} = \frac{0.000,315,674,58}{0.077,936,7} = 0.004,050,3$. There-

fore r will be, nearly, $(= 0.004,050,3 + 1.0587) = 1.062,750,3$; or 1.062,750,3 will be a second near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, that will be nearer to it's true value than either 1.0587, or 1.0665. Q. E. I.

This number 1.062,750,3, thus found for a second near value of r , or the greater root of the said trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, is less than it's true value 1.063,235,9, by the difference 0.000,485,6, which is less than the 2189th part of 1.063,235,9. This degree of exactness is much greater than that of either of the two former numbers 1.0587 and 1.0665.

The Investigation of a third near Value of r , or the greater Root of the trinomial Equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, by a Process of Mr. Raphson's Method of Approximation.

Art. 91. If this number 1.062,750,3, thus obtained for a second near value of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, is not thought to be sufficiently exact, we may proceed to find a third near value of it by Mr. Raphson's method of approximation, which will be as exact as any one need desire. This may be done in the following manner.

Let the said second near value of r , just now obtained, to wit, 1.062,750,3, be substituted, instead of r , in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, in order to discover whether the result of such substitution will be greater, or less, than 0.500,000,0, or the absolute term of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, and consequently whether 1.062,750,3 will be less, or greater, than the greater root of the said equation.

Now,

Now, if r is $= 1.062,750,3$, we shall have $r^{10} = \overline{1.062,750,3}^{10}$, and $r^{31} = \overline{1.062,750,3}^{31}$, and $r^{32} = \overline{1.062,750,3}^{32}$; and consequently the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be $= 0.500,000,0 \times \overline{1.062,750,3}^{10} + \overline{1.062,750,3}^{31} - \overline{1.062,750,3}^{32}$. We must therefore find the values of $\overline{1.062,750,3}^{10}$, and $\overline{1.062,750,3}^{31}$, and $\overline{1.062,750,3}^{32}$; which may be done as follows.

The logarithm of $1.062,750,3$ is $= 0.026,431,2$. Therefore the logarithm of $\overline{1.062,750,3}^{10}$ will be $(= 10 \times 0.026,431,2) = 0.264,312,0$; which is the logarithm of the number $1.837,858,2$. Therefore $\overline{1.062,750,3}^{10}$ will be $= 1.837,858,2$. Therefore $0.500,000,0 \times \overline{1.062,750,3}^{10}$ will be $(= 0.500,000,0 \times 1.837,858,2 = \frac{1}{2} \times 1.837,858,2) = 0.918,929,1$.

And the logarithm of $\overline{1.062,750,3}^{31}$ will be $(= 31 \times 0.026,431,2) = 0.819,367,2$; which is the logarithm of the number $6.597,315,1$. Therefore $\overline{1.062,750,3}^{31}$ will be $= 6.597,315,1$. Therefore $\overline{1.062,750,3}^{32}$ will be $(= \overline{1.062,750,3}^{31} \times 1.062,750,3 = 6.597,315,1 \times 1.062,750,3) = 7.011,298,6$.

Therefore the trinomial quantity $0.500,000,0 \times \overline{1.062,750,3}^{10} + \overline{1.062,750,3}^{31} - \overline{1.062,750,3}^{32}$ will be $= 0.918,929,1 + 6.597,315,1 - 7.011,298,6 = 7.516,244,2 - 7.011,298,6 = 0.504,945,6$; which is a little greater than $0.500,000,0$, or the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. Therefore $1.062,750,3$ must be somewhat less than the greater root of that equation.

Now let w be put for the excess of r , or the greater root of the said equation above $1.062,750,3$, or let r be $= 1.062,750,3 + w$. And let the binomial quantity $1.062,750,3 + w$ be substituted instead of r in the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.

Then, since r is $= 1.062,750,3 + w$, we shall have $r^{10} = \overline{1.062,750,3 + w}^{10}$ ($=$, by the binomial theorem, to $\overline{1.062,750,3}^{10} + 10 \times \frac{\overline{1.062,750,3}^9}{1.062,750,3} \times w + \&c = 1.837,858,2 + 10 \times \frac{1.837,858,2}{1.062,750,3} \times w + \&c = 1.837,858,2 + 10 \times 1.729,341,5 \times w + \&c) = 1.837,858,2 + 17.293,415,0 \times w + \&c$, and consequently $0.500,000,0 \times r^{10} (= \frac{1}{2} \times r^{10}) = \frac{1}{2} \times 1.837,858,2 + \frac{1}{2} \times 17.293,415,0 \times w + \&c = 0.918,929,1 + 8.646,707,5 \times w + \&c$.

And r^{31} will be $= \overline{1.062,750,3 + w}^{31}$ ($=$, by the binomial theorem, to $\overline{1.062,750,3}^{31} + 31 \times \frac{\overline{1.062,750,3}^{31}}{1.062,750,3} \times w + \&c = 6.597,315,1 + 31 \times \frac{6.597,315,1}{1.062,750,3} \times w + \&c = 6.597,315,1 + 31 \times 6.207,775,3 \times w + \&c$)
 $= 6.597,315,1 + 192.441,034,3 \times w + \&c$; and r^{32} will be $= \overline{1.062,750,3 + w}^{32}$ ($= \overline{1.062,750,3}^{32} + 32 \times \frac{\overline{1.062,750,3}^{32}}{1.062,750,3} \times w + \&c$)
 $= 7.011,298,6 + 32 \times 6.597,315,1 \times w + \&c = 7.011,298,6 + 211.114,083,2 \times w + \&c$.

Therefore the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ will be $=$ the compound quantity

$$\begin{aligned}
 & \left\{ \begin{array}{l} c.918,929,1 + 8.646,707,5 \times w + \&c \\ + 6.597,315,1 + 192.441,034,3 \times w + \&c \\ - 7.011,298,6 - 211.114,083,2 \times w - \&c \end{array} \right\} \\
 & = \left\{ \begin{array}{l} 7.516,244,2 + 201.087,741,8 \times w + \&c \\ - 7.011,298,6 - 211.114,083,2 \times w - \&c \end{array} \right\} \\
 & = 0.504,945,6 - 10.026,341,4 \times w - \&c.
 \end{aligned}$$

But the trinomial $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ is $= 0.500,000,0$.

Therefore the compound quantity $0.504,945,6 - 10.026,341,4 \times w - \&c$ will also be $= 0.500,000,0$.

Therefore (adding $10.026,341,4 \times w$ to both sides,) we shall have $0.504,945,6 - \&c = 0.500,000,0 + 10.026,341,4 \times w$, and (subtracting $0.500,000,0$ from both sides,) $10.026,341,4 \times w = 0.004,945,6 - \&c$, and consequently $w = \frac{0.004,945,6}{10.026,341,4} - \&c = 0.000,493,2 - \&c$; that is,

w will be nearly equal to, but somewhat less than, $0.000,493,2$. Therefore r , or $1.062,750,3 + w$, will be nearly equal to, but somewhat less than, $1.062,750,3 + 0.000,493,2$, or $1.063,243,5$; that is, the third near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, that is obtained by this process of Mr. Raphson's method of approximation, will be $1.063,243,5$. Q. E. I.

This number $1.063,243,5$ is greater than the true value of r , or the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, (which is $1.063,235,9$.) by the very small quantity $0.000,007,6$, which is less than the 139,899th part of $1.063,235,9$, or the true value of the said greater root. This equation may therefore now be considered as resolved to a great degree of exactness.

Art. 92. I have now gone through the examination of all the three equations (relating to the investigation of r , or the rate of Interest of Money, allowed in bargains that have been made concerning annuities granted for given terms of years,) for the resolution of which Dr. Halley has given us certain Algebræick expressions derived from Mr. De Moivre's Theorem for raising the powers of a multinomial quantity, upon a supposition that the said Theorem is true in the case of fractional powers, and of powers that are both fractional and negative, or of the reciprocals of fractional powers, as well as in the case of integral and affirmative powers, in which alone it has hitherto been demonstrated: and I have shewn how all these equations may be resolved, to the same, or indeed to any, degree of exactness, by other methods, grounded on more clear and certain principles, and obtained by Algebræick operations of much less intricacy and complication than those which were employed in the investigation of Dr. Halley's expressions. But the full explanation of these methods of resolution has unavoidably extended the subject through a greater number of pages than I could have wished. And therefore I will now endeavour to recapitulate the results of the reasonings used in the foregoing articles, and to compress them into as narrow a compass as possible, and will deduce from them such practical directions as shall be necessary to enable the reader to apply the said results with ease and expedition to the resolution of the three equations to which they relate.

Of the Resolution of the Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ by the three Methods set forth above in Art. 5, 6, 7, 8, 9, &c. - - - 32.

Art. 93. In this equation the lesser value of r is $= 1$, as is shewn above in art. 5, page 362. And the greater value of r is that which is necessary to be found in order to the solution of the Problem stated in art. 3 and 4, from which the said equation was deduced. This greater value of r may be found by three different methods, which have been explained in the foregoing articles.

Of the Resolution of the said Equation by the first of the said three Methods.

To find it by the first of those methods, we may proceed as follows.

Art. 94. Compute the value of $\frac{z}{a}$, the co-efficient of r , and call it P ; and compute the value of the absolute term $\frac{z}{a} - 1$, and call it Q .

3 O 2

Then

Then will the equation $\frac{z}{a} \times r - r^t = \frac{a}{z} - 1$ be converted into the equation $P \times r - r^t = Q$.

Compute the value of $\sqrt[t-1]{P}$, or $P^{\frac{1}{t-1}}$, and call it N; and compute the value of $\sqrt[t-1]{\frac{P}{t}}$, or $\frac{P}{t}^{\frac{1}{t-1}}$, and call it M.

Then will the greater root of the equation $P \times r - r^t = Q$ be less than N, but greater than M; or N will be the greater limit, and M will be the lesser limit, of the magnitude of the said greater root.

Let d be $= M - 1$.

Then will $M + d$ be greater than the greater root of the equation $P \times r - r^t = Q$; and consequently the said greater root will be of an intermediate magnitude between $M + d$ and M.

Let b be taken $= M + d$, and be employed as a first near value of r , or the greater root of the equation $P \times r - r^t = Q$; and, for that purpose, let it be substituted instead of r in the binomial quantity $P \times r - r^t$; and let the quantity $P \times b - b^t$, (or the value of the binomial quantity $P \times r - r^t$ resulting from such substitution,) be called B.

Then, since b , (or $M + d$), is greater than r , (or the greater root of the equation $P \times r - r^t = Q$), the quantity B will be less than the absolute term Q, and consequently may be subtracted from it.

From b subtract $\frac{b}{200}$, and call the remainder c ; and let c be substituted instead of r in the binomial quantity $P \times r - r^t$; and let the quantity $P \times c - c^t$, (or the value of the binomial quantity $P \times r - r^t$ resulting from such substitution,) be called C.

Then, since b is greater than c , the quantity B will be less than the quantity C, and consequently may be subtracted from it.

Now make the following proportion; to wit, As $C - B$ is to $Q - B$, so very nearly, will $b - c$ be to $b - r$. Therefore $b - r$ will be, nearly, $= \frac{Q - B \times b - c}{C - B}$, and b will be, nearly, $= r + \frac{Q - B \times b - c}{C - B}$, and r will be, nearly,

nearly, $= b - \text{the fraction } \frac{Q - B \times b - c}{C - B}$; that is, $b - \text{the fraction } \frac{Q - B \times b - c}{C - B}$ will be a second near value of r , or the greater root of the equation $P \times r - r^t = Q$, that will be much nearer to it's true value than it's first near value, b , or $M + d$, was. Q. E. I.

Now let g be put $= b - \text{the fraction } \frac{Q - B \times b - c}{C - B}$, and let g be substituted instead of r in the binomial quantity $P \times r - r^t$; and, if the quantity $P \times g - g^t$, (or the value of $P \times r - r^t$ resulting from such substitution,) is greater than Q , (or the absolute term of the equation $P \times r - r^t = Q$;) it will follow that g will be less than the greater root of the said equation; but, if the said result is less than Q , the quantity g will be greater than the said greater root. Therefore, in the first case, if we wish to obtain a value of the said greater root that shall be more exact than g , we must substitute $g + w$ instead of r in the equation $P \times r - r^t = Q$; and in the second case we must substitute $g - w$ instead of r in the said equation; and must resolve the transformed equation arising from such substitution as if it were a mere simple equation, agreeably to the directions of Mr. Raphson's method of approximation: and the value of $g + w$, or $g - w$, obtained by the said resolution, will be a third near value of r , or the greater root of the equation $P \times r - r^t = Q$, as exact as need to be desired. Q. E. I.

Art. 95. In this description of the first method of resolving the equation $P \times r - r^t = Q$ so as to find it's greater root, I have supposed $M + d$ to be taken for the first near value of r , or the said greater root, and to be made use of in finding a second near value of the said greater root by the Differential method of approximation. And this I have done, because, in general, it will be found that in an equation of this form, $P \times r - r^t = Q$, the quantity $M + d$ will be near enough to the true value of the said greater root to make it fit to be so employed. And we found it to be so in the case of the numeral equation $17.8614 \times r - r^{12.5} = 16.8614$, which we resolved in this manner in art. 13, 14, and 15, pages 372, 373, &c. - - - 376, where $M + d$ was $= 1.063,045,0$, and the true value of the said greater root is $= 1.06$, which differs from $1.063,045,0$, by only $0.003,045,0$, or less than the 348th part of 1.06 , or the said true value. And, if we had not known already what the true value of the said greater root was, we might have concluded that the number $1.063,045,0$, or $M + d$, would be but little greater than the said true value, because

because the value of the binomial quantity $P \times r - r^{12.5}$, or $17.8614 \times r - r^{12.5}$, arising from the substitution of 1.063,045,0 in it's terms instead of r , to wit, the quantity 16.840,160,6 (which is obtained in art. 14, page 374,) is but little less than 16.8614, or the absolute term of the equation $17.8614 \times r - r^{12.5} = 16.8614$, their difference being $= 0.021,239,4$, which is only the 790th part of the absolute term 16.8614. But it sometimes happens that the quantity $M + d$ (which is always greater than the greater root of the equation

$P \times r - r^f = Q$.) exceeds it in too great a degree to be fit to be taken for a first near value of it, or to be employed in discovering a second and more exact value of it by means of the Differential method of approximation; of which we have had an example in the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, which has been resolved above in art. 17, 18, 19, 20, and 21, pages 378, 379, &c - - - 384. For in that equation the true value of r , or it's greater root, is 1.06, and the value of M is $= 1.038,797,1$, and the value of $M - 1$, or d , is $= 0.038,797,1$, and the value of $M + d$ is $(= 1.038,797,1 + 0.038,797,1) = 1.077,594,2$, which exceeds 1.06, or the true value of the greater root of the equation, by 0.017,594,2; which excess is the 60th part of the said true value, instead of being (as in the former equation $17.8614r - r^{12.5} = 16.8614$.) only the 348th part of the said true value. And "that this number 1.077,594,2 would be too great to be a convenient first near value of r , or the greater root of the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$," might be discovered, without a previous knowledge that the true value of the said greater root was 1.06, by substituting 1.077,594,2 instead of r in the binomial quantity $967.648,136,7 \times r - r^{70}$, and observing that the value of the said quantity resulting from such substitution, to wit, 855.722,019,7 (which is obtained in page 379,) is very considerably less than 966.648,136,7, the absolute term of the said equation. In these cases, therefore, it will be expedient to choose some number less than $M + d$ for our first near value of r , and to call it b , and substitute it instead of r in the binomial quantity $P \times r - r^f$, in order to discover whether the quantity $P \times b - b^f$, (or the value of $P \times r - r^f$ resulting from such substitution,) will be greater, or less, than Q , or the absolute term of the equation $P \times r - r^f = Q$, and consequently whether b will be less, or greater, than the true value of r , or the greater root of that equation, and also whether the difference of the said quantity $P \times b - b^f$ from Q will be small or great. And in some cases it will be found convenient to choose $M + \frac{d}{2}$, instead of $M + d$, for the said first near value of r ; as was done above in page 380, where we chose $(1.038,797,1 + \frac{0.038,797,1}{2})$, or $1.038,797,1 + 0.019,398,5$, or) 1.058,195,6 for our first near value of r ; and afterwards proceeded, (in art. 19,

art. 19, pages 380, 381, 382,) to derive from it a second near value of r by the Differential method of approximation, which second near value of r was 1.059,733,2; and then made use of the said second value, 1.059,733,2, to obtain a third near value of r by a process of Mr. Raphson's method of approximation, which was 1.059,997,3, which is exceedingly near to it's true value 1.06. But it may sometimes happen that some other number, less than $M + d$, but greater than $M + \frac{d}{2}$, (such as $M + \frac{2d}{3}$), will be nearer than $M + \frac{d}{2}$ to the true value of the greater root of the equation $P \times r - r^t = Q$, and consequently fitter to be taken for it's first near value and employed in obtaining a second near value of it by means of the Differential method. And therefore in the choice of the said first near value of r , from a knowledge of it's limits $M + d$ and M , the calculator must exercise his own judgement and sagacity, as no general rule can be laid down that will answer equally well in all cases. But it will always be very easy (by means of the said limits and the comparison of the value of the binomial quantity $P \times r - r^t$ resulting from the substitution of $M + d$ in it's terms, instead of r , with Q , or the absolute term of the equation $P \times r - r^t = Q$), to find a number, less than $M + d$, that will be sufficiently near to the true value of r , or the greater root of the said equation, to be chosen for it's first near value and employed with success to discover a second and more exact near value of it by means of the Differential method of approximation.

Art. 96. And, when some quantity, less than $M + d$, has been pitched upon as a proper quantity to be made use of as a first near value of r , or employed in the discovery of a second and more exact value of r by the Differential method of approximation, and denoted by the letter b , and it shall have been found that the binomial quantity $P \times b - b^t$, (or the value of the binomial quantity $P \times r - r^t$ arising from the substitution of b in it's terms instead of r), is less than Q , or the absolute term of the equation $P \times r - r^t = Q$, and consequently that b (though less than $M + d$), will yet be greater than r , or the greater root of the equation $P \times r - r^t = Q$, we must take $c = b - \frac{b}{200}$, and substitute c instead of r in the binomial quantity $P \times r - r^t$, and denote $P \times c - c^t$, (or the value of the binomial quantity $P \times r - r^t$ resulting from such substitution,) by the capital letter C , and $P \times b - b^t$ by the capital letter B , as above in art. 94, and proceed, exactly as in the said art. 94, to make the proportion therein set forth, of the four quantities $C - B$, $Q - B$, $b - c$, and $b - r$; by which we shall obtain b — the fraction

$\frac{Q - B \times \overline{b - c}}{C - B}$ for a second near value of r , or the greater root of the equation $P \times r - r^t = Q$, that will be much nearer to it's true value than the former near value, b , was. Q. E. I.

But, if the quantity b , which we have chosen for our first near value of r , or the greater root of the equation $P \times r - r^t = Q$, is not only less than $M + d$, but likewise than the true value of r , or the said greater root, (as it must be if B , or the binomial quantity $P \times b - b^t$, is greater than the absolute term Q .) it will then be proper to make $c = b + \frac{b}{200}$, instead of making it $= b - \frac{b}{200}$, and to denote the quantity $P \times c - c^t$ (as before,) by the capital letter C , and to make the following proportion, to wit, As $B - C$ is to $B - Q$, so, very nearly, will $c - b$ be to $r - b$; whence it will follow that $r - b$ will be $= \frac{B - Q \times \overline{c - b}}{B - C}$, and consequently that r will be $= b +$ the fraction $\frac{B - Q \times \overline{c - b}}{B - C}$, or that $b +$ the fraction $\frac{B - Q \times \overline{c - b}}{B - C}$ will be a second near value of r , or the greater root of the equation $P \times r - r^t = Q$, that will be much nearer to the true value of the said greater root than it's former near value b was. Q. E. I.

Of this method of obtaining a second near value of r , or the greater root of the equation $P \times r - r^t = Q$, we have had an example above in art. 19, pages 380, 381, and 382, where, in resolving the equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$, we took $M + \frac{d}{2}$, or $1.058,195,6$, for our first near value of r , or the greater root of the said equation, and derived from it, by the Differential method of approximation, the number $1.059,733,2$ for a second, and much more exact, value of the said greater root.

When we have obtained in this manner a second near value of the greater root of the equation $P \times r - r^t = Q$, we may, if still greater exactness is required, find a third and more exact value of it by a process of Mr. Raphson's method of approximation.

This is the substance of what has been above delivered, concerning this first and, in my opinion, best method of resolving the equation $\frac{x}{a} \times r - r^t = \frac{x}{a} - 1$, or $P \times r - r^t = Q$, by approximation.

Of

Of the Resolution of the said Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or $P \times r - r^t = Q$,

by the second of the three foregoing Methods, which is given above in Art. 23, 24, and 25, Pages 386, 387, 388, &c, - - - - - 392.

Art. 97. Let b be a first near value of r , or the greater root of the equation $P \times r - r^t = Q$, that is either equal to $M + d$, or, though less than $M + d$, is yet greater than the true value of the said greater root; which will appear by substituting it instead of r in the binomial quantity $P \times r - r^t$, and observing that $P \times b - b^t$, (or the value of the said binomial quantity resulting from such substitution,) is less than Q , or the absolute term of the equation $P \times r - r^t = Q$. Such a first near value of r , or the said greater root, must be found by means of it's limits $M + d$ and M , in the same manner as in the foregoing resolution of the said equation by the first method.

Then, since b is greater than r , or the greater root of the said equation, let w be = the excess of b above r , or the said greater root. And we shall

$$\text{have } r = b - w = b \times \overline{1 - \frac{w}{b}}.$$

$$\text{Let } x \text{ be } = \frac{w}{b}.$$

And we shall have $r (= b \times \overline{1 - \frac{w}{b}} = b \times \overline{1 - x}) = b - bx$, and $P \times r (= P \times \overline{b - bx}) = P \times b - Pb \times x$.

And we shall likewise have $r^t (= \overline{b \times \overline{1 - x}})^t = b^t \times \overline{1 - x}^t = b^t \times$ the series $1 - tx + t \times \frac{t-1}{2} \times x^2 - \&c) =$ the series $b^t - t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 - \&c$.

Therefore $Pr - r^t$ will be $(= Pb - Pb \times x - \text{the series } b^t - t \times b^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 - \&c = Pb - Pb \times x - b^t + t \times b^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c) = Pb - b^t - Pb \times x + tb^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c.$

But $Pr - r^t$ is $= Q.$

Therefore $Pb - b^t - Pb \times x + tb^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$ will also be $= Q.$

Therefore (subtracting $Pb - b^t$ from both sides,) we shall have $tb^t \times x - Pb \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = Q + b^t - Pb$, and $2tb^t \times x - 2Pb \times x - t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = 2Q + 2b^t - 2Pb$, or $\overline{2tb^t - 2Pb} \times x - \overline{t-1} \times b^t \times x^2 + \&c = 2Q + 2b^t - 2Pb$, and $\frac{2tb^t - 2Pb}{t-1} \times x - x^2 + \&c = \frac{2Q + 2b^t - 2Pb}{t-1}.$

Now let G be $= \frac{tb^t - Pb}{t-1} \times b^t$, and H be $= \frac{2Q + 2b^t - 2Pb}{t-1} \times b^t.$

And we shall have $2G \times x - x^2 + \&c = H.$

Therefore (subtracting both sides from GG ,) we shall have $GG - 2Gx + x^2 = GG - H$, and consequently $G - x = \sqrt{GG - H}$, and $G = \sqrt{GG - H} + x$, and $x = G - \sqrt{GG - H}.$

Therefore w (which is $= b \times x$,) will be $= b \times G - \sqrt{GG - H}$, and r , or $b - w$, will be $= b - b \times G + \sqrt{GG - H}$; that is, the greater root of the equation $Pr - r^t = Q$ will be nearly equal to $b - b \times G + \sqrt{GG - H}$, or the said quantity $b - b \times G + \sqrt{GG - H}$ will be a second near value of the said greater root.

Q. E. I.

Art. 98. But, if b is not only less than $M + d$, but less also than r , or the greater root of the equation $Pr - r^t = Q$ (which will be the case when, upon substituting b instead of r in the binomial quantity $Pr - r^t$, it shall be found that

that $Pb - b^t$, or the value of $Pr - r^t$ resulting from such substitution, is greater than Q , or the absolute term of the said equation,) we must proceed as follows.

Let w be the excess of r ; or the said greater root, above b . And we shall then have $r = b + w = b \times 1 + \frac{w}{b}$.

Let x be put $= \frac{w}{b}$. And we shall then have $r (= b \times 1 + \frac{w}{b} = b \times 1 + x) = b + bx$, and consequently $Pr (= P \times \overline{b + bx}) = Pb + Pb \times x$.

And we shall likewise have $r^t (= \overline{b \times 1 + x})^t = b^t \times \overline{1 + x}^t = b^t \times$ the series $1 + tx + t \times \frac{t-1}{2} \times x^2 + \&c) =$ the series $b^t + tb^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$.

Therefore $Pr - r^t$ will be $(= Pb + Pb \times x - b^t - tb^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c) = Pb - b^t + Pb \times x - tb^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c$.

But $Pr - r^t$ is $= Q$.

Therefore $Pb - b^t + Pb \times x - tb^t \times x - t \times \frac{t-1}{2} \times b^t \times x^2 - \&c$ will also be $= Q$.

Therefore (adding $tb^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$ to both sides,) we shall have $Pb - b^t + Pb \times x = Q + tb^t \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$; and (subtracting $Pb \times x$ from both sides,) we shall have $Pb - b^t = Q + tb^t \times x - Pb \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$; and (subtracting Q from both sides,) we shall have $Pb - b^t - Q = tb^t \times x - Pb \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c$, or (ranging the terms of the equation in the more usual manner, with the absolute term, or known quantity, $Pb - b^t - Q$ on the right hand,) $tb^t \times x - Pb \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = Pb - b^t - Q$.

Now tb^t is greater than Pb , as may be shewn in the following manner.

The quantity b is always greater than M , or the lower limit of the magnitude of the greater root of the equation $Pr - r^t = Q$. But M is $= \frac{P}{t} \sqrt[t-1]{t-1}$.

Therefore b must be greater than $\frac{P}{t} \sqrt[t-1]{t-1}$. Therefore b^{t-1} must be greater than $\frac{P}{t} \sqrt[t-1]{t-1}^{t-1}$, or than $\frac{P}{t}$, or than $\frac{P}{t}$. Therefore $t \times b^{t-1}$ will be greater than $t \times \frac{P}{t}$, or than P , and $t \times b^{t-1} \times b$, or tb^t , will be greater than $P \times b$, or Pb . Q. E. D.

Therefore the equation $tb^t \times x - Pb \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = Pb - b^t - Q$ will be $(tb^t - Pb) \times x + t \times \frac{t-1}{2} \times b^t \times x^2 + \&c = Pb - b^t - Q$. Therefore (multiplying all the terms into 2,) we shall have $2 \times (tb^t - Pb) \times x + t \times (t-1) \times b^t \times x^2 + \&c = 2Pb - 2b^t - 2Q$, and (dividing all the terms by $t \times (t-1) \times b^t$, we shall have $\frac{2 \times (tb^t - Pb)}{t \times (t-1) \times b^t} \times x + x^2 + \&c = \frac{2Pb - 2b^t - 2Q}{t \times (t-1) \times b^t}$, or $\frac{2 \times (tb^t - Pb)}{(t-1) \times b^t} \times x + x^2 + \&c = \frac{2Pb - 2b^t - 2Q}{(t-1) \times b^t}$.

Now let G be $= \frac{tb^t - Pb}{(t-1) \times b^t}$, and H be $= \frac{2Pb - 2b^t - 2Q}{(t-1) \times b^t}$.

And we shall then have $2G \times x + x^2 + \&c = H$.

Therefore (adding GG to both sides,) we shall have $GG + 2G \times x + x^2 = H + GG$, and consequently $G + x = \sqrt{H + GG}$, and $x = \sqrt{H + GG} - G$.

Therefore w (which is $= b \times x$), will be $= b \times \sqrt{H + GG} - G$, and consequently r , or $b + w$, will be $= b + b \times \sqrt{H + GG} - G$, or $b + b \times \sqrt{H + GG}^{\frac{1}{2}} - G$; that is, the greater root of the equation $Pr - r^t = Q$ will be $= b + b \times \sqrt{H + GG}^{\frac{1}{2}} - G$, or the said quantity $b + b \times \sqrt{H + GG}^{\frac{1}{2}} - G$ will be a second near value of the said greater root. Q. E. I.

Of

Of this method of resolving an equation of this form, $Pr - r^t = Q$, an example has been given above in art. 25, pages 391, 392, in the case of the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$; where, taking 1.058,195,6 for b , or the first near value of r , or the greater root of the said equation, we found that x would be $= 0.001,703,3$, and that w , or $b \times x$, would be $(= 1.058,195,6 \times 0.001,703,3) = 0.001,802,4$, and that r , or $b + w$, would be $(= 1.058,195,6 + 0.001,802,4) = 1.059\,998,0$; which is exceedingly near to 1.06, or the true value of the said greater root. So that this second method of resolving equations of this form seems to be a very exact one.

Art. 99. The directions given in art. 97 for finding a second near value of r , or the greater root of the equation $Pr - r^t = Q$, when b , or it's first near value, is either equal to $M + d$, or less than $M + d$, but yet greater than the said greater root, or when the binomial quantity $Pb - b^t$ is less than the absolute term Q , may be reduced to a narrower compass in the following manner.

Let G be $= \frac{tb^t - Pb}{n-1 \times b^t}$, and H be $= \frac{2Q + 2b^t - 2Pb}{n-1 \times b^t}$, and let w be $= b \times \sqrt{G - \sqrt{GG - H}}$. Then will $b - w$, or $b - b \times \sqrt{G - \sqrt{GG - H}}$, be a second near value of r , or the said greater root, that will be much nearer to it's true value than it's first value b was. Q. E. I.

Art. 100. And the directions given in art. 98 for finding a second near value of r , or the greater root of the said equation $Pr - r^t = Q$, when b , or it's first near value, is less than the said greater root, or when the binomial quantity $Pb - b^t$ is greater than the absolute term Q , may be reduced to a narrower compass in the following manner.

Let G be $= \frac{tb^t - Pb}{n-1 \times b^t}$, and H be $= \frac{2Pb - 2b^t - 2Q}{n-1 \times b^t}$; and let w be $= b \times \sqrt{\sqrt{H + GG} - G}$, or $b \times \sqrt{H + GG}^{\frac{1}{2}} - G$. Then will $b + w$, or $b + b \times \sqrt{H + GG}^{\frac{1}{2}} - G$, be a second near value of r , or the said greater root, that will be much nearer to it's true value than it's first value b was.

Q. E. I.

Of

Of the Resolution of the said Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, or $Pr - r^t = Q$ by the last of the three foregoing Methods, which is given above in Art. 26, 27, 28, and 29, Pages 393, 394, &c. - - - - 398.

Art. 101. Let p be a first near value of r , or the greater root of the equation $Pr - r^t = Q$ that is less than the true value of the said greater root; so that the binomial quantity $Pp - p^t$ will be greater than the absolute term Q . Such a first near value of r must be found by means of it's limits $M + d$ and M in the same manner as the first near value of it, denoted by b , is to be found in the two foregoing methods of resolving this equation.

Further, let r be $= p + w$; and let y be $= \frac{w}{p}$; whence we shall have $w = py$, and $r (= p + w) = p + py$.

Then, by the reasonings used above in art. 28 and 29, pages 394, 395, 396, and 397, we shall obtain the following equation, to wit, $y + t \times \frac{t-1}{2} \times$

$$\begin{aligned} & \frac{p^t}{tp^t - Pp} \times y^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp} \times y^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \\ & \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp} \times y^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{tp^t - Pp} \times y^5 \\ & + \&c = \frac{Pp - p^t - Q}{tp^t - Pp}. \end{aligned}$$

Now let q be put for the absolute term $\frac{Pp - p^t - Q}{tp^t - Pp}$ of this equation; and

let b be put for $t \times \frac{t-1}{2} \times \frac{p^t}{tp^t - Pp}$, or the numeral co-efficient of y^2 ;

and c be put for $t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{p^t}{tp^t - Pp}$, or $b \times \frac{t-2}{3}$, or the numeral co-efficient of y^3 ;

and d be put for $t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{p^t}{tp^t - Pp}$, or $c \times \frac{t-3}{4}$, or the numeral co-efficient of y^4 ; and e be put for

$t \times$

$$t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{p^t}{tp^t - Pp}, \text{ or } d \times \frac{t-4}{5}, \text{ or the nu-}$$

meral coefficient of y^5 . And the foregoing equation will be converted into the equation $y + by^2 + cy^3 + dy^4 + ey^5 + \&c = q$. And therefore, by virtue of the first Theorem of Sir Isaac Newton concerning the reversion of infinite serieses in his Letter to Mr. Oldenburgh, dated Oct. 24, 1676, we shall have $y = q - b \times q^2 + \overline{2bb - c} \times q^3 - \overline{5b^3 - 5bc + d} \times q^4 + \overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times q^5 - \&c$.

Therefore these five terms of this series, or so many of them as shall be judged to be necessary in order to obtain the value of y to a sufficient degree of exactness, must be computed; and the value of y thereby obtained must be multiplied into p , in order to obtain the value of w . And, when the value of w has been thus found, $p + w$ will be a second near value of r , or the greater root of the equation $Pr - r^t = Q$, that will be much nearer to it's true value than it's former near value p was. Q. E. I.

An example of the resolution of an equation of the aforesaid form $Pr - r^t = Q$ by this third method has been given above in art. 30, pages 398, 399, and 400, in the case of the numeral equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$; where, taking 1.058,195,6 for p , or the first near value of r , or the greater root of the said equation, we found that y would be $= 0.001,699,4$, and w , or $p \times y$, would be $(= 1.058,195,6 \times 0.001,699,4) = 0.001,798,2$, and that r , or $p + w$, would be $(= 1.058,195,6 + 0.001,798,2) = 1.059,993,8$; which is very nearly equal to 1.06, or the true value of the said greater root.

And in resolving this equation $967.648,136,7 \times r - r^{70} = 966.648,136,7$ we computed only the four first terms of the series which is equal to y . If we had computed the fifth term $\overline{3c^2 - 21bbc + 6bd + 14b^4 - e} \times q^5$, the result would have been still more exact.

Of the Resolution of the Equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, by the Method set forth above in Art. 37, 38, 39, &c, - - - 63, Pages 413, 414, 415, &c, - - - - 436.

Art. 102. In this equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ the lesser value of r is $= 1$, as is shewn in art. 37, page 413. And the greater value of r is that which

which is necessary to be found in order to the solution of the Problem stated in art. 33, pages 408 and 409, from which the said equation was deduced.

To find this greater root of the said equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ by the method set forth in this Appendix, we may proceed as follows.

Compute the value of $\frac{z+a}{z}$, the co-efficient of r^t , and call it P; and compute the value of the absolute term $\frac{a}{z}$, and call it Q.

Then will the equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$ be converted into the equation $Pr^t - r^{t+1} = Q$, in which the notation is as simple as possible.

In the next place we must compute the value of $\frac{t}{t-1} \times P$, and call it M.

Then will r , or the greater root of the equation $Pr^t - r^{t+1} = Q$ be less than P, but greater than $\frac{t}{t+1} \times P$, or M; or P and M will be the two limits of the magnitude of the said greater root.

And hence it follows that an arithmetical mean quantity between these two limits P and M, or P and $\frac{t}{t+1} \times P$, or $\frac{t+1}{t+1} \times P$ and $\frac{t}{t+1} \times P$, to wit,

the quantity $\frac{t + \frac{1}{2}}{t+1} \times P$, or $\frac{2t+1}{2t+2} \times P$, will be nearly equal to the said greater root, or may be taken for it's first near value, and employed with success to discover a second and more exact value of it by the Differential method, or Mr. Raphson's method, or some other method, of approximation. And thus we may easily obtain a good first near value of the greater root of this equation $Pr^t - r^{t+1} = Q$ even without a previous knowledge of the lesser root of this equation.

But, as in the present case we know that the lesser root of the equation $Pr^t - r^{t+1} = Q$ is $= 1$, it will be expedient to make use of that knowledge to discover another near value of r , or the greater root of the said equation, which will often be as near to it's true value as the quantity $\frac{2t+1}{2t+2} \times P$. For this purpose subtract 1 (the lesser root of this equation,) from M, or $\frac{t}{t-1} \times P$, and call the remainder d . And $M + \frac{d}{2}$ will often be found to be very nearly equal to the greater root of the said equation.

Now

Now let b be put for $\frac{2t+1}{2t+2} \times P$, and be substituted instead of r in the binomial quantity $Pr^t - r^{t+1}$; and let $P \times b^t - b^{t+1}$, or the result of the said substitution, be called B . And, in like manner, let c be put for $M + \frac{d}{2}$, and be substituted instead of r in the binomial quantity $Pr^t - r^{t+1}$; and let $P \times c^t - c^{t+1}$, or the result of the said substitution, be denoted by the capital letter C .

Then say, (according to the directions of the Differential method of approximation,) As the difference of the two known quantities B and C is to the difference of the two known quantities Q and C , so, very nearly, will the difference of the two known quantities b and c (corresponding to B and C ,) be to the difference of the unknown quantity r , (or the greater root of the said equation $Pr^t - r^{t+1} = Q$,) and the known quantity c , which correspond to the absolute term Q and the quantity C . And by means of this proportion we shall obtain a second near value of r , or the said greater root, that will be much nearer to it's true value than either of the two quantities $\frac{2t+1}{2t+2} \times P$ and $M + \frac{d}{2}$ was.

Q. E. I.

Having thus found a second near value of r , or the greater root of the equation $Pr^t - r^{t+1} = Q$, by means of the Differential method of approximation, call the said value e , and substitute it instead of r in the binomial quantity $Pr^t - r^{t+1}$, and let $P e^t - e^{t+1}$, or the result of the said substitution, be called E . Then, if E is less than Q , or the absolute term of the equation $Pr^t - r^{t+1} = Q$, the quantity e will be greater than r , or the greater root of the said equation; and, if E is greater than Q , the said quantity e will be less than r , or the said greater root. And, if E is very nearly equal to Q , as, for example, if it differs from Q by less than the 100th part of Q , we may conclude that e will be very nearly equal to r , or the said greater root, and so nearly as to make it unnecessary to seek for a third and more exact value of it. But, if it is deemed expedient to seek for a third and more exact value of the said greater root, we may easily find such more exact value of it by putting w for the unknown difference between e and the true value of r , or the said greater root, and substituting $e + w$, (if e is less than r ,) or $e - w$, (if e is greater than r ,) instead of r in the equation $Pr^t - r^{t+1} = Q$, and resolving the transformed equation resulting from such substitution, as if it were a mere simple

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equation,

equation, according to the directions of Mr. Raphson's method of approximation. And the value of $e + w$, or $e - w$, obtained by this means will be a third near value of r , or the greater root of the equation $Pr^t - r^{t+1} = Q$, that will be much more exact than e , it's second value. Q. E. I.

Art. 103. Two numeral equations of this form, $Pr^t - r^{t+1} = Q$ have been resolved in the foregoing articles by the processes just now described, with some little variation in the manner of obtaining the second near value of r by means of the Differential method of approximation in the first of them from that contained in the foregoing directions; to wit, the equations $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$ and $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, in the first of which the exact value of r , or the greater root of the equation, is 1.068,14, and in the second it is 1.06.

Art. 104. The former of these equations, to wit, $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, is resolved in art. 49, 50, 51, 52, and 53, pages 424, 425, 426, &c, - - - 430, by means of the following processes.

We, first, (in art. 49,) computed the expression $\frac{2t+1}{2t+2} \times P$, or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$, and found it to be $= 1.066,115,6$, and concluded that the said number 1.066,115,6 would be nearly equal to r , or the greater root of the said equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$.

We then computed $M + \frac{d}{z}$, or $\frac{3t}{2t+2} \times P - \frac{1}{z}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{z}$, (which has been shewn above, in art. 45, page 421, to be $= M + \frac{d}{z}$), and found it to be $= 1.061,983,3$, and concluded that the said number 1.061,983,3 would also be nearly equal to r , or the greater root of the said equation.

We then substituted the number 1.066,115,6 instead of r , in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$; and we found the value of the said binomial quantity resulting from this substitution to be $= 0.095,109,5$ &c, which is greater than 0.090,909,0, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$. And hence we concluded that the number 1.066,115,6 would be less than the true value of r , or the greater root of the said equation.

Having thus found that 1.066,115,6 was less than the said greater root, we concluded that the other near value of r , or the said greater root, (which was derived

derived from the expression $M + \frac{d}{z}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{z}$,) to wit, the number 1.061,983,3, which is considerably less than 1.066,115,6, must be in the same degree more remote from the true value of the said greater root than the number 1.066,115,6; and therefore we did not think fit to substitute the number 1.061,983,3, instead of r , in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, (as we had done the number 1.066,115,6,) but increased the number 1.066,115,6 by the very small quantity 0.001,377,4, (which is only the third part of 0.004,132,3, or of the difference of the two numbers 1.066,115,6 and 1.061,983,3,) whereby we obtained the number 1.067,493,0, for another near value of r , or the said greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, that would be much nearer than 1.061,983,3 to the true value of the said greater root. And this new number 1.067,493,0 we substituted, instead of r , in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, and found the value of the said quantity resulting from such substitution to be $= 0.092,294,9$ &c; which is a little greater than 0.090,909,0, or the absolute term of the said equation: whence it followed that 1.067,493,0 must be a little less than the true value of r , or the said greater root.

Having thus obtained two pretty near values of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, to wit, the numbers 1.066,115,6 and 1.067,493,0, and having also found the correspondent values of the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, to wit, 0.095,109,5 and 0.092,294,9, we proceeded to find a second near value of r , or the said greater root, that would be much nearer to its true value than either 1.066,115,6 or 1.067,493,0, by the Differential method of approximation, by means of the following proportion; As 0.095,109,5 — 0.092,294,9 is to 0.092,294,9 — 0.090,909,0, so, very nearly, will 1.067,493,0 — 1.066,115,6 be to r — 1.067,493,0, or as 0.002,814,6 is to 0.001,385,9, so, very nearly, will 0.001,377,4 be to r — 1.067,493,0. And hence it followed that r — 1.067,493,0 would be, very nearly, $(= \frac{0.001,385,9 \times 0.001,377,4}{0.002,814,6} = \frac{0.000,001,377,4}{0.002,814,6}) = 0.000,678,2$, and consequently that r would be, very nearly, $(= 0.000,678,2 + 1.067,493,0) = 1.068,171,2$. Therefore the second near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, will be 1.068,171,2. Q. E. I.

We then, in art. 51, substituted this last number 1.068,171,2 instead of r in the binomial quantity $1.090,909,0 \times r^{21} - r^{22}$, and found that the value of the said quantity resulting from such substitution would be $= 0.090,824,9$ &c; which is a little less than 0.090,909,0, or the absolute term of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$: and hence we concluded that 1.068,171,2 would be a little greater than r , or the greater root of the said equation.

We then, in art. 52, pages 428, 429, put w for the excess of 1.068,171,2 above r , or the true value of the said greater root, and substituted the binomial quantity 1.068,171,2 — w instead of r in the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, and resolved the transformed equation retaking from such substitution as if it had been a mere simple equation, agreeably to the directions of Mr. Raphson's method of approximation, and thereby found w to be $= 0.000,038,0$, and consequently r , or 1.068,171,2 — w , to be ($= 1.068,171,2 - 0.000,038,0$) $= 1.068,133,2$; or obtained the number 1.068,133,2 for a third near value of r , or the greater root of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$. Q. E. I.

Art. 105. The second of these equations, to wit, the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, is resolved in art. 54, 55, 56, &c, - - - 63, pages 431, 432, 433, &c, - - - 436, by means of the following processes.

In this equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, we have $t = 40$, and P , or $\frac{z+a}{z}$, $= 1.066,461,5$, and Q , or $\frac{a}{z} = 0.066,461,5$. And hence we found $\frac{2t+1}{2t+2} \times P$, or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$, to be $= 1.053,455,8$, which we therefore considered as a first near value of r , or the greater root of this equation.

We then computed the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, and found it to be $= 1.060,675,3$, which we therefore considered as being also a first near value of r , or the greater root of the said equation.

These two near values of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, were obtained with great ease; and the latter of them, to wit, the number 1.060,675,3 (which was derived from the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$), is exceedingly near the true value of the said greater root, which we know to be 1.06. But this will not always be the case; as we have seen in the foregoing equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, in which the number 1.061,983,3 (derived from the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$), was much more distant from 1.068,14 (which was the true value of r , or the greater root of that equation,) than the number 1.066,115,6, which was derived from the expression $\frac{2t+1}{2t+2} \times P$, or $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$. And I am inclined to think that in many equations of this form $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, or $P r^t - r^{t+1} = Q$, the former

former near value of r , that is derived from the expression $\frac{2t+1}{2t+2} \times \frac{z+a}{z}$, or $\frac{2t+1}{2t+2} \times P$, will be nearer the truth than the latter near value of it, which is derived from the expression $M + \frac{d}{2}$, or $\frac{3t}{2t+2} \times \frac{z+a}{z} - \frac{1}{2}$, or $\frac{3t}{2t+2} \times P - \frac{1}{2}$.

Having thus obtained the two numbers 1.053,455,8 and 1.060,675,3 for two probable near values of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, we substituted, (in art. 60, page 432,) first, the number 1.053,455,8, and, afterwards, the number 1.060,675,3, instead of r , in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$; and we found the value of the said quantity resulting from the first of these substitutions to be $= 0.105,420,5$ &c, and the value of it resulting from the latter substitution to be $= 0.061,050,5$ &c.

The former of these values, to wit, 0.105,420,5 &c, is much greater than 0.066,461,5, or the absolute term of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and consequently the number 1.053,455,8 (which corresponds to the said value,) must be considerably less than r , or the greater root of the said equation. And the latter of these values, to wit, 0.061,050,5 &c, is somewhat less than the absolute term 0.066,461,5; and consequently the number 1.060,675,3 (which corresponds to the said value,) must be somewhat greater than r , or the greater root of the said equation. But it will be much nearer than the number 1.053,455,8 to the true value of the said greater root. And therefore, if we had resolved to make use of only one of the said two numbers 1.053,455,8 and 1.060,675,3 as a ground-work for a further approach to the true value of r , or the said greater root, by a process of the Differential method of approximation, the latter number 1.060,675,3 ought certainly to have been chosen for that purpose. But then it would have been necessary to diminish the said number 1.060,675,3 (which we know to be somewhat greater than the truth,) by some small quantity, (as, for instance, by a 200th part of itself, or by $\frac{1.060,675,3}{200}$, or 0.005,303,3,) and then to substitute

the remainder, (to wit, 1.055,472,0,) instead of r , in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$, in order to obtain the value of the said binomial quantity arising from such substitution, which value would have been necessary, as well as the number 0.061,050,5 &c (or the value of the said binomial quantity resulting from the substitution of 1.060,675,3 in it's terms, instead of r), towards the application of the Differential method of approximation. Now these substitutions are (as we have seen,) rather tedious and troublesome operations. And therefore, to avoid the trouble of making this new substitution, I thought it expedient to make use of the number 1.053,455,8 (though less exact than I could have wished,) as well as the number 1.060,675,3, together with the

the two numbers 0.105,420,5 &c and 0.061,050,5, &c, (resulting from the substitution of the said numbers 1.053,455,8 and 1.060,675,3, instead of r , in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$;) in the application of the Differential method of approximation, in art. 61, page 434. And we thereby obtained ($1.060,675,3 - 0.000,880,4$, or) 1.059,794,9 for a second near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$.

Q. E. I.

We then (in art. 62, page 434,) substituted this number 1.059,794,9 instead of r , in the binomial quantity $1.066,461,5 \times r^{40} - r^{41}$, and found the value of the said quantity resulting from such substitution to be $= 0.068,041,8$ &c, which is somewhat greater than 0.066,461,5, or the absolute term of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$; and we thence concluded that the number 1.059,794,9 must be somewhat less than the true value of r , or the greater root of the said equation.

We then, (in art. 63,) in order to obtain a third near value of the said greater root, that should be still more exact than 1.059,794,9, or the second near value of it obtained by means of the Differential method of approximation, put w for the unknown excess of the said greater root above 1.059,794,9, and substituted the binomial quantity $1.059,794,9 + w$, instead of r , in the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, and resolved the transformed equation resulting from such substitution, as if it had been a mere simple equation, agreeably to the directions of Mr. Raphson's method of approximation, and thereby found that w would be nearly $= 0.000,242,2$, and consequently that r , or $1.059,794,9 + w$, would be, nearly, $(= 1.059,794,9 + 0.000,242,2) = 1.060,037,1$, or that 1.060,037,1 would be a third near value of r , or the greater root of the equation $1.066,461,5 \times r^{40} - r^{41} = 0.066,461,5$, that would be very near the truth.

Q. E. I.

Of the Resolution of the trinomial Equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ by the Method set forth above in Art. 67, 68, 69, &c, - - - 81, Pages 439, 440, 441, &c, - - - 455.

Art. 106. In this equation (which has two roots,) the lesser root is $= 1$. For, if r is $= 1$, we shall have $r^x = 1$, and $r^{x+t} = 1$, and $r^{x+t+1} = 1$, and consequently $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} (= \frac{a}{z} \times 1 + 1 - 1 = \frac{a}{z} \times 1) = \frac{a}{z}$, which is the absolute term of the equation $\frac{a}{z} \times r^x +$

$+ r^{x+t} - r^{x+t+1} = \frac{a}{z}$. But this lesser root of this equation will not enable us to solve the Problem of Dr. Halley, stated in art. 64, page 436, from which the said trinomial equation was deduced. And therefore it will be necessary to find the second, or greater, root of that equation. And for this purpose it will be expedient in the first place to find the limits of the magnitude of the said greater root.

Now it is shewn above, in art. 78, 79, that the greater root of the said equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ must be less than the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, (which evidently has but one root,) and greater than the root of the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1}$, which likewise has evidently but one root; that is, the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ will be the greater limit, and the root of the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1}$ will be the lesser limit, of the magnitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$.

Further, it is shewn, in the same art. 79, that the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ is, nearly, $= 1 + \frac{\sqrt{\frac{4/a}{z} + 1} - 1}{2t}$. Let this root, or the greater limit of the magnitude of r , or the greater root of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, be called A.

In the next place we must resolve the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1}$, and call it's root M. This may be done as follows.

Put $f = \frac{x+t}{x+t+1}$, the co-efficient of r^t ; and put $g =$ the absolute term $\frac{x}{x+t+1}$. And the equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1}$ will thereby be converted into the equation $r^{t+1} - fr^t = g$.

Now

Now let $1 + v$ be substituted instead of r in this equation. And we may thence derive the quadratick equation $vv + 2 \times \frac{t + 1 - ft}{t^2 - ft^2 + t + ft} = \frac{2g + 2f - 2}{t^2 - ft^2 + t + ft}$.

Now let b be put $= t + 1 - ft$, and i be $= 2g + 2f - 2$, and k be $= t^2 - ft^2 + t + ft$. And we shall then have $vv + 2 \times \frac{b}{k} = \frac{i}{k}$. Therefore (adding $\frac{b^2}{k^2}$ to both sides,) we shall have $vv + 2 \times \frac{b}{k} + \frac{b^2}{k^2} (= \frac{i}{k} + \frac{b^2}{k^2} = \frac{ik}{k^2} + \frac{b^2}{k^2}) = \frac{ik + b^2}{k^2}$; and (extracting the square-roots of both sides,) $v + \frac{b}{k} = \frac{\sqrt{ik + b^2}}{k}$, and consequently $v = \frac{\sqrt{ik + b^2}}{k} - \frac{b}{k}$, or $\frac{\sqrt{ik + b^2} - b}{k}$, and r , or $1 + v$, $= 1 + \frac{\sqrt{ik + b^2} - b}{k}$; that is, the root of the binomial equation $r^{t+1} - \left[\frac{x+t}{x+t+1} \times r^t = \frac{x}{x+t+1} \right]$ will be, nearly, $= 1 + \frac{\sqrt{ik + b^2} - b}{k}$. Therefore M, or the lower limit of the magnitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, will be, nearly, $= 1 + \frac{\sqrt{ik + b^2} - b}{k}$. Q. E. I.

We have therefore now found that A, or the higher limit of the magnitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ will be, nearly, $= 1 + \frac{\sqrt{\left[\frac{4ta}{z} + 1 \right]} - 1}{2t}$, and that M, or the lower limit of the magnitude of the said greater root will be, nearly, $= 1 + \frac{\sqrt{ik + b^2} - b}{k}$.

Art. 107. Having thus found a near value of M, the lower limit of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, we must substitute it instead of r in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, and compare the trinomial quantity $\frac{a}{z} \times M^x + M^{x+t} - M^{x+t+1}$, resulting from such substitution, with $\frac{a}{z}$, or the absolute term of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, and from that comparison derive
a con-

a conjectural first near value of the greater root of the said trinomial equation, and call it b .

And then we must subtract 1 (or the lesser root of the said trinomial equation,) from M , and put $d =$ the remainder $M - 1$. And $M + \frac{d}{2}$ will be another first near value of the said greater root, which will not differ much from its true value. Call this near value of it c .

Then substitute b and c successively, instead of r , in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, and call the value of the said trinomial quantity resulting from the first substitution B , and the value of it resulting from the second substitution C ; and find a second near value of the greater root of the said trinomial equation by the Differential method of approximation by means of the following proportion; As the difference of B and C is to the difference of B and $\frac{a}{z}$, so, very nearly, will be the difference of b and c to the difference of b and r , or the greater root of the said trinomial equation. And we shall thereby obtain a second near value of the said greater root that will be more exact than either b or c . Q. E. I.

Let this second near value of r be called e ; and let it be substituted, instead of r , in the trinomial quantity $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, and let the value of the said quantity resulting from such substitution be called E .

Then, if the quantity E differs but little from $\frac{a}{z}$, (or the absolute term of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1}$, as, for example, if it differs from it by less than a 100th part of $\frac{a}{z}$, we may conclude that e will be very nearly equal to the true value of r , or the greater root of the said trinomial equation. But, if it differs from it by a greater quantity, or, in general, if a doubt is entertained whether e is sufficiently near to the true value of the said greater root, we must then have recourse to Mr. Raphson's method of approximation to obtain a third near value of the said greater root that shall be much nearer to its true value than e was; and, for this purpose, we must put w for the unknown difference between e and the said true value; and, if E is greater than the absolute term $\frac{a}{z}$, and consequently e is less than the true value of r , we must substitute $e + w$ instead of r in the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$; and, if E is less than the absolute term $\frac{a}{z}$, and conse-

quently e is greater than the true value of r , we must substitute $e - w$ instead of r in the said trinomial equation; and we must then resolve the transformed equation thereby obtained, (of which w will be the root,) as if it were a mere simple equation, omitting all the terms of it that involve any higher power of w than it's simple power, or w itself. And we shall thereby obtain a third near value of r , or the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, that will be much nearer than e to it's true value, and as exact as any one would desire.

Q. E. I.

Art. 108. A numeral equation of this form $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, to wit, the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, has been resolved in art. 84, 85, 86, &c, - - - 91, pages 456, 457, 458, &c, - - - 466, by the processes just now described. The several steps of this resolution were as follows.

In art. 84, (page 456,) it is shewn that in this case the general binomial equation $r^{t+1} - r^t = \frac{a}{z}$ will be converted into the numeral equation $r^{22} - r^{21} = 0.500,000$, and that the general binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$ will be converted into the numeral equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$. Therefore A, or the higher limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ will be the root of the binomial equation $r^{22} - r^{21} = 0.500,000,0$; and M, or the lower limit of the greater root of the said trinomial equation, will be the root of the binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$. In order, therefore, to discover the said limits A and M, it is necessary to resolve the two binomial equations $r^{22} - r^{21} = 0.500,000,0$ and $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$.

In art. 85, page 457, the value of A, or the greater of the said limits, is determined by computing the expression $1 + \frac{\sqrt{\frac{4za}{z} + 1} - 1}{2t}$, to which it had been shewn above in art. 79 that the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ would be nearly equal. And by this computation of the expression $1 + \frac{\sqrt{\frac{4za}{z} + 1} - 1}{2t}$ it appears that A, or the said greater limit, will be nearly =

1.1323.

Q. E. I.

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In art. 86, pages 457, 458, the value of M , or the lesser of the said limits, is determined by resolving the second binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, of which it is the root. This resolution is performed in the following manner.

We substituted $1 + v$ instead of r in the terms of the said binomial equation by means of Sir Isaac Newton's binomial Theorem, and by that substitution transformed it into another equation of which v was the root, which (by omitting all the terms involving v^3, v^4, v^5 , &c, as being of inconsiderable magnitude,) was ultimately reduced to the quadratic equation $vv + 0.0600 \times v = 0.004,535,1$; and then, by resolving this quadratic equation, we found v to be $= 0.0437$, and consequently $1 + v$, or r , or the root of the binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, to be nearly $= 1.0437$; that is, we found M , or the lower limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,000,0$, to be nearly $= 1.0437$. Q. E. I.

This value of M might also have been obtained by computing the Algebraick expression $1 + \frac{\sqrt{ik + b^2} - b}{k}$, given above in art. 106, page 488, in which b is $= t + 1 - ft$, and i is $= 2g + 2f - 2$, and k is $= t^2 - ft^2 + t + ft$, and f is $= \frac{x + t}{x + t + 1}$, and g is $= \frac{x}{x + t + 1} \times \frac{a}{z}$.

Having thus discovered that A , or the higher limit of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ is $= 1.1323$, and that its lower limit, M , is $= 1.0437$, we proceeded, in art. 87, page 459, to substitute 1.0437 , or the value of M , instead of r in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, and we found the value of the said trinomial quantity resulting from such substitution to be $= 0.602,317,9$; which therefore we concluded to be the greatest possible magnitude of the said trinomial quantity during the increase of r from 0 to A , or 1.1323 , which is the greatest possible magnitude of r in the said trinomial quantity.

We then, in art. 88, page 460, obtained a first near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$ by making the following conjectural supposition, to wit, that the whole decrement of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ from $0.602,317,9$ to 0, while r increases from M , or 1.0437 , to 1.1323 , its greatest possible value in the said trinomial quantity, (or that which it has when the said trinomial quantity is $= 0$.) will be to the decrement of the said trinomial quantity from $0.602,317,9$, to $0.500,000,0$, (which is its value when r is equal to the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$.) in nearly the same proportion as the whole increment of r from M , or 1.0437 , to its greatest value A , or 1.1323 , (which increment is contemporary with the decrement of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ from $0.602,317,9$ to 0,) to the increment of r from M , or

1.0437, to the value of the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, (which increment is contemporary with the decrement of the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$ from 0.602,317,9 to 0.500,000,0,) that is, that the whole decrement 0.602,317,9 — 0, or 0.602,317,9, will be to the decrement 0.602,317,9 — 0.500,000,0, or 0.102,317,9, in nearly the same proportion as the whole increment $A - M$, or 1.1323 — 1.0437, or 0.0886, is to the increment $r - M$, or $r - 1.0437$; from which proportion it followed that $r - 1.0437$ would be, nearly, = $\left(\frac{0.0886 \times 0.102,317,9}{0.602,317,9} = \frac{0.009,065,365,94}{0.602,317,9} \right) = 0.0150$, and consequently that r would be nearly $(= 0.0150 + 1.0437) = 1.0587$; or that 1.0587 would, probably, be not very far distant from the true value of the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. And thus we obtained the number 1.0587 for a first near value of the said greater root.

We then, in art. 89, page 461, subtracted 1, or the lesser root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, from M , or 1.0437, and called the remainder thence arising d , and added $\frac{d}{2}$, or $\frac{0.0437}{2}$, or 0.0218, to M , or 1.0437; by which we obtained $(M + \frac{d}{2}$, or $1.0437 + 0.0218$, or) 1.0665 for another near value of r , or the greater root of the said equation.

And, having thus obtained the two numbers 1.0587 and 1.0665 for first near values of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, we employed them both (in art. 90, pages 462, 463, 464,) in procuring another near value of the said greater root by means of the Differential method of approximation; and for that purpose we substituted, first, the number 1.0587, and then the number 1.0665, instead of r , in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$; and we found the value of the said trinomial quantity resulting from the first of these substitutions to be $= 0.540,471,1$, and the value of it resulting from the latter substitution to be $= 0.462,534,4$. And, having found these values, we made the following proportion; to wit, As $0.540,471,1 - 0.462,534,4$ is to $0.540,471,1 - 0.500,000,0$, so, very nearly, will $1.0665 - 1.0587$ be to $r - 1.0587$; that is, as 0.077,936,7 is to 0.040,471,1, so, very nearly, will 0.0078 be to $r - 1.0587$. And from this proportion we had $r - 1.0587 (= \frac{0.040,471,1 \times 0.0078}{0.077,936,7} = \frac{0.000,315,674,58}{0.077,936,7}) = 0.004,050,3$, and consequently $r (= 0.004,050,3 + 1.0587) = 1.062,750,3$; that is, we obtained the number 1.062,750,3 for a second near value of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, that is much nearer to it's true value than either of the two former numbers 1.0587 and 1.0665.

Q. E. I.

And,

And, having thus found 1.062,750,3. for a second near value of r , or the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, we substituted it, (in art. 91, page 464,) instead of r , in the trinomial quantity $0.500,000,0 \times r^{10} + r^{31} - r^{32}$, and found the value of the said trinomial quantity resulting from such substitution to be $= 0.504,945,6$; which is a little greater than 0.500,000,0, or the absolute term of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$. And we thence concluded that 1.062,750,3 must be somewhat less than the greater root of that equation.

We then, in order to find a third near value of r , or the greater root of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, that should be still nearer to it's true value than the last value 1.062,750,3 was, put w for the unknown excess of the said greater root above 1.062,750,3, and substituted the binomial quantity 1.062,750,3 + w , instead of r , in the said equation, omitting all the terms that involved any higher powers of w than it's simple power, or w itself, agreeably to the directions of Mr. Raphson's method of approximation: and by this substitution we transformed the said equation into the equation $0.504,945,6 - 10.026,341,4 \times w = 0.500,000,0$, which, when properly reduced, becomes $10.026,341,4 \times w = 0.004,945,6$. And, by the resolution of this simple equation, we found w to be $= 0.000,493,2$, and consequently r , or 1.062,750,3 + w , to be $(= 1.062,750,3 + 0.000,493,2) = 1.063,243,5$; that is, we obtained the number 1.063,243,5 for a third near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, that will be much nearer to it's true value than the foregoing near value, 1.062,750,3, was. Q. E. I.

Art. 109. It has been observed in the preceeding article, page 491, that the value of M , or the lower limit of the magnitude of r , or the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, might have been obtained by computing the Algebräick expression $1 + \frac{\sqrt{ik + b^2} - b}{k}$, given above in art. 106, page 488, in which b is $= t + 1 - ft$, and i is $= 2g + 2f - 2$, and k is $= t^2 - ft^2 + t + ft$, and f is $= \frac{x + t}{x + t + 1}$, and g is $= \frac{x}{x + t + 1} \times \frac{a}{z}$. And I should have determined the said value by computing the said expression, if I had not thought (as is hinted above in art. 74, page 450,) that it might be obtained with more ease and expedition by resolving the particular numeral equation $r^{32} - \frac{31}{32} \times r^{21} = \frac{10}{32} \times 0.500,000,0$ or $r^{32} - 0.968,750,0 \times r^{21} = 0.156,250,0$, (which answers to the general equation $r^{t+1} - \frac{x+t}{x+t+1} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$ obtained in art. 79,) by a new application of the same reasonings as we had before employed in art. 73, (pages

(pages 448, 449, and 450,) to resolve the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, or $x^{n+1} - fx^n = g$. But, that we may be the better able to judge, which of these two methods of obtaining the value of M , or the root of the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$, is the easier and more convenient, I will now, before I finally quit the consideration of the equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, compute the value of M by means of the said general expression $1 + \frac{\sqrt{ik+b^2}-b}{k}$. This computation will be as follows.

In the case of the binomial equation $r^{t+1} - \sqrt{\frac{x+t}{x+t+1}} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$, when x is $= 10$, and t is $= 21$, as they are in the numeral equation $r^{21+1} - \sqrt{\frac{10+21}{10+21+1}} \times r^{21} = \frac{10}{10+21+1} \times 0.500,000,0$, or $r^{22} - \frac{31}{32} \times r^{21} = \frac{10}{32} \times 0.500,000,0$, or $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, we shall have $f (= \frac{x+t}{x+t+1} = \frac{10+21}{10+21+1} = \frac{31}{32}) = 0.968,750,0$, and $g (= \frac{x}{x+t+1} \times \frac{a}{z} = \frac{10}{10+21+1} \times 0.500,000,0 = \frac{10}{32} \times 0.500,000,0) = 0.156,250,0$.

Therefore ft will be $(= 0.968,750,0 \times 21) = 20.343,750,0$, and ft^2 will be $(= ft \times t = 20.343,750,0 \times 21) = 427.218,750,0$. Therefore b , or $t+1-ft$, will be $(= 21+1-20.343,750,0 = 22-20.343,750,0) = 1.656,250,0$; and i , or $2g+2f-2$, will be $(= 2 \times 0.156,250,0 + 2 \times 0.968,750,0 - 2 = 0.312,500,0 + 1.937,500,0 - 2 = 2.250,000,0 - 2) = 0.250,000,0$; and k , or t^2-ft^2+t+ft , will be $(= 21^2 - 427.218,750,0 + 21 + 20.343,750,0 = 441 - 427.218,750,0 + 21 + 20.343,750,0 = 482.343,750,0 - 427.218,750,0) = 55.125,000,0$.

Therefore ik will be $(= 0.250,000,0 \times 55.125,000,0 = \frac{55.125,000,0}{4}) = 13.781,250,0$; and b^2 will be $(= 1.656,250,0^2) = 2.743,164,062,500,00$, and $ik+b^2$ will be $(= 13.781,250,0 + 2.743,164,062,500,00) = 16.524,414,062,500,00$. Therefore $\sqrt{ik+b^2}$ will be $(= \sqrt{16.524,414,062,500,00}) = 4.065,023,2$, and $\sqrt{ik+b^2}-b$ will be $(= 4.065,023,2 - 1.656,250,0) = 2.408,773,2$, and $\frac{\sqrt{ik+b^2}-b}{k}$ will be $(= \frac{2.408,773,2}{55.125,000,0}) = 0.043,$

0.043,696,5. Therefore $1 + \frac{\sqrt{ik + b^2} - b}{k}$ will be ($= 1 + 0.043,696,5$) $= 1.043,696,5$; that is, M, or the root of the binomial equation $r^{22} - 0.968,750,0 \times r^{21} = 0.156,250,0$, or $r^{t+1} - \frac{x+t}{x+t+1} \times r^t = \frac{x}{x+t+1} \times \frac{a}{z}$, will be $= 1.043,696,5$, or, very nearly, 1.0437; which is the value of M found above by the other method of obtaining it, in art. 86, pages 457, 458.

End of the Recapitulation (begun in Art. 93, Page 467,) of the Methods of resolving

the three Equations $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, $\frac{z+a}{z} \times r^t - r^{t+1}$

$$= \frac{a}{z}, \text{ and } \frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z},$$

contained in this Appendix to Dr. Halley's

Discourse on Compound Interest.

Another Problem relating to Compound Interest, that is not mentioned by Dr. Halley in his foregoing Discourse on that Subject, but is nearly connected with the last of the three foregoing Problems relating to the Prolongation of an Annuity for an additional Number of Years beyond the Term for which it was originally granted.

Art. 110. Instead of supposing the annuity a , that has been originally granted for a certain number of years of which t years are still to come, to be prolonged for another term of x years which is to begin at the expiration of the said first term, let us suppose that the said annuity is to be made perpetual at the end of the said first term, in consideration of a sum of z pounds that has been agreed upon between the grantor of the annuity and the purchaser of it as the price of such prolongation, or perpetuation. And let it be required, from the knowledge of the quantity, or annual value, of the annuity so granted in perpetuity, after the expiration of the said first term, and of the price z that has been paid for the said prolongation, or perpetuation, of it, to find the rate of Interest that has been allowed to the purchaser for his money.

This Problem may be considered as the extreme case of the last-mentioned Problem of Dr. Halley concerning the prolongation of a given annuity for a certain number of years beyond the original term for which it was granted, and may

may be solved by means of an equation that is derived from the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, which relates to that Problem. The manner of deriving such an equation from the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ is as follows.

Let all the terms of the equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$ be divided by r^x . And we shall then have $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$. Now let

x , or the number of years during which the annuity is to be prolonged beyond the original term, of which t years are still unexpired, be, not (as in the foregoing example,) a short term of 10 years, but some exceedingly long term, as, for instance, a term of ten thousand millions of years. Then it is evident that r^x will be a very great number, and consequently that $\frac{a}{z \times r^x}$ will be a

very small quantity in comparison to $\frac{a}{z}$ and r^t and r^{t+1} , and therefore that the magnitude of r in the equation $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$ will be nearly

the same as it would be in the equation $\frac{a}{z} + r^t - r^{t+1} = 0$; in which case

we should have $\frac{a}{z} + r^t = r^{t+1}$, and $\frac{a}{z} = r^{t+1} - r^t$, or (if we range the terms of this equation in the usual manner, with the absolute term, or known quantity, on the right hand,) $r^{t+1} - r^t = \frac{a}{z}$. And, if we suppose the

number x to be absolutely infinite, or the annuity to be continued for ever, instead of being prolonged for any certain number of years, however great, the quantity $\frac{a}{z \times r^x}$ will not be extremely small, as before, but will be absolutely

$= 0$, and the equation $\frac{a}{z} + r^t - r^{t+1} = \frac{a}{z \times r^x}$ will be converted into the equation $\frac{a}{z} + r^t - r^{t+1} = 0$, and consequently $r^{t+1} - r^t$ will be exactly $= \frac{a}{z}$. And therefore the value of r , or the rate of Interest of Money allowed

the purchaser of the perpetuity of the annuity a in the Problem just now proposed, and which the said Problem requires us to find, will be the value of r in the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, or the root of the said equation.

Art. III.

Art. 111. This Problem may also be reduced to the binomial equation $r^{t+1} - r^t = \frac{a}{z}$ in another manner, to wit, as follows.

It is observed in art. 66, page 438, that the value of an annuity of 1 pound a year for t years is known to be $= \frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$. Therefore the value of

an annuity of 1l. a year for $t+x$ years will be $\frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$. There-

fore the excess of the value of an annuity of 1l. a year for $t+x$ years above the value of an annuity of 1l. a year for only t years will be equal to the excess of

$\frac{1l.}{r-1} - \frac{1l.}{r^{t+x} \times r-1}$ above $\frac{1l.}{r-1} - \frac{1l.}{r^t \times r-1}$, that is, to $\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$.

But the excess of the value of an annuity of 1l. *per annum* for $t+x$ years above the value of an annuity of 1l. *per annum* for only t years is equal to the value of an annuity of 1l. a year for x years to commence at the end of t years; or to be added to an annuity of 1l. a year already granted for a term of t years. Therefore the value of such an additional annuity for x years, to be added to a like annuity of

1l. a year for t years already granted, will be $= \frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1}$.

And consequently the value of an annuity of a pounds a year for x years, to be added to a like annuity of a pounds a year for t years already granted, will be

a times the former value, or will be $= a \times \left[\frac{1l.}{r^t \times r-1} - \frac{1l.}{r^{t+x} \times r-1} \right] =$

$\frac{a}{r^t \times r-1} - \frac{a}{r^{t+x} \times r-1}$; that is, according to Dr. Halley's notation, z , (or

the price paid for the prolongation of the annuity of a pounds for the additional term of x years, beyond the term of t years for which it was originally

granted,) will be $= \frac{al.}{r^t \times r-1} - \frac{al.}{r^{t+x} \times r-1}$.

Now let x , or the number of years in the said additional term, become extremely great, as, for instance, equal to ten thousand millions of years.

And it is evident that in such case the quantity r^{t+x} will be extremely great, and consequently the sum $\frac{al.}{r^{t+x} \times r-1}$ will be extremely small. And there-

fore, if x , or the said number of years, becomes infinite, or the said additional term of years is converted into a perpetuity, the said sum $\frac{al.}{r^{t+x} \times r-1}$ will be-

come equal to nothing, and the quantity $\frac{al.}{r^t \times r-1} - \frac{al.}{r^{t+x} \times r-1}$ will become $= \frac{al.}{r^t \times r-1} - 0$, or $\frac{al.}{r^t \times r-1}$. So that $\frac{al.}{r^t \times r-1}$ will be the value of a perpetual annuity of a pounds a year that is to commence at a future period at the expiration of t years.

Art. 112. But the value of such a perpetual annuity that is to commence after a given number of years, or (as it is often expressed,) of the reversion of an estate in fee-simple after the expiration of a given number of years, may be found in a more direct manner, and without any reference to the foregoing Problem of Dr. Halley and the equation $\frac{a}{z} \times r^x + r^t - r^{t+1} = \frac{a}{z}$, (that was derived from it,) by the following train of reasoning.

The sum of 1 pound being put out at compound Interest will, in the space of one year, increase to r , and in the space of two years to r^2 , and in the space of three years to r^3 , and in the space of four years to r^4 , and, in general, in the space of t years to r^t . And hence it follows that the present value of 1 pound to be received at the end of one year will be $\frac{1l.}{r}$; and the present value of the same sum to be received at the end of two years will be $\frac{1}{r^2}$; and it's present value, when it is to be received only at the end of three years, will be $\frac{1}{r^3}$; and it's present value, when it is to be received only at the end of four years, will be $\frac{1}{r^4}$; and it's present value, when it is to be received only at the end of t years, will be $\frac{1}{r^t}$. There-

fore the sum of all the present values of several equal sums of one pound each, which are to be received at the end of one year, two years, three years, four years, and the several following years to t years inclusively, will be equal to the series $\frac{1l.}{r} + \frac{1l.}{r^2} + \frac{1l.}{r^3} + \frac{1l.}{r^4} + \&c - - - - - \frac{1l.}{r^t}$; that is, the present value of an annuity of 1 pound a year, payable at the end of every year, that is to begin immediately, (or so that the first payment of it shall be made at the end of only one year,) and that is to continue during t years, (or so that the last payment of it shall be made at the end of the t th year,) will be equal to the series $\frac{1l.}{r} + \frac{1l.}{r^2} + \frac{1l.}{r^3} + \frac{1l.}{r^4} + \&c - - - - - + \frac{1l.}{r^t}$, consisting of t terms.

And, if the said successive payments of one pound a year are to continue
for

for ever, the present value of such a perpetual annuity will be equal to the said series $\frac{1l}{r} + \frac{1l}{r^2} + \frac{1l}{r^3} + \frac{1l}{r^4} + \frac{1l}{r^5} + \frac{1l}{r^6} + \frac{1l}{r^7} + \&c$ continued *ad infinitum*.

Now it has been shewn above in page 237, that, if A, B, C, D, and E be a series of quantities of any kind in continued geometrical proportion, of which A is the greatest, and E is the least, the sum of all the terms of such geometrical progression will be equal to the quotient that arises by dividing the excess of the square of it's greatest term A above the product of the multiplication of it's greatest term but one, to wit, B, into it's least term E, by the excess of it's greatest term above it's greatest term but one, or

will be $= \frac{AA - BE}{A - B}$. Therefore the series $\frac{1l}{r} + \frac{1l}{r^2} + \frac{1l}{r^3} + \frac{1l}{r^4} + \&c$
 $\dots + \frac{1l}{r^t}$, (consisting of t terms, of which $\frac{1l}{r}$ is the greatest, and $\frac{1l}{r^t}$ is the greatest but one, and $\frac{1l}{r^t}$ is the least,) will be $= \frac{\frac{1}{r} \times \frac{1}{r} - \frac{1}{r^t} \times \frac{1}{r^t}}{\frac{1}{r} - \frac{1}{r^t}}$

$$\left(= \frac{\frac{1}{r} - \frac{1}{r} \times \frac{1}{r^t}}{1 - \frac{1}{r}} = \frac{\frac{1}{r} - \frac{1}{r^{t+1}}}{1 - \frac{1}{r}} = \frac{\frac{r^{t+1} - r}{r^{t+2}}}{\frac{r-1}{r}} = \frac{\frac{r^{t+1} - r}{r^{t+2}}}{\frac{r-1}{r} \times \frac{r^{t+1}}{r^{t+2}}} = \right.$$

$$\left. \frac{r^{t+1} - r}{r-1 \times r^{t+1}} \right) = \frac{r^t - 1}{r^t \times r - 1}. \text{ And the series } \frac{1l}{r} + \frac{1l}{r^2} + \frac{1l}{r^3} + \frac{1l}{r^4} + \frac{1l}{r^5}$$

$$+ \frac{1l}{r^6} + \frac{1l}{r^7} + \&c \text{ ad infinitum will be } \left(= \frac{AA - BE}{A - B} = \frac{AA - B \times 0}{A - B} = \frac{AA}{A - B} \right.$$

$$\left. = \frac{\frac{1}{r} \times \frac{1}{r}}{\frac{1}{r} - \frac{1}{r^2}} = \frac{\frac{1}{r}}{1 - \frac{1}{r}} = \frac{\frac{1}{r}}{\frac{r-1}{r}} \right) = \frac{1}{r-1}. \text{ Therefore the excess of}$$

$$\text{the latter series above the former will be } = \frac{1}{r-1} - \left(\frac{r^t - 1}{r^t \times r - 1} \right) \left(= \frac{r^t}{r^t \times r - 1} \right.$$

$$\left. - \frac{\sqrt{r^t - 1}}{r^t \times r - 1} = \frac{r^t - \sqrt{r^t - 1}}{r^t \times r - 1} = \frac{r^t - r^t + 1}{r^t \times r - 1} \right) = \frac{1}{r^t \times r - 1}.$$

But the said excess of the latter series above the former series is the excess of the value of a perpetual annuity of 1*l.* a year, that is to commence immediately, (or of which the first payment is to be received at the end of a year,) above the value of a like annuity of one pound a year that is also to commence immediately, but is to continue only during *t* years, and consequently must be equal to the value of a perpetual annuity of one pound a year that is to commence at the expiration of *t* years, or of which the first payment is to be received at the end of *t* + 1 years. Therefore the value of a perpetual annuity of one pound a year, that is to commence at the end of *t* years, or of which the first payment is to be received at the end of *t* + 1 years, will be $= \frac{1l.}{r^t \times r - 1}$.

And consequently the value of a perpetual annuity of *a* pounds a year, that is to commence at the end of *t* years, or of which the first payment is to be received at the end of *t* + 1 years, will be equal to *a* times $\frac{1l.}{r^t \times r - 1}$, or to

$$\frac{al.}{r^t \times r - 1} \quad Q. E. D.$$

Art. 113. Now let *z* be put for the value of a perpetual annuity of *a* pounds a year that is to commence at the end of *t* years, or of which the first payment is to be received at the end of *t* + 1 years, or for the value of a reversion of an estate in fee-simple after the expiration of *t* years. And we shall then have

$$z = \frac{al.}{r^t \times r - 1} \quad \text{Therefore (multiplying both sides into } r - 1,) \text{ we shall}$$

$$\text{have } (r - 1) \times z = \frac{al.}{r^t}, \text{ that is, } zr - z \text{ will be } = \frac{al.}{r^t}; \text{ and (multiplying}$$

$$\text{all the terms into } r^t,) \text{ we shall have } zr^{t+1} - zr^t = a, \text{ and consequently (dividing all the terms by } z,) \text{ } r^{t+1} - r^t = \frac{a}{z}.$$

Therefore, if *z* is the sum of money that has been paid by a purchaser for a perpetual annuity of *a* pounds *per annum* that is to commence after the expiration of *t* years, or for the reversion of an estate in fee-simple, after the expiration of a term of *t* years, and it is required, from our knowledge of *a*, the number of pounds in the said annuity, or in the net rent of the said estate, and of *t*, the number of years at the expiration of which the said annuity is to commence, and of *z*, the number of pounds in the sum of money that has been paid for the said annuity, to determine *r*, or the rate of Interest that has been allowed to the purchaser for his money in the sale of the said annuity, the said Problem may be solved, and the said rate of interest may be discovered by resolving the binomial equation

$$r^{t+1} - r^t = \frac{a}{z} \quad \text{This equation I shall therefore now proceed to resolve.}$$

Of

Of the Resolution of the Equation $r^{t+1} - r^t = \frac{a}{z}$.

Art. 114. As in this binomial equation $r^{t+1} - r^t = \frac{a}{z}$ the unknown term r^t , which is marked with the sign $-$, or is subtracted from the other unknown term r^{t+1} , is a lower power of r , or the root of the equation, than the term r^{t+1} , from which it is subtracted, it is evident that this equation can have but one root, or, in the language of modern Algebraists, one real and affirmative root. This root may be investigated in the following manner.

In the first place, let us suppose r , or the rate of Interest sought, to be nearly equal to 1.05, or, in general, to some known rate of Interest taken by conjecture, and denoted by the letter b ; and let 1.05, or b , be substituted, instead of r , in the binomial quantity $r^{t+1} - r^t$; and let the value of the said quantity resulting from such substitution be called B.

Then, if B is greater than $r^{t+1} - r^t$, or, (it's equal,) the absolute term $\frac{a}{z}$, we may conclude that 1.05, or b , will be greater than r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$; and, if B is less than $r^{t+1} - r^t$, or $\frac{a}{z}$, we may conclude that 1.05, or b , will be less than r , or the root of the said equation.

Then, having thus taken 1.05, or, in general, b , for a first near value of r , and found, (by the substitution of it in the binomial quantity $r^{t+1} - r^t$, and by comparing B, or the result of the said substitution, with the absolute term $\frac{a}{z}$,) whether 1.05, or b , is greater or less than the true value of r , we may proceed to find a second near value of r , that shall be more exact than 1.05, or b , by a process of the Differential method of approximation, in the manner following.

If b is less than r , let it be increased by the addition of the small quantity $\frac{b}{200}$, and let the sum $b + \frac{b}{200}$ be called c . Then let c be substituted, instead of r , in the binomial quantity $r^{t+1} - r^t$; and let the value of the said binomial quantity resulting from such substitution be denoted by the capital letter C. And, having thus found the two quantities B and C, (or the two values of the binomial

binomial quantity $r^{t+1} - r^t$ resulting from the substitution of b and c in it's terms instead of r ,) make the following proportion; to wit, As the difference of B and C is to the difference of B and $\frac{a}{z}$ (the absolute term of the equation $r^{t+1} - r^t = \frac{a}{z}$,) so, very nearly, will the difference of b and c be to the difference of b and r ; or, as $C - B$ is to $\frac{a}{z} - B$, so, very nearly, will $c - b$, (or $b + \frac{b}{200} - b$,) or $\frac{b}{200}$, be to $r - b$; whence we shall have $r - b$, nearly, $= \frac{\frac{b}{200} \times \overline{\frac{a}{z} - B}}{C - B}$, and consequently r , nearly, $= b +$ the fraction $\frac{\frac{b}{200} \times \overline{\frac{a}{z} - B}}{C - B}$; that is, $b +$ the fraction $\frac{\frac{b}{200} \times \overline{\frac{a}{z} - B}}{C - B}$ will be a second near value of r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$, that will be more exact than it's first near value b . Q. E. I.

And, if b is greater than r , we must diminish b by subtracting $\frac{b}{200}$ from it, and put the remainder $b - \frac{b}{200} = c$. We must then substitute c instead of r in the binomial quantity $r^{t+1} - r^t$, and denote the value of the said quantity resulting from such substitution by the capital letter C , and make the following proportion; to wit, As $B - C$ is to $B - \frac{a}{z}$, so, very nearly, will $b - c$, (or $b - \left[b - \frac{b}{200}\right]$, or $b - b + \frac{b}{200}$,) or $\frac{b}{200}$, be to $b - r$; whence we shall have $b - r$, nearly, $= \frac{\frac{b}{200} \times \overline{B - \frac{a}{z}}}{B - C}$, and consequently b , nearly, $= r +$ the fraction $\frac{\frac{b}{200} \times \overline{B - \frac{a}{z}}}{B - C}$, and r , nearly, $= b -$ the fraction $\frac{\frac{b}{200} \times \overline{B - \frac{a}{z}}}{B - C}$; that is, $b -$ the fraction $\frac{\frac{b}{200} \times \overline{B - \frac{a}{z}}}{B - C}$ will be a second near value of r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$, that will be more exact than it's first near value b . Q. E. I.

If

If a still greater degree of exactness is desired, it may be obtained by putting e for the second near value of r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$, (obtained by the said process of the differential method of approximation,) and substituting e , instead of r , in the binomial quantity $r^{t+1} - r^t$, in order to discover whether the value of the said binomial quantity resulting from this substitution will be greater, or less, than $\frac{a}{z}$, (or the absolute term of the said equation,) and consequently whether e will be greater, or less, than the true value of r in the said equation, and then putting w for the unknown difference between e and r , and substituting either $e - w$, or $e + w$, (according as e shall be greater, or less, than r ,) instead of r , in the equation $r^{t+1} - r^t = \frac{a}{z}$, and resolving the transformed equation, thereby obtained, as if it were a mere simple equation, agreeably to the directions of Mr. Raphson's method of approximation. And the value of $e - w$, or $e + w$, thereby found will be a third near value of r , that will be much more exact than it's second near value e .

Q. E. I.

An Example of the foregoing Method of resolving the Equation

$$r^{t+1} - r^t = \frac{a}{z}.$$

Art. 115. I will now proceed to illustrate the foregoing method of resolving the equation $r^{t+1} - r^t = \frac{a}{z}$ by an Example. And, that we may be the better able to judge of the degree of exactness with which each of the three successive near values of r will approach to it's true value, I will chuse for the said example a numeral equation, in which we shall know before-hand the exact value of it's root r , and then will make gradual approaches to it's true value in the method above-described, in the same manner as if we had not had such previous knowledge of the said true value.

Let us therefore suppose that the Interest of Money is 6 per cent, or that r is $= 1.06$. And let it be required to find the sum of money that ought to be paid for the purchase of a perpetual annuity of 100 pounds a year that is to commence at the end of 40 years, or of which the first payment will be due at the end of 41 years.

In this Problem a is = 100*l.*, and t is = 40 years; and consequently $\frac{a}{r^t \times r - 1}$, which (by what is shewn above in art. 112,) is the price that ought to be paid for the said perpetual, but remote, annuity, will be = $\frac{100*l.*}{1.06^{40} \times 1.06 - 1}$
 (= $\frac{100*l.*}{1.06^{40} \times 0.06} = \frac{100 \times 100*l.*}{1.06^{40} \times 100 \times 0.06} = \frac{10000*l.*}{1.06^{40} \times 6} = \frac{10000*l.*}{6} \times \frac{1}{1.06^{40}} = \frac{1666.666,666,6*l.*}{1.06^{40}}$). We must therefore find the value of $\frac{1666.666,666,6*l.*}{1.06^{40}}$, and, for that purpose, must first compute the value of 1.06^{40} .

Now the logarithm of 1.06 is = 0.025,3059. Therefore the logarithm of 1.06^{40} will be (= $40 \times 0.025,305,9$) = 1.012,236,0; which is the logarithm of the number 10.285,751,1. Therefore 1.06^{40} will be = 10.285,751,1, and consequently $\frac{1666.666,666,6*l.*}{1.06^{40}}$ will be = $\frac{1666.666,666,6*l.*}{10.285,751,1} = 162.036,457,0*l.*$; that is, z , or the price of the said perpetual, but remote, annuity will be = 162.036,457,0*l.*, or 162*l.* os. 8 $\frac{3}{4}$ *d.* Q. E. I.

And hence it follows, *et converso*, that, if 162*l.* os. 8 $\frac{3}{4}$ *d.*, or 162.036,457,0*l.*, is the price paid for the purchase of a perpetual annuity of 100 pounds a year, that is to commence at the end of 40 years, the rate of Interest that has been allowed to the purchaser in the sale of the said annuity must have been 6 per cent, or r will be = 1.06.

Art. 116. We will now proceed to solve this second Problem (which is the converse of the Problem just now solved,) in the same manner as we should solve it if we had not already solved the former Problem, in which r was supposed to be = 1.06, and z was obtained by computing the expression

$$\frac{a}{r^t \times r - 1}.$$

It has been shewn above, in art. 113, that the solution of this Problem may be reduced to the resolution of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$. We must therefore endeavour to resolve the said equation $r^{t+1} - r^t = \frac{a}{z}$, when t is = 40 years, and a is = 100 pounds, and z is = 162.036,457,0*l.*; in which case we shall have $t + 1$ (= $40 + 1$) = 41, and $\frac{a}{z}$ (= $\frac{100*l.*}{162.036,457,0}$) = 0.617,145,0, and the general equation $r^{t+1} - r^t = \frac{a}{z}$ will be converted into

into the particular, or numeral, equation $r^{41} - r^{40} = 0.617,145,0$. This therefore is the equation which is now to be resolved by the method described in art. 114.

The Resolution of the Equation $r^{41} - r^{40} = 0.617,145,0$ by the Method described in Art. 114.

Art. 117. Let r be supposed to be nearly $= 1.05$; and let 1.05 be substituted instead of r in the binomial quantity $r^{41} - r^{40}$, that we may thereby discover whether the value of the said binomial quantity resulting from such substitution will be greater, or will be less, than the true value of $r^{41} - r^{40}$, or (it's equal) $0.617,145,0$, the absolute term of the equation $r^{41} - r^{40} = 0.617,145,0$, which is to be resolved, and consequently whether 1.05 will be greater, or will be less, than the true value of r in that equation. This substitution may be made as follows.

The logarithm of 1.05 is $= 0.021,189,3$. Therefore the logarithm of $\overline{1.05}^{40}$ will be $(= 40 \times 0.021,189,3) = 0.847,572,0$; which is the logarithm of the number $7.040,000,0$. Therefore $\overline{1.05}^{40}$ will be $= 7.040,000,0$, and consequently $\overline{1.05}^{41}$ will be $(= \overline{1.05}^{40} \times 1.05 = 7.040,000,0 \times 1.05) = 7.392,000,0$. Therefore the binomial quantity $\overline{1.05}^{41} - \overline{1.05}^{40}$ will be $(= 7.392,000,0 - 7.040,000,0) = 0.352,000,0$. This number is less than $0.617,145,0$, or the absolute term of the equation $r^{41} - r^{40} = 0.617,145,0$. And therefore 1.05 must be less than the true value of r in the said equation.

Having thus discovered that 1.05 is less than r , let us increase 1.05 by the small quantity $\frac{1.05}{200}$, or $0.005,25$, and substitute the sum thereby obtained, to wit, the number $1.055,250,0$, instead of r , in the binomial quantity $r^{41} - r^{40}$. This substitution may be made as follows.

The logarithm of $1.055,250,0$ is $= 0.023,355,3$. Therefore the logarithm of $\overline{1.055,250,0}^{40}$ will be $(= 40 \times 0.023,355,3) = 0.934,212,0$; which is the logarithm of the number $8.594,329,4$. Therefore $\overline{1.055,250,0}^{40}$ is $= 8.594,329,4$. Therefore $\overline{1.055,250,0}^{41}$ will be $(= \overline{1.055,250,0}^{40} \times 1.055,250,0 = 8.594,329,4 \times 1.055,250,0) = 9.069,166,1$; and consequently the binomial quantity $\overline{1.055,250,0}^{41} - \overline{1.055,250,0}^{40}$ will be $(= 9.069,166,1 - 8.594,329,4) = 0.474,836,7$.

Now let 0.352,000,0, (or the value of the binomial quantity $r^{41} - r^{40}$ resulting from the substitution of 1.05 in it's terms instead of r ;) be called B; and let 0.474,836,7, (or the value of the binomial quantity $r^{41} - r^{40}$ resulting from the substitution of 1.055,250,0 in it's terms instead of r ;) be called C; and let us make the following proportion; to wit, As $C - B$ is to $\frac{a}{z} - C$, so, very nearly, will 1.055,250,0 - 1.05 be to $r - 1.055,250,0$; or as 0.474,836,7 - 0.352,000,0 is to 0.617,145,0 - 0.474,836,7, so, very nearly, will 0.005,250,0 be to $r - 1.055,250,0$; that is, As 0.122,836,7 is to 0.142,308,3, so, very nearly, will 0.005,250,0 be to $r - 1.055,250,0$.

And we shall thence have $r - 1.055,250,0$, nearly, $(= \frac{0.142,308,3 \times 0.005,250,0}{0.122,836,7}) = \frac{0.000,747,118,575}{0.122,836,7} = 0.006,082,2$, and consequently r , nearly, $(= 0.006,082,2 + 1.055,250,0) = 1.061,332,2$; that is, 1.061,332,2 will be a second near value of r , or the root of the equation $r^{41} - r^{40} = 0.617,145,0$, that will approach much nearer to it's true value, than either of the two former numbers 1.05 and 1.055,250,0 did.

Q. E. I.

Now let this number 1.061,332,2 be substituted instead of r in the binomial quantity $r^{41} - r^{40}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or will be less, than 0.617,145,0, or the absolute term of the equation $r^{41} - r^{40} = 0.617,145,0$, and consequently to determine whether the said number 1.061,332,2 is greater, or is less, than the true value of r , or the root of the equation $r^{41} - r^{40} = 0.617,145,0$. This substitution may be made as follows.

The logarithm of 1.061,332,2 is = 0.025,851,3. Therefore the logarithm of $1.061,332,2^{40}$ will be $(= 40 \times 0.025,851,3) = 1.034,052,0$; which is the logarithm of the number 10.815,634,3. Therefore $1.061,332,2^{40}$ will be = 10.815,634,3. Therefore $1.061,332,2^{41}$ will be $(= 1.061,332,2^{40} \times 1.061,332,2 = 10.815,634,3 \times 1.061,332,2) = 11.478,980,9$, and consequently the binomial quantity $1.061,332,2^{41} - 1.061,332,2^{40}$ will be $(= 11.478,980,9 - 10.815,634,3) = 0.663,346,6$. This number is greater than 0.617,145,0, or the absolute term of the equation $r^{41} - r^{40} = 0.617,145,0$. Therefore 1.061,332,2 must be greater than r , or the root of that equation.

Q. E. I.

Having thus discovered that 1.061,332,2 is greater than r , let it's unknown excess above r be denoted by w , and let the binomial quantity $1.061,332,2 - w$ be substituted instead of r in the equation $r^{41} - r^{40} = 0.617,145,0$.

Then,

Then, since r is $= 1.061,332,2 - w$, we shall have $r^{41} =$
 $\overline{1.061,332,2 - w}^{41} =$ (by the binomial theorem,) to $\overline{1.061,332,2}^{41} - 41$
 $\times \overline{1.061,332,2}^{40} \times w + \&c = 11.478,980,9 - 41 \times 10.815,634,3 \times w$
 $+ \&c = 11.478,980,9 - 443.441,006,3 \times w + \&c$; and we shall have
 $r^{40} = \overline{1.061,332,2 - w}^{40} =$ (by the binomial theorem,) to $\overline{1.061,332,2}^{40}$
 $- 40 \times \overline{1.061,332,2}^{39} \times w + \&c = 10.815,634,3 - 40 \times \frac{10.815,634,3}{1.061,332,2} \times w$
 $+ \&c = 10.815,634,3 - 40 \times 10.190,621,0 \times w + \&c = 10.815,634,3$
 $- 407.624,840,0 \times w + \&c.$

Therefore the binomial quantity $r^{41} - r^{40}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 11.478,980,9 - 443.441,006,3 \times w + \&c \\ - 10.815,634,3 + 407.624,840,0 \times w - \&c \end{array} \right\} =$$

$$0.663,346,6 - 35.816,166,3 \times w + \&c.$$

But the binomial quantity $r^{41} - r^{40}$ is $= 0.617,145,0.$

Therefore the compound quantity $0.663,346,6 - 35.816,166,3 \times w + \&c$ will also be $= 0.617,145,0.$ Therefore $0.663,346,6$ will be $= 0.617,145,0 + 35.816,166,3 \times w$, and $35.816,166,3 \times w$ will be $(= 0.663,346,6 - 0.617,145,0) = 0.046,201,6$, and w will be $(= \frac{0.046,201,6}{35.816,166,3}) = 0.001,289,9.$ Therefore r , or $1.061,332,2 - w$, will be $(= 1.061,332,2 - 0.001,289,9) = 1.060,042,3$; that is, $1.060,042,3$ will be a third near value of r , or the root of the binomial equation $r^{41} - r^{40} = 0.617,145,0$, that will be much nearer to its true value than $1.061,332,2$, or the second near value of it was. Q. E. I.

This number $1.060,042,3$ exceeds the true value of r , (which we know to be 1.06 , or $1.060,000,0$;) by only the small quantity $0.000,042,3$, which is less than the 25,059th part of the said true value 1.06 . This is a great degree of exactness, and sufficient for all useful purposes; and therefore I shall not carry the investigation of the value of r any further. But, if greater exactness were to be desired, it might easily be obtained by going through one more process of Mr. Raphson's method of approximation.

Art. 118. The equation $r^{t+1} - r^t = \frac{a}{x}$, (to which this last Problem concerning a perpetual, but remote, annuity, or a reversion of an estate in fee-simple, is reduced, is the very same equation which was obtained above in art. 79, page 452, for the determination of A , or the higher limit of the magnitude

nitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^t - r^{t+1} = \frac{a}{z}$; and therefore it's root will be nearly = the expression $1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$, which is shewn in art. 71, pages 446, 447, and art. 79, pages 452, 453, to be nearly equal to, but somewhat greater than, the root of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$. We may therefore obtain a first near value of r , or the root of the numeral equation $r^{41} - r^{40} = 0.617,145,0$, (which we have just now been resolving,) by computing the said expression

$$1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}, \text{ which may be done as follows.}$$

Another Resolution of the foregoing Equation $r^{41} - r^{40} = 0.617,145,0$ by means

$$\text{of the Expression } 1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}, \text{ obtained above in Art. 71 and 79.}$$

Art. 119. In the numeral equation $r^{t+1} - r^t = 0.617,145,0$ we have $t = 40$, and $\frac{a}{z} = 0.617,145,0$, and consequently $t \times \frac{a}{z} (= 40 \times 0.617,145,0) = 24.685,800,0$, and $\frac{4ta}{z} (= 4 \times 24.685,800,0) = 98.743,200,0$, and $\frac{4ta}{z} + 1 (= 98.743,200,0 + 1) = 99.743,200,0$, and $\sqrt{\frac{4ta}{z} + 1} (= \sqrt{99.743,200,0}) = 9.987,1$, and $\sqrt{\frac{4ta}{z} + 1} - 1 (= 9.987,1 - 1) = 8.987,1$, and $\frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t} (= \frac{8.987,1}{2 \times 40} = \frac{8.987,1}{80}) = 0.112,3$. Therefore $1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$ will be $(= 1 + 0.112,3) = 1.112,3$; which will therefore be

be a first near value of r , or the root of the numeral equation $r^{41} - r^{40} = 0.617,145,0$. Q. E. I.

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Art. 120. This number 1.112,3, obtained for a first near value by computing

the expression $1 + \frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$, is much more remote from it's true value 1.06 than the number 1.05 was, which we took by a mere conjecture for our first near value of r in the foregoing resolution of the numeral equation $r^{41} - r^{40} = 0.617,145,0$ in art. 117. And therefore it would not be advise-

able to make use of this expression $1 + \frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$ in the resolution of this equation.

But it may be worth while to inquire, how it comes to pass that the ex-

pression $1 + \frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$, when applied to the resolution of this equation $r^{41} - r^{40} = 0.617,145,0$, gives us a first near value of r that is so much greater than the truth. And for this purpose it will be necessary for us to resume the consideration of the binomial equation $x^{n+1} - x^n = P$, which is resolved above in art. 71, pages 446, 447, in an imperfect manner by means of the binomial theorem, and it's root x is, in consequence of such resolution, declared to be nearly equal to, but less than, the expression $1 + \frac{\sqrt{4nP + 1} - 1}{2n}$.

Art. 121. Now in art. 71 the expression $1 + \frac{\sqrt{4nP + 1} - 1}{2n}$ was obtained by the following train of reasoning.

If v is considerably less than 1, and consequently the powers of v , to wit, v, v^2, v^3, v^4, v^5 , &c form a decreasing progression of terms that decrease with a great degree of swiftness, let $1 + v$ be substituted instead of x in the equation $x^{n+1} - x^n = P$.

Then

Then we shall have, in the 1st place, $x^{n+1} = \overline{1+v}^{n+1} =$, by the binomial theorem, to the series $1 + \overline{n+1} \times v + \overline{n+1} \times \frac{\overline{n+1}-1}{2} \times v^2$

$$+ \overline{n+1} \times \frac{\overline{n+1}-1}{2} \times \frac{\overline{n+1}-2}{3} \times v^3$$

$$+ \overline{n+1} \times \frac{\overline{n+1}-1}{2} \times \frac{\overline{n+1}-2}{3} \times \frac{\overline{n+1}-3}{4} \times v^4$$

$$+ \overline{n+1} \times \frac{\overline{n+1}-1}{2} \times \frac{\overline{n+1}-2}{3} \times \frac{\overline{n+1}-3}{4} \times \frac{\overline{n+1}-4}{5} \times v^5$$

$$+ \&c, \text{ continued to } \overline{n+1} + 1, \text{ or } n+2, \text{ terms,} = \text{the series}$$

$$1 + \overline{n+1} \times v + \overline{n+1} \times \frac{n}{2} \times v^2$$

$$+ \overline{n+1} \times \frac{n}{2} \times \frac{n-1}{3} \times v^3 + \overline{n+1} \times \frac{n}{2} \times \frac{n-1}{3} \times \frac{n-2}{4} \times v^4$$

$$+ \overline{n+1} \times \frac{n}{2} \times \frac{n-1}{3} \times \frac{n-2}{4} \times \frac{n-3}{5} \times v^5$$

$$+ \&c \text{ continued to } n+2 \text{ terms;}$$

or (putting B, C, D, E, F, &c for the co-efficients of $v, v^2, v^3, v^4, v^5, \&c$, respectively,) we shall have $x^{n+1} =$ the series $1 + Bv + Cv^2 + Dv^3 + Ev^4 + Fv^5 + \&c$ continued to $n+2$ terms, $=$ the series $1 + \overline{n+1} \times v + \frac{n}{2} \times Bv^2 + \frac{n-1}{3} \times Cv^3 + \frac{n-2}{4} \times Dv^4 + \frac{n-3}{5} \times Ev^5 + \&c$ continued to $n+2$ terms, or $=$ to the series $1 + \overline{n+1} \times v + \frac{n}{2} \times \overline{n+1} \times v^2 + Dv^3 + Ev^4 + Fv^5 + \&c$, continued to $n+2$ terms, or to the series $1 + nv + v + \frac{nn}{2} \times v^2 + \frac{n}{2} \times v^3 + Dv^3 + Ev^4 + Fv^5 + \&c$, continued to $n+2$ terms.

And, in the second place, we shall have $x^n = \overline{1+v}^n =$, by the binomial theorem, to the series $1 + n \times v + n \times \frac{n-1}{2} \times v^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times v^3$

$$+ n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times v^4 + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$$

$$\times \frac{n-4}{5} \times v^5 + \&c, \text{ continued to } n+1 \text{ terms;}$$

or

or (putting b, c, d, e, f , &c for the co-efficients of v, v^2, v^3, v^4, v^5 , &c, respectively,) we shall have $x^n =$ the series $1 + bv + cv^2 + dv^3 + ev^4 + fv^5 + \&c$, continued to $n + 1$ terms, $=$ the series $1 + nv + \frac{n-1}{2} \times bv^2 + \frac{n-2}{3} \times cv^3 + \frac{n-3}{4} \times dv^4 + \frac{n-4}{5} \times ev^5 + \&c$, continued to $n + 1$ terms, $=$ the series $1 + nv + \frac{n-1}{2} \times n \times v^2 + dv^3 + ev^4 + fv^5$ continued to $n + 1$ terms, $=$ the series $1 + nv + \frac{nn}{2} \times v^2 - \frac{n}{2} \times v^3 + dv^3 + ev^4 + fv^5 + \&c$ continued to $n + 1$ terms.

Therefore the binomial quantity $x^{n+1} - x^n$ will be $=$ the series $1 + nv + v + \frac{nn}{2} \times v^2 + \frac{n}{2} \times v^3 + Dv^3 + Ev^4 + Fv^5 + \&c$ continued to $n + 2$ terms, $-$ the series $1 + nv + \frac{nn}{2} \times v^2 - \frac{n}{2} \times v^3 + dv^3 + ev^4 + fv^5 + \&c$, continued to $n + 1$ terms, $=$ the series $1 - 1 + nv + v - nv + \frac{nn}{2} \times v^2 + \frac{n}{2} \times v^3 - \frac{nn}{2} \times v^2 + \frac{n}{2} \times v^3 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c =$ the series $v + \frac{2n}{2} \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c =$ the series $v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$.

But the binomial quantity $x^{n+1} - x^n$ is $= P$.

Therefore the series $v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$ will also be $= P$.

And thus far the reasoning used in art. 71 is just and accurate.

Art. 122. But afterwards I supposed all the terms of this series which come after the two first terms $v + n \times v^2$, to be much less than the said two terms on account of the decrease of the powers of v , which makes v^3, v^4, v^5 , &c be much less than v and v^2 , and will therefore contribute greatly to the diminution of the several terms Dv^3, Ev^4, Fv^5 , &c, and dv^3, ev^4 , and fv^5 , &c, in which the said powers of v are involved: and I thence concluded that the whole series $v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$ would be but a little greater than it's two first terms $v + n \times v^2$, and consequently that $v + n \times v^2$ would be nearly equal to, though somewhat less than, P (to which the said whole series is equal,) and therefore that $\frac{v}{n} + v^2$ would be nearly equal

to, though somewhat less than, $\frac{P}{n}$; and then, by resolving this quadratick equation $\frac{v}{n} + v^2 = \frac{P}{n}$, or $v^2 + \frac{v}{n} = \frac{P}{n}$, I found that v would be nearly equal to, though somewhat less than, the expression $\frac{\sqrt{4nP+1}-1}{2n}$, and consequently that $1+v$, or x , would be nearly equal to, though somewhat less than, the expression $1 + \frac{\sqrt{4nP+1}-1}{2n}$; from which expression the expression $1 + \frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$ was afterwards derived for a near value of r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$ (which is of the same form as the equation $x^{n+1} - x^n = P$), that would be somewhat greater than it's true value.

Art. 123. But it must be observed that the decrease of the terms Dv^3 , Ev^4 , Fv^5 , &c, and dv^3 , ev^4 , fv^5 , &c, in the series $v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$ (which is $= P$), does not depend solely on the decrease of the powers of v , to wit, v^3 , v^4 , v^5 , &c, that are involved in them, but likewise on the magnitude of the co-efficients of those powers, to wit, the co-efficients D , d , E , e , F , f , &c; and that these co-efficients will, in the first terms Dv^3 , Ev^4 , Fv^5 , &c, and dv^3 , ev^4 , fv^5 , &c, immediately following the terms $v + n \times v^2$, continually increase, (though in a proportion continually growing less and less,) and that, when n is a great number, this increase of these co-efficients will be so great as very much to counter-act the diminution of these terms arising from the decrease of the powers of v , (to wit, v^3 , v^4 , v^5 , &c,) and to make the said terms either not decrease at all, or decrease very slowly. And then, it is evident, the rejection of all these terms, as being very small and inconsiderable in comparison of the two first terms v and $n \times v^2$, and the supposition that the said two terms are nearly equal to the whole series, and consequently to P , will not be well-grounded, and will consequently give us a value of v that will be *much* greater than it's true value; as we have found to be the case above in art. 119, where the value of v obtained by the computation

of the expression $\frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$ was 0.112,3, which is *much* greater than it's true value, which we know to be 0.06.

Art. 124. But, that this may appear the more clearly, I will now proceed to compute the values of these co-efficients D , E , F , and d , e , f , when n is supposed to be $= 40$, as it is in the foregoing equation $r^{41} - r^{40} = 0.617,145,0$.
Now

Now on this supposition, that n is $= 40$, we shall have

$$B (= n + 1 = 40 + 1) = 41,$$

$$\text{and } C (= \frac{n}{2} \times B = \frac{40}{2} \times 41 = 20 \times 41) = 820,$$

$$\text{and } D (= \frac{n-1}{3} \times C = \frac{40-1}{3} \times C = \frac{39}{3} \times C = 13 \times C = 13 \times 820) = 10,660,$$

$$\text{and } E (= \frac{n-2}{4} \times D = \frac{40-2}{4} \times D = \frac{38}{4} \times D = \frac{19}{2} \times D = \frac{19}{2} \times 10,660 = 19 \times 5,330) = 101,270,$$

$$\text{and } F (= \frac{n-3}{5} \times E = \frac{40-3}{5} \times E = \frac{37}{5} \times E = \frac{37}{5} \times 101,270 = 37 \times 20,254) = 749,398;$$

and we shall have $b (= n) = 40$,

$$\text{and } c (= \frac{n-1}{2} \times b = \frac{40-1}{2} \times b = \frac{39}{2} \times b = \frac{39}{2} \times 40 = 39 \times 20) = 780,$$

$$\text{and } d (= \frac{n-2}{3} \times c = \frac{40-2}{3} \times c = \frac{38}{3} \times c = \frac{38}{3} \times 780 = 38 \times 260) = 9,880,$$

$$\text{and } e (= \frac{n-3}{4} \times d = \frac{40-3}{4} \times d = \frac{37}{4} \times d = \frac{37}{4} \times 9,880 = 37 \times 2,470) = 91,390,$$

$$\text{and } f (= \frac{n-4}{5} \times e = \frac{40-4}{5} \times e = \frac{36}{5} \times e = \frac{36}{5} \times 91,390 = 36 \times 18,278) = 658,008.$$

Therefore the series $v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$ will, in this case, be $= v + 40 \times v^2 + 10,660 \times v^3 - 9,880 \times v^3 + 101,270 \times v^4 - 91,390 \times v^4 + 749,398 \times v^5 - 658,008 \times v^5 + \&c = v + 40 \times v^2 + 780 \times v^3 + 9880 \times v^4 + 91,390 \times v^5 + \&c$; in which series the co-efficients of v^3 , v^4 , and v^5 , to wit, the numbers 780, 9,880, and 91,390, increase in very great proportions and very much contribute to increase the magnitude of the terms to which they belong, and to counter-act the diminution of those terms arising from the decrease of the powers of v , (to wit, v^3 , v^4 , and v^5), into which they are multiplied. And hence it happens that in this case these terms cannot be rejected as small and inconsiderable in comparison of the two first terms $v + n \times v^2$, without causing the value of v , resulting from such a rejection of them, to be much greater than it's true value.

Art. 125. But the effect of this increase of the numbers 780, 9,880, and 91,390, (which are the co-efficients of v^3 , v^4 , and v^5 in the foregoing series,) in counter-acting the diminution of the terms $780 \times v^3$, $9,880 \times v^4$, and $91,390 \times v^5$, arising from the decrease of the powers of v which are involved

in them, (to wit, v^3 , v^4 , and v^5 ,) may be made still more evident by substituting, instead of those powers of v , their respective values, which we are in this case enabled to do by means of our previous knowledge that in the foregoing equation $r^{41} - r^{40} = 0.617,145,0$, the true value of r is 1.06, and consequently that the true value of v is 0.06, or $\frac{6}{100}$. This substitution may be made as follows.

Since v is = 0.06, we shall have v^2 ($= \overline{0.06^2}$) = 0.003,6, and v^3 ($= 0.06 \times 0.003,6$) = 0.000,216, and v^4 ($= 0.06 \times 0.000,216$) = 0.000,012,96, and v^5 ($= 0.06 \times 0.000,012,96$) = 0.000,000,777,6. Therefore $40 \times v^2$ will be ($= 40 \times 0.003,6$) = 0.144,0, and $780 \times v^3$ will be ($= 780 \times 0.000,216$) = 0.168,480, and $9,880 \times v^4$ will be ($= 9,880 \times 0.000,012,96$) = 0.128,044,80, and $91,390 \times v^5$ will be ($= 91,390 \times 0.000,000,777,6$) = 0.071,064,864,0; and the series $v + 40v^2 + 780v^3 + 9,880v^4 + 91,390 \times v^5 + \&c$ will become = 0.06 + 0.144,0 + 0.168,480 + 0.128,044,80 + 0.071,064,864,0 + $\&c$, in which the 3d, 4th, and 5th terms are so far from being very small and inconsiderable in comparison of the first and second terms, that each of them is greater than the first term 0.06, or v . In this case therefore the value of v obtained by the rejection of all the terms of this series, except the two first, or by supposing those two terms alone to be equal to the whole series, and consequently to P , must necessarily be much greater than it's true value.

Art. 126. But, if n in the equation $x^{n+1} - x^n = P$, or t in the equation $x^{t+1} - x^t = \frac{a}{z}$, is much less than 40, as, for example, if it is 10, or 11, or 12, the numeral coefficients D , E , F , $\&c$, and d , e , f , $\&c$, of v^3 , v^4 , v^5 , $\&c$, in the series $1 + v + n \times v^2 + Dv^3 - dv^3 + Ev^4 - ev^4 + Fv^5 - fv^5 + \&c$ will have much less effect in counter-acting the diminution of the terms to which they belong, arising from the decrease of v^3 , v^4 , v^5 , $\&c$, than they had in the foregoing case of the equation $r^{41} - r^{40} = 0.617,145,0$; and in such a case the value of v arising from the computation of the expression

$$\frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t},$$

(and which is derived from the supposition that $v + n \times v^2$, or $v + t \times v^2$, the two first terms of the said series, are nearly equal to the whole series,) will be much nearer to it's true value than in the foregoing example, and may, perhaps, afford a tolerably good first near value of v ; and

$$1 + \frac{\sqrt{\frac{4^t a}{z} + 1} - 1}{2t}$$

consequently the expression may, in such a case, afford a tolerably good first near value of $1 + v$, or r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$. There will, however, be often a good deal of difficulty in determining in what cases, and with what magnitudes of the index t of the unknown

unknown quantity r , the said quadratick expression $1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$ will approach pretty nearly to the true value of r , or exceed it by only a small difference, and will therefore be fit to be adopted as a first near value of it. And therefore I should, in most cases, think it better to neglect that quadratick expression, and take 1.05, or some other rate of Interest, (chosen either upon some good ground, or even by a mere conjecture,) for a first near value of r , or the root of the equation $r^{t+1} - r^t = \frac{a}{z}$, and then to proceed to find a second near value of it by means of the Differential method of approximation, and, lastly, to find a third near value of it by a process of Mr. Raphson's method of approximation; as is done above in art. 117 in resolving the equation $r^{41} - r^{40} = 0.617,145,0$.

End of the Scholium begun in Art. 120, Page 509.

Art. 127. But here it may be observed that, in art. 79, page 452, I have, in treating of the binomial equation $r^{t+1} - r^t = \frac{a}{z}$, (by the resolution of which we may obtain the value of A , or the higher limit of the magnitude of the greater root of the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$),

recommended the quadratick expression $1 + \frac{\sqrt{\frac{4ta}{z} + 1} - 1}{2t}$ as a tolerably near value of it's root, and consequently of the quantity A ; and that, in treating of the binomial equation $r^{t+1} - \frac{x+t}{x+t+1} \times r^t = \frac{x}{x+t+1}$ (by the resolution of which we may obtain the value of M , or the lower limit of the magnitude of the greater root of the said trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$), I have recommended a like quadratick expression, to wit, the expression

$1 + \frac{\sqrt{ik + b^2} - b}{k}$, as a tolerably near value of it's root, and consequently of the quantity M . And it may be asked, whether I have altered my opinion upon this subject, and no longer recommend the use of these quadratick expressions for obtaining near values of the roots of these two binomial equations, or of the said limits A and M .—In answer to this question I will acknowledge, that I have altered my opinion on this subject, and that,—though the values of the quantities A and M that were obtained by means of those two quadratick expressions in art. 85 and 86, pages 457 and 458, were found to be sufficiently exact to enable us to derive from them a pretty good first near value of the greater root of the trinomial equation $0.500,000,0 \times r^{10} + r^{31} - r^{32} = 0.500,000,0$, of which root the quantities A and M are the limits, and therefore those expressions may sometimes be employed with advantage;—yet, upon the whole,

whole, (after having attentively considered the observations that have been set forth in the foregoing Scholium, and the disagreeable uncertainty we shall often labour under with respect to these quadratick expressions, in not being able to determine whether they will, or will not, give us tolerably near values of the roots of the said two binomial equations, or of the quantities A and M, to which the said roots are equal,) I now think it would, in most cases, be adviseable not to have recourse to these expressions for the resolution of those binomial equations, but to resolve them in the manner that has been described above in art. 114, and exemplified in art. 117 in the resolution of the equation $r^{41} - r^{40} = 0.617,145,0$.

Of two other Methods of resolving the Equation $r^{t+1} - r^t = \frac{a}{z}$ with great Exactness.

Art. 128. The method of resolving the equation $r^{t+1} - r^t = \frac{a}{z}$ which has been described above in art. 114, and exemplified in art. 117 in the resolution of the equation $r^{41} - r^{40} = 0.617,145,0$, appears to me the best method that can be taken for that purpose; though the said equation may also be resolved with as great exactness, by means of Sir Isaac Newton's Binomial Theorem, by two other methods similar to the second and third methods by which it has been shewn above (in art. 23, 24, 25, &c, - - - 30, pages 386, 387, 388, &c - - - 401,) that the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ may be resolved. But these methods seem to be much inferiour to the method above recommended in simplicity and perspicuity, and therefore I do not think it necessary to enter upon a description of them.

CONCLUSION.

Art. 129. Having therefore fully examined the three equations $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and $\frac{a}{z} \times r^t + r^t - r^{t+1} = \frac{a}{z}$, for the resolution of which Dr. Halley invented his approximating Algebræick expressions derived from Mr. De Moivre's Multinomial Theorem, and also the equation $r^{t+1} - r^t = \frac{a}{z}$ relating to the purchase of a perpetual, but remote, annuity, or of the reversion of an estate in fee-simple, and shewn how all these equations may be conveniently resolved without having recourse to those abstruse expressions, I shall here put an end to this Appendix to Dr. Halley's said Discourse on Compound Interest.

End of the Appendix to the foregoing Discourse of Dr. Halley on Compound Interest.

A L E T.

A LETTER

FROM

THE LATE CELEBRATED ALGEBRÄIST,
MR. ABRAHAM DE MOIVRE, F. R. S.

TO

DR. EDMUND HALLEY, F. R. S.

CONCERNING

ONE OF THE ALGEBRÄICK EXPRESSIONS GIVEN BY DR. HALLEY
IN HIS DISCOURSE ON COMPOUND INTEREST

REPRINTED ABOVE IN PAGES 219, 220, &c - - - 230 OF THIS VOLUME.

A LETTER

FROM

THE STATE DEPARTMENT AND THE

INS. AFFAIRS DE. MONTG. F. R. S.

TO

DR. EDWARD HALL, F. R. S.

CONCERNING

ON THE ALGEBRAIC EXPRESSION GIVEN BY DR. HALL
IN HIS DISCUSS ON COMPOUND INTEREST

PRINTED AND SOLD BY JAMES GILG, 210, N. 2ND ST. OF THIS TOWN.

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TO

DR. EDMUND HALLEY, F. R. S.

CONCERNING

*One of the Algebräick Expressions given by Dr. HALLEY in his Discourse
on Compound Interest reprinted above in Pages 219,
220, &c - - - 230 of this Volume.*

Article 1. **M**Y learned friend Mr. Nicholas Vilant, Professor of Mathematics in the University of St. Andrew's in Scotland, has informed me that he has met with a very curious Letter of the celebrated Algebräist, Mr. Abraham De Moivre, to Dr. Edmund Halley, concerning one of the abstruse Algebräick expressions given by the latter in his *Discourse on Compound Interest*, which have been the subject of the long Notes on that Discourse that have been printed in the foregoing part of this Volume. He met with this Letter many years ago in a periodical publication that came out in the year 1761, intitled *The Mathematical Magazine, and Philosophical Repository*, which was undertaken by Mr. G. Witchell, and Mr. T. Mofs, and some other persons, and printed for J. Wilkie at the Bible in Saint Paul's Church-yard. This publication, he informs me, was a very useful one: the part of it called *The Mathematical Magazine* contained many curious papers taken from foreign Literary Memoirs and Journals, or furnished by learned foreign correspondents; and *The Philosophical Repository* promised to be a most valuable work, and was to have comprehended a new and universal Dictionary of pure and mixt Mathematics. But, not meeting with encouragement, it was soon discontinued, only five numbers of it having been published; in one of which, in pages 41, 42, and 43, the said Letter of Mr. De Moivre is to be found. Mr. Witchell, the principal

principal undertaker of this publication, had originally been a journey-man watch-maker, but, notwithstanding the want of a learned education, had (like many other men who have become eminent in science,) attained a great degree of skill and knowledge in the Mathematicks, and was much patronized by the learned Dr. Bevis (a literary gentleman of great note at that time,) who probably communicated to him the said Letter of Mr. De Moivre to be published in his Magazine, where it was printed (as it is supposed,) for the first, and, perhaps, the only, time: for Mr. Vilant never met with it in any other book. The other undertaker of that useful Magazine, Mr. T. Mofs, was (as Mr. Vilant informs me,) an Excise-officer, and had been a writer on the subject of *Gauging* with a deserved reputation. Mr. Witchell was considered as a person of such respectable conduct and abilities that, in the latter part of his life, he was appointed Head Master of the Royal Mathematical Academy at Portsmouth. From this Magazine Professor *Vilant* has been so kind as to copy this Letter, and to send me the Copy of it, to be published on the present occasion. The Letter itself has no date to it, but was probably written soon after the first publication of Dr. Halley's Discourse on Compound Interest, (which was in the year 1705, in the Introduction to the first edition of Sherwin's Tables of Logarithms,) when that Discourse, from the accuracy of the abstruse and undemonstrated Algebræick expressions, given in it, of the values of the rate of Interest of Money in those Problems in which that rate was the unknown quantity, must naturally have attracted the attention of Mathematicians in general, and particularly of so excellent an Algebræist as Mr. De Moivre, and have excited them to endeavour to find out the investigation of them. The letter is as follows.

TO DR. EDMUND HALLEY.

Sir,

I believe I have hit upon the right method of demonstrating your Theorem for finding the rate of Interest in Annuities; and of continuing it, if there were occasion for it.

Let n be the number of years: let s represent so many years purchase as the annuity is worth, present money: x the rate of Interest. The equation expressing their relation is

$$\frac{x^n - 1}{x^n \times x - 1} = s.$$

Let x be made $= 1 + z$. Therefore $\frac{(1+z)^n - 1}{(1+z)^n \times z}$ will be $= s$.

Let both the numerator and the denominator of the first part of the equation be reduced into infinite series. And we shall have

$$\frac{nz + n \times \frac{n-1}{2} \times zz + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times z^3 + \&c}{z + nzz + n \times \frac{n-1}{2} \times z^3 + \&c} = s; \text{ and (dividing both the numerator and the denominator of the fraction by } z, \text{) we shall}$$

$$\text{have } \frac{n + n \times \frac{n-1}{2} \times z + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times zz + \&c}{1 + nz + n \times \frac{n-1}{2} \times zz + \&c} = s; \text{ and (taking the quote of the numerator by the denominator,) we shall have } n \text{ multiplied}$$

into $1 - \frac{n+1}{2} \times z + \frac{n+1}{2} \times \frac{n+2}{3} \times zz - \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times z^3 + \&c = s$.

And, because the co-efficient of the second term of this series is $-\frac{n+1}{2}$, I am very naturally directed to raise that series to the power $-\frac{2}{n+1}$, which is the reciprocal of it.

Now this series, thus raised, will be $1 + z - \frac{n-1}{12} \times zz + \&c$; which being continued to the cube, I found the co-efficient of the cube to be 0. The co-efficient of the biquadrate I did not pursue, it seeming, at first view, to be something intricate.

$$\text{We have therefore } n \times \left(1 + z - \frac{n-1}{12} \times zz \right)^{-\frac{n+1}{2}} = s, \text{ and therefore}$$

$$\left(1 + z - \frac{n-1}{12} \times zz \right)^{-\frac{n+1}{2}} = \frac{s}{n}, \text{ and consequently}$$

$$\left(1 + z - \frac{n-1}{12} \times zz \right)^{\frac{n+1}{2}} = \frac{n}{s}, \text{ and } 1 + z - \frac{n-1}{12} \times zz = \left(\frac{n}{s} \right)^{\frac{2}{n+1}},$$

$$\text{and } z - \frac{n-1}{12} \times zz = \left(\frac{n}{s} \right)^{\frac{2}{n+1}} - 1.$$

Let $\left(\frac{n}{s} \right)^{\frac{2}{n+1}} - 1$ be called y .

Therefore $z - \frac{n-1}{12} \times zz$ will be $= y$.

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Let $\frac{n-1}{12}$ be called c . And we shall have $z = \frac{1}{2c} - \sqrt{\frac{1}{4cc} - \frac{y}{c}}$, and consequently the rate of Interest $= 1 + \frac{1}{2c} - \sqrt{\frac{1}{4cc} - \frac{y}{c}}$.

Now make, in this Theorem, $\frac{1}{2c} = b$; that is, $\frac{6}{n-1} = b$. And we shall have the rate of Interest $= 1 + b - \sqrt{bb - 2by}$; which is your very Theorem.

Now I take this to be extremely near the truth, when n is small. For it goes so far as the cube exactly; and it partakes of the biquadrate and of all other powers.

I am your very humble servant,

ABRAHAM DE MOIVRE.

Art. 2. In the foregoing Letter of Mr. De Moivre the notation is different from that used by Dr. Halley; who uses t , instead of n , for the number of years during which the annuity is to continue; and z , instead of s , for the present value of the annuity, or the price that has been paid for it by the purchaser; and r , instead of x , for the rate of Interest which has been allowed the purchaser for his money, which rate is the unknown quantity that is to be found. If Dr. Halley's notation be substituted instead of Mr. De Moivre's in the investigation contained in the foregoing Letter, the said investigation will be as follows.

The foregoing Investigation of the Expression $1 + b - \sqrt{bb - 2by}$, or $1 + b - (bb - 2by)^{\frac{1}{2}}$, by Mr. De Moivre, expressed in Dr. Halley's Notation.

Let t be the number of years: let z represent so many years purchase as the annuity is worth, present money: r the rate of Interest. The equation expressing their relation is $\frac{r^t - 1}{r^t \times r - 1} = z$.

Let r be made $= 1 + v$. Therefore $\frac{(1+v)^t - 1}{(1+v)^t \times v}$ will be $= z$.

Let both the numerator and the denominator of the first part of the equation be reduced into infinite series. And we shall have

$$\frac{tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + \&c}{v + tv + t \times \frac{t-1}{2} \times v^2 + \&c} = z; \text{ and (dividing}$$

both the numerator and the denominator of the fraction by v), we shall have

$$\frac{t + t \times \frac{t-1}{2} \times v + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \&c}{1 + tv + t \times \frac{t-1}{2} \times vv + \&c} = z; \text{ and (taking}$$

the quote of the numerator by the denominator,) we shall have t multiplied

$$\text{into } 1 - \left[\frac{t+1}{2} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \left[\frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 \right. \right. \\ \left. \left. + \&c \right] = z.$$

And, because the co-efficient of the second term of this series is $-\left[\frac{t+1}{2} \right]$, I am very naturally directed to raise that series to the power $-\left[\frac{2}{t+1} \right]$, which is the reciprocal of it.

Now this series, thus raised, will be $1 + v - \left[\frac{t-1}{12} \times vv \&c \right]$; which being continued to the cube, I found the co-efficient of the cube to be 0. The co-efficient of the biquadrate I did not pursue, it seeming, at first view, to be something intricate.

$$\text{We have therefore } t \times \left[1 + v - \left[\frac{t-1}{12} \times vv \right] - \left[\frac{t+1}{2} \right] = z, \text{ and therefore} \right. \\ \left. 1 + v - \left[\frac{t-1}{12} \times vv \&c \right] - \left[\frac{t+1}{2} \right] = \frac{z}{t}, \text{ and } 1 + v - \left[\frac{t-1}{12} \times vv \right]^{\frac{t+1}{2}} = \right. \\ \left. \frac{t}{z}, \text{ and } 1 + v - \left[\frac{t-1}{12} \times vv \right] = \left[\frac{t}{z} \right]^{\frac{2}{t+1}}, \text{ and } v - \left[\frac{t-1}{12} \times vv \right] = \left[\frac{t}{z} \right]^{\frac{2}{t+1}} - 1.$$

$$\text{Let } \left[\frac{t}{z} \right]^{\frac{2}{t+1}} - 1 \text{ be called } y.$$

$$\text{Therefore } v - \left[\frac{t-1}{12} \times vv \right] \text{ will be } = y.$$

Let $\frac{t-1}{12}$ be called c . And we shall have $v = \frac{1}{2c} - \sqrt{\frac{1}{4cc} - \frac{y}{c}}$, and consequently the rate of Interest $= 1 + \frac{1}{2c} - \sqrt{\frac{1}{4cc} - \frac{y}{c}}$.

Now make, in this Theorem, $\frac{1}{2a} = b$; that is, $\frac{6}{t-1} = b$. And we shall have the rate of Interest $= 1 + b - \sqrt{bb - 2by}$; which is your very Theorem.

Now I take this to be extremely near the truth, when t is small. For it goes so far as the cube exactly; and it partakes of the biquadrate and of all other powers.

I am your very humble servant,

ABRAHAM DE MOIVRE.

Art. 3. In the foregoing investigation Mr. De Moivre supposes the annuity granted for a term of t years to be an annuity of only one pound a year; which is the reason why no mention is made of the fraction $\frac{at^t}{z}$, which Dr. Halley directs us to raise to the power of which the index is $\frac{2}{t+1}$. For, when a is $= 1$ l., the fraction $\frac{at^t}{z}$ (mentioned by Dr. Halley,) becomes $(= 1 \times \frac{t^t}{z}) = \frac{t^t}{z}$, which is the fraction mentioned by Mr. De Moivre, and which he directs us to raise to the power of which $\frac{2}{t+1}$ is the index.

Remarks on the foregoing Letter of Mr. De Moivre.

Art. 4. This investigation of Mr. De Moivre relates to the second of those abstruse and intricate Algebræick expressions, given us by Dr. Halley, which are obtained by means of the Multinomial Theorem of Mr. De Moivre. The investigation of this second expression $1 + b - \sqrt{bb - 2by}$, or $1 + b - (bb - 2by)^{\frac{1}{2}}$, that had been communicated to me by Mr. William Morgan, (the learned Actuary of the Office for granting Equitable Assurances on Lives,) is, in substance, the same with this investigation of it by Mr. De Moivre, and has been set forth and explained at great length above in the Third of the preceding long Notes on Dr. Halley's Discourse on Compound Interest, in pages 302, 303, &c, - - - - 310, of the present Volume, to which I refer the reader. And, indeed, of

of these two investigations of this expression, Mr. Morgan's is rather the easier, because it enables us to obtain the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ (in page 303), by only converting the fraction $\frac{1}{1+v}$, or the quantity $(1+v)^{-1}$, into an infinite series by means of the binomial theorem in the case of integral and negative powers, or of the reciprocals of integral and affirmative powers, whereas in Mr. De Moivre's investigation the same series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ is obtained by means of a most laborious and difficult operation of Algebraick division of one series by another, to wit, of the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \&c$ by the series $1 + tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + \&c$, the quotient of which division is the said series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$. The quotient of this division may be found in the following manner.

Art. 5. The co-efficient $\frac{t-1}{2} \times \frac{t-2}{3}$ is $= \frac{t^2 - 3t + 2}{6}$, and the co-efficient $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4}$ is $(= \frac{t^2 - 3t + 2}{6} \times \frac{t-3}{4}) = \frac{t^3 - 6t^2 + 11t - 6}{24}$.

Therefore the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \&c$ is $= 1 + \frac{t-1}{2} \times v + \frac{t^2 - 3t + 2}{6} \times vv + \frac{t^3 - 6t^2 + 11t - 6}{24} \times v^3 + \&c$, and the series $1 + tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + \&c$ is $= 1 + tv + \frac{t^2 - t}{2} \times vv + \frac{t^3 - 3t^2 + 2t}{6} \times v^3 + \&c$. We must therefore divide the series $1 + \frac{t-1}{2} \times v + \frac{t^2 - 3t + 2}{6} \times vv + \frac{t^3 - 6t^2 + 11t - 6}{24} \times v^3 + \&c$ by the series $1 + tv + \frac{t^2 - t}{2} \times vv + \frac{t^3 - 3t^2 + 2t}{6} \times v^3 + \&c$ which may be done as follows.

The

The Division of the Series $1 + \frac{t-1}{2} \times v + \frac{t^2-3t+2}{6} \times vv + \frac{t^3-6t^2+11t-6}{24} \times v^3 + \&c$ by the Series $1 + tv + \frac{t^2-t}{2} \times vv + \frac{t^3-3t^2+2t}{6} \times v^3 + \&c$.

The Divisor.

$$\left. \begin{aligned} 1 + tv + \frac{t^2-t}{2} \times vv \\ + \frac{t^3-3t^2+2t}{6} \times v^3 + \&c \end{aligned} \right\}$$

The Quotient.

$$\left. \begin{aligned} 1 - \frac{t+1}{2} \times v + \frac{t^2+3t+2}{6} \times vv \\ - \frac{t^3+6t^2+11t+6}{24} \times v^3 \&c \end{aligned} \right\}$$

The Dividend.

$$1 + \frac{t-1}{2} \times v + \frac{t^2-3t+2}{6} \times vv + \frac{t^3-6t^2+11t-6}{24} \times v^3 + \&c$$

$$1 + tv + \frac{t^2-t}{2} \times vv + \frac{t^3-3t^2+2t}{6} \times v^3 + \&c$$

$$* - \frac{t+1}{2} \times v - \frac{2t^2+2}{6} \times vv - \frac{3t^3+6t^2+3t-6}{24} \times v^3$$

$$- \frac{t+1}{2} \times v - \frac{t^2-t}{2} \times vv - \frac{t^3+6t^2+11t+6}{24} \times v^3$$

$$* + \frac{t^2+3t+2}{6} \times vv + \frac{3t^3+6t^2-3t-6}{24} \times v^3$$

$$+ \frac{t^2+3t+2}{6} \times vv + \frac{t^3+3t^2+2t}{6} \times v^3$$

$$* - \frac{t^3-6t^2-11t-6}{24} \times v^3$$

$$- \frac{t^3-6t^2-11t-6}{24} \times v^3$$

Therefore

Therefore the quotient of the foregoing division, or the quotient required to be found in Mr. De Moivre's investigation of Dr. Halley's expression, is the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t^2+3t+2}{6} \times vv - \frac{t^3+6t^2+11t+6}{24} \times v^3$ &c, which is equal to the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3$ &c, which is the series given as the quotient of the said division in Mr. De Moivre's investigation.

Art. 6. In the subsequent part of Mr. De Moivre's investigation the steps are nearly the same as in Mr. Morgan's investigation, given above in the Third Note on Dr. Halley's Discourse, pages 303, 304, 305, &c, - - - 310. For, having (by the foregoing operation of division,) obtained the equation $t \times$ the

series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv + \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c, = z$. Mr. De Moivre raises the said series to the power of which $-\sqrt{\frac{2}{t+1}}$ is the index. This is done by his own multinomial theorem, and produces the series $1 + v - \sqrt{\frac{-1}{12}} \times vv + 0 \times v^3$ &c.

But, since $t \times$ the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \frac{t+1}{2} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ is $= z$, we shall have the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c = \frac{z}{t}$. Therefore the $\frac{-2}{t+1}$ th power of the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv - \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 - \&c$ will be $= \frac{z}{t} \sqrt{\frac{-2}{t+1}}$.

But the $\frac{-2}{t+1}$ th power of the series $1 - \sqrt{\frac{t+1}{2}} \times v + \frac{t+1}{2} \times \frac{t+2}{3} \times vv + \sqrt{\frac{t+1}{2}} \times \frac{t+2}{3} \times \frac{t+3}{4} \times v^3 + \&c$ is $=$ the series $1 + v - \sqrt{\frac{-1}{12}} \times vv + 0 \times v^3$ &c.

Therefore

Therefore the said series $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c$ will also be

$$= \frac{z}{t} \sqrt{\frac{2}{t+1}}.$$

But $\frac{z}{t} \sqrt{\frac{2}{t+1}}$ is $(= \frac{1}{\frac{z}{t} \sqrt{\frac{2}{t+1}}} = \frac{1}{\frac{z \sqrt{2}}{t \sqrt{t+1}}} = 1 \times \text{the fraction } \frac{t \sqrt{t+1}}{z \sqrt{2}})$

$$= \frac{t}{z} \sqrt{\frac{2}{t+1}}.$$

Therefore the series $1 + v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c$ is $= \frac{t}{z} \sqrt{\frac{2}{t+1}}.$

Therefore $v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c$ will be $= \frac{t}{z} \sqrt{\frac{2}{t+1}} - 1.$

Let y be put $= \frac{t}{z} \sqrt{\frac{2}{t+1}} - 1.$

And we shall then have $v - \sqrt{\frac{t-1}{12}} \times vv + 0 \times v^3 \&c = y$, or
 $v - \sqrt{\frac{t-1}{12}} \times vv$, nearly, $= y.$

Let $\frac{t-1}{12}$ be called $c.$

And we shall then have $v - c \times vv$, nearly $= y.$

Therefore $\frac{v}{c} - vv$, or $\frac{1}{c} \times v - vv$, will be $= \frac{y}{c}$; and (subtracting both sides of this equation from $\frac{1}{4cc}$, or the square of $\frac{1}{2c}$, or half of $\frac{1}{c}$, the co-efficient of v , which square is always greater than $\frac{1}{c} \times v - vv$, or $v \times \left[\frac{1}{c} - v \right]$, we shall have $\frac{1}{4cc} - \frac{1}{c} \times v + vv (= \frac{1}{4cc} - \frac{y}{c} = \frac{1}{4cc} - \frac{4cy}{4cc}) = \frac{1-4cy}{4cc}$, and (extracting the square-roots of both sides,) $\frac{1}{2c} - v = \frac{\sqrt{1-4cy}}{2c}$, and (adding v to both sides,) $\frac{1}{2c} = \frac{\sqrt{1-4cy}}{2c} + v$, and (subtracting $\frac{\sqrt{1-4cy}}{2c}$ from both sides,) $v = \frac{1}{2c} - \frac{\sqrt{1-4cy}}{2c} = \frac{1 - \sqrt{1-4cy}}{2c}.$

Now

Now make $b = \frac{1}{2c}$, or $= \frac{6}{t-1}$.

And we shall have $2b = \frac{1}{c}$, and $2by (= \frac{1}{c} \times y) = \frac{y}{c}$, and $bb - 2by$
 $(= \frac{1}{4cc} - \frac{y}{c} = \frac{1}{4cc} - \frac{4cy}{4cc}) = \frac{1-4cy}{4cc}$, and consequently $\sqrt{bb - 2by} (=$
 $\sqrt{\frac{1-4cy}{4cc}}) = \frac{\sqrt{1-4cy}}{2c}$, and $b - \sqrt{bb - 2by} (= \frac{1}{2c} - \frac{\sqrt{1-4cy}}{2c}) =$
 $\frac{1-\sqrt{1-4cy}}{2c}$, which has been shewn to be $= v$. Therefore v will be $= b$
 $- \sqrt{bb - 2by}$, and r , or $1 + v$, will be $= 1 + b - \sqrt{bb - 2by}$; which is
 Dr. Halley's expression. Q. E. I.

Art. 7. The foregoing expression $1 + b - \sqrt{bb - 2by}$ given by Dr. Halley for a near value of the rate of Interest sought, or of the greater root of the binomial equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, is more exact than one would naturally expect from a single process of an approximation of this nature; as has appeared in the Third of the foregoing Notes upon Dr. Halley's Discourse above-mentioned, in the resolution of the equation $1.090,909,0 \times r^{21} - r^{22} = 0.090,909,0$, in art. 7, pages 311, 312, where this expression gives us 1.068,327,5 for the value of r , or the greater root of the said equation, instead of 1.068,14, which is it's true value, and with which the number 1.068,327,5 agrees in the four first figures 1.068. This great degree of exactness of the expression $1 + b - \sqrt{bb - 2by}$ seems to be owing to the fortunate circumstance of having the co-efficient of v^3 in the series $1 + v - \left[\frac{t-1}{12} \times vv + 0 \times v^3 \right. \&c.$ (obtained by means of Mr. De Moivre's Multinomial Theorem,) equal to 0, which makes the retention of the three first terms of the said series, to wit, the terms $1 + v - \left[\frac{t-1}{12} \times vv \right]$, as a near value of the whole, give us as near a value of the whole series as we should have had by retaining the four first terms of the series, if the fourth term, involving v^3 , had had some finite number for it's co-efficient. This I call a *fortunate circumstance*, because I can hardly suppose that Dr. Halley could have foreseen, or have had any reason to conjecture before-hand, that the co-efficient of v^3 in the said fourth term would have been $= 0$. And Mr. De Moivre, in the last paragraph of his foregoing Letter, seems also to ascribe the great exactness of this expression to this circumstance.

Art. 8. After considering the foregoing investigation, given by Mr. De Moivre, of the expression $1 + b - \sqrt{bb - 2by}$ found by Dr. Halley for the value of r , or the greater root of the binomial equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and the investigations given by Mr. Morgan of the three different expressions $1 + \sqrt{bb + 2by} - b$, $1 + b - \sqrt{bb - 2by}$, and $1 + b - \sqrt{bb - 2by}$, or $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, and $1 + b - \sqrt{bb - 2by}^{\frac{1}{2}}$, found by Dr. Halley for the values of r , or the greater roots of the binomial equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, the binomial equation $\frac{z+a}{z} \times r^t - r^{t+1} = \frac{a}{z}$, and the trinomial equation $\frac{a}{z} \times r^x + r^{x+t} - r^{x+t+1} = \frac{a}{z}$, (which investigations of Mr. Morgan have been very fully set forth above in the Notes to Dr. Halley's Discourse on Compound Interest,) we may venture to form a conjecture concerning the method in which Dr. Halley himself proceeded to investigate one of these expressions, though he himself thought fit to conceal his investigations of them, and even the foregoing Letter of Mr. De Moivre, (in which he had discovered and communicated to Dr. Halley an investigation of the second of those expressions,) which was found amongst Dr. Halley's papers after his death, and was published for the first time, 19 years after it, in the year 1761, in the *Mathematical Magazine and Philosophical Repository* above-mentioned. I shall therefore now proceed to state the manner in which it seems probable to me that Dr. Halley proceeded in his investigation of the first of these expressions, to wit, the expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, which he has given us for a near value of r , or the greater root of the binomial equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

A Conjecture concerning the Manner in which Dr. Edmund Halley investigated the Expression $1 + \sqrt{bb + 2by}^{\frac{1}{2}} - b$, or $1 + \sqrt{bb + 2by} - b$, which he has given us for the Value of r , or the greater Root of the Binomial Equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Art. 9. In the first place, then, in considering the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$ it must, probably, have occurred to Dr. Halley, that, since r^t , or the higher power of the unknown quantity r in this equation, is marked with the sign $-$, or is subtracted from the other term $\frac{z}{a} \times r$, this equation must have two roots, or, in the language of Dr. Halley and other modern Algebraists, two real and affirmative roots. He would therefore have set himself to inquire into the magnitudes, or values, of these two roots.

Secondly, in inquiring into the magnitudes of these two roots, he would soon observe that one of these roots must be equal to 1. For, if r is $= 1$, r^t will be $(1^t) = 1$, and $\frac{z}{a} \times r - r^t$ will be $(= \frac{z}{a} \times 1 - 1) = \frac{z}{a} - 1$, which is the absolute term of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; whence it follows that 1 must be one of the roots of the said equation.

In the third place he would conclude that this could not be the root that he wanted, or that would solve the Problem from which this equation was derived; because in that Problem r was put for the value of one pound, together with it's interest for one year, or one other time, at the end of which the interest was to be paid, and therefore must always be greater than 1*l.*, or 1, however small either that interest, or the time at which it is to be paid, may be. And hence he would conclude that 1 must be the lesser root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, and that it's other, or greater, root would be the value of r which should solve the Problem from which this equation had been derived. And he would therefore endeavour to find this greater root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$.

Art. 10. Now in this stage of his investigation he might have inquired into the limits of the magnitude of the greater root of this equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, in the manner set forth above in pages 363, 364, &c. --- 370; And, if he had done so, he would have found that this greater root must be

less than $\left(\frac{z}{a}\right)^{\frac{1}{t-1}}$, or $\sqrt[t-1]{\frac{z}{a}}$, but greater than $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$, or $\sqrt[t-1]{\frac{z}{at}}$; which lesser limit of the magnitude of this greater root is at the same time the greater limit of the magnitude of the lesser root of the equation, and therefore is greater than the lesser root of it, or 1, and consequently is nearer than 1 to the true value of r , or the greater root sought, and is therefore fitter than 1 to be taken for a first near value of r , or the said greater root, and to be made the ground-work of a further approach to the true value of the said greater root by Mr. Raphson's, or some other, method of approximation. And by supposing

this quantity $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$, (which is the lesser limit of the magnitude of the greater root sought,) to be an arithmetical mean quantity between the lesser root 1 and the greater root r , which is sought, he might have obtained the quantity $\left(\frac{z}{at}\right)^{\frac{1}{t-1}} + \left(\frac{z}{at}\right)^{\frac{1}{t-1}} - 1$, or $2 \times \left(\frac{z}{at}\right)^{\frac{1}{t-1}} - 1$, or (putting M for $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$, and d for $M - 1$, or $\left(\frac{z}{at}\right)^{\frac{1}{t-1}} - 1$), $M + M - 1$, or $2M - 1$, or $M + d$, for a first near value of r , which would be a little greater than it's true value, but nearer to it than either the lesser root 1, or the lower limit of the greater root, to wit, $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$, or M .

But Dr. Halley seems to have contented himself with making use of 1, or the lesser root of the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$, as a first near value of r , or it's greater root, and to have put v for the excess of the greater root r above the lesser root 1, whereby he had $r = 1 + v$.

He then substituted $1 + v$ instead of r in the equation $\frac{z}{a} \times r - r^t = \frac{z}{a} - 1$; which was thereby transformed into the equation $\frac{z}{a} \times \overline{1 + v} - \overline{1 + v}^t = \frac{z}{a} - 1$, or (by the binomial theorem,) $\frac{z}{a} \times \overline{1 + v} -$ the series $1 + tv + t \times \frac{t-1}{2} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times$

$\times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c,$
 continued to $t+1$ terms, $= \frac{z}{a} - 1$, or $\frac{z}{a} \times 1 + \frac{zv}{a} -$ the said series $=$
 $\frac{z}{a} - 1$; whence (adding the said series to both sides,) we shall have $\frac{z}{a} +$
 $\frac{zv}{a} = \frac{z}{a} - 1 +$ the said series; and (adding 1 to both sides,) we shall have
 $\frac{z}{a} + \frac{zv}{a} + 1 = \frac{z}{a} +$ the said series; and (subtracting $\frac{z}{a}$ from both sides,) we shall have
 $\frac{zv}{a} + 1 =$ the said series $1 + tv + t \times \frac{t-1}{2} \times vv + t$
 $\times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c,$ continued to $t+1$ terms; and (sub-
 tracting 1 from both sides,) we shall have $\frac{zv}{a} =$ the series $tv + t \times \frac{t-1}{2} \times vv$
 $+ t \times \frac{t-1}{2} \times \frac{t-2}{3} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^4 + t \times \frac{t-1}{2}$
 $\times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^5 + \&c,$ continued to t terms; and (dividing all the
 terms by v ,) we shall have $\frac{z}{a} =$ the series $t + t \times \frac{t-1}{2} \times v + t \times \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times vv + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4}$
 $\times \frac{t-4}{5} \times v^4 + \&c,$ continued to t terms; and (dividing all the terms by t ,) we shall have
 $\frac{z}{at} =$ the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2}$
 $\times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c,$ conti-
 nued to t terms; or (in the more usual manner of ranging the terms of an
 equation, with the absolute term, or known quantity, on the right-hand side
 of the mark of equality,) we shall have the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5}$
 $\times v^4 + \&c,$ continued to t terms, $= \frac{z}{at}$.

Art. 11. Having thus obtained this equation $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times$
 $\frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5}$
 $\times v^4 + \&c,$ continued to t terms, $= \frac{z}{at}$, Dr. Halley's next business would
 be to resolve it, or to find the value of the unknown quantity v , which is
 involved in it, and which is the interest of 1 pound for a single year, or other
 time at the end of which the interest is to be paid.

Now, if this interest were extremely small, (as, for example, if it were only 1 per cent, in which case v would be $= \frac{1}{100} l.$ or $0.01 l.$) and if (besides the smallness of this interest,) the term of years, denoted by t , during which the annuity was to continue, was a very short one, (as, for example, only 5 years,) the several terms $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3$ and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4$, &c, (that involve the powers v^3, v^4, v^5, v^6 , &c) to the end of the series, would be so small in comparison of the two terms $\frac{t-1}{2} \times v$ and $\frac{t-1}{2} \times \frac{t-2}{3} \times vv$, which involve the two first powers of v , that they might *safely* (or without producing any important error in the conclusion resulting from such an operation,) be neglected, and the three first terms of the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ might be considered as being equal to the whole of the said series, and consequently as being equal to the absolute term, or known quantity, $\frac{z}{at}$; and then the value of v , the unknown quantity, might be easily found by resolving the quadratick equation $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv = \frac{z}{at}$. For, if the term of years denoted by t was only 5 years, we should have $\frac{t-1}{2} (= \frac{5-1}{2} = \frac{4}{2}) = 2$, and $\frac{t-1}{2} \times \frac{t-2}{3} (= 2 \times \frac{t-2}{3} = 2 \times \frac{5-2}{3} = 2 \times \frac{3}{3}) = 2$, and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} (= 2 \times \frac{t-3}{4} = 2 \times \frac{5-3}{4} = 2 \times \frac{2}{4} = \frac{4}{4}) = 1$, and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} (= 1 \times \frac{t-4}{5} = 1 \times \frac{5-4}{5} = 1 \times \frac{1}{5}) = \frac{1}{5}$, and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times \frac{t-5}{6} (= \frac{1}{5} \times \frac{t-5}{6} = \frac{1}{5} \times \frac{5-5}{6} = \frac{1}{5} \times \frac{0}{6}) = 0$; and consequently all the following terms of the series would be equal to 0 likewise, and the whole series would consist of the five terms $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4$, or $1 + 2 \times v + 2 \times vv + 1 \times v^3 + \frac{1}{5} \times v^4$, and the equation obtained above for determining the value of v , to wit, the equation $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c = \frac{z}{at}$ would become $1 + 2v + 2vv + v^3 + \frac{v^4}{5} = \frac{z}{at}$. Now if, in this equation, v was less than $\frac{1}{100}$, it

it is evident that the term v^3 would be less than the 100th part of vv , and consequently than the 200th part of $2vv$, and that $\frac{v^4}{5}$ would be less than the 100,000th part of $2vv$; and therefore we might, in such a case, safely, or without erring much from the truth, consider the three terms $1 + 2v + 2vv$ as being equal to the whole five terms $1 + 2v + 2vv + v^3 + \frac{v^4}{5}$, and consequently as being equal to the absolute term $\frac{z}{at}$; and then the value of v might be obtained to a considerable degree of exactness by the easy operation of resolving the quadratick equation $1 + 2v + 2vv = \frac{z}{at}$, or $2v + 2vv = \frac{z}{at} - 1$, or $v + vv = \frac{z}{2at} - \frac{1}{2}$. But this is a case that seldom, or never, happens; because the interest of money is usually, not 1 per cent, or a hundredth part of the principal, but 3 per cent, or 4 per cent, or 5 per cent, and, in Dr. Halley's time, was 6 per cent; that is, v will be equal, not to 0.01, but to 0.03, or 0.04, or 0.05, or 0.06, the powers of which will decrease much less swiftly than the powers of 0.01. Thus, for example, if v is = 0.06, we shall have $vv = 0.0036$, and $v^3 = 0.000216$, and $v^4 = 0.00001296$; and consequently v^3 , or the fourth term of the series $1 + 2v + 2vv + v^3 + \frac{v^4}{5}$, will not be, as before, equal to only a 200th part of the third term $2vv$, but will be equal to $(2vv \times \frac{v}{2} = 2vv \times \frac{0.06}{2} = 2vv \times \frac{6}{100 \times 2} = 2vv \times \frac{6}{200}$, or to) $\frac{6}{200}$, or about $\frac{1}{33}$ part of the said third term, and therefore the two terms $v^3 + \frac{v^4}{5}$ cannot in this case be neglected, and the equation $1 + 2v + 2vv + v^3 + \frac{v^4}{5} = \frac{z}{at}$ be considered as being nearly the same with the quadratick equation $1 + 2v + 2vv = \frac{z}{at}$, without making the value of v resulting from such a supposition, and obtained by resolving the said quadratick equation, be much greater than its true value. And, if it should also happen that the term of years during which the annuity is to continue, and which is denoted by t , should not be a short term of only 5 years, but a term of 21, or more, years, (as is very frequently the case,) it is evident that in this case the co-efficients of the several powers of v in the aforefaid series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ will, in the first part of the series, continually increase, and consequently tend to increase the terms of which they are the co-efficients, and to counter-act the diminution of the said terms arising from the continual decrease of the powers of v . Thus, for example, if t is = 21, we shall have $\frac{t-1}{2}$ (=

$\frac{21-1}{2} = \frac{20}{2} = 10$, and $\frac{t-1}{2} \times \frac{t-2}{3} (= 10 \times \frac{t-2}{3} = 10 \times \frac{21-2}{3} = 10 \times \frac{19}{3} = \frac{190}{3}) = 63.33$, and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} (= 63.33 \times \frac{t-3}{4} = 63.33 \times \frac{21-3}{4} = 63.33 \times \frac{18}{4} = 63.33 \times \frac{9}{2} = \frac{569.97}{2}) = 284.98$, and $\frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} (= 284.98 \times \frac{t-4}{5} = 284.98 \times \frac{21-4}{5} = 284.98 \times \frac{17}{5} = \frac{4844.66}{5}) = 968.93$; and consequently the series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ will in this case be $= 1 + 10 \times v + 63.33 \times vv + 284.98 \times v^3 + 968.93 \times v^4 + \&c$, in which the co-efficients of the powers of v continually increase, and consequently counter-act the diminution of the terms to which they belong arising from the decrease of v , vv , v^3 , and v^4 , or the several powers of v . In this case therefore it would be by no means adviseable to neglect the terms that involve the cube, and the fourth power, and other following powers, of v , as being very small in comparison of the terms $10 \times v$ and $63.33 \times vv$; and, if we were to do so, for the sake of resolving the equation as a quadratick equation, the value of v thereby obtained would be very much greater than it's true value.

These observations must, no doubt, have occurred to Dr. Halley, and must have induced him to lay aside any intention which he might, perhaps, have at first been inclined to adopt, of neglecting all the terms of the aforesaid series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, that follow the third term $\frac{t-1}{2} \times \frac{t-2}{3} \times vv$, as being very small in comparison of the said third term, and resolving the equation as if it were a mere quadratick equation, involving only the three first terms of the said series, to wit, $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv$. And he would then, probably, have used his best endeavours to convert the aforesaid equation $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c = \frac{z}{at}$ into some other equation, (involving the same unknown quantity v ;) in which the co-efficients of the powers of v should not be increasing quantities, as they are in the foregoing equation.

Now, when he was in pursuit of such a method of converting the aforesaid equation into another equation better fitted for his purpose, it seems reasonable to suppose that he would have made the following reflections.

Though

Though in the series $1 + m \times x + m \times \frac{m-1}{2} \times x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times x^3 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times x^4 + \&c$ continued to $m + 1$ terms, (which is equal to the m th power of the binomial quantity $1 + x$ when m is a whole number,) the co-efficients of the several powers of x , to wit, x , x^2 , x^3 , x^4 , x^5 , x^6 , &c, will, if m is a whole number (as 21, or 30, or 40,) in the first half of the terms of the series continually increase, yet, if a fraction, as $\frac{1}{n}$, (in which n is supposed to be any whole number, or mixt number, whatsoever,) be substituted in the said series instead of the whole number m , the co-efficients of the powers of x in the new series thence arising, to wit, the series $1 + \frac{1}{n} \times x + \frac{1}{n} \times \frac{1}{n} - 1 \times x^2 + \frac{1}{n} \times \frac{1}{n} - 1 \times \frac{1}{n} - 2 \times x^3 + \frac{1}{n} \times \frac{1}{n} - 1 \times \frac{1}{n} - 2 \times \frac{1}{n} - 3 \times x^4 + \&c$, or $1 + \frac{1}{n} \times x + \frac{1}{n} \times \frac{1-n}{2n} \times x^2 + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times x^3 + \frac{1}{n} \times \frac{1-n}{2n} \times \frac{1-2n}{3n} \times \frac{1-3n}{4n} \times x^4 + \&c$, or $1 + \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \&c$, will decrease from the very beginning of the series. For the several generating fractions $\frac{n-1}{2n}$, $\frac{2n-1}{3n}$, $\frac{3n-1}{4n}$, &c, by the continual multiplication of which into $\frac{1}{n}$, (the co-efficient of x ,) the co-efficients of x^2 , x^3 , x^4 , &c are produced, are less than the fractions $\frac{n-0}{2n}$, $\frac{2n-0}{3n}$, $\frac{3n-0}{4n}$, &c, respectively, or than the fractions $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, &c, or than $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, &c, and therefore will always be less than 1, and consequently, by being multiplied into the co-efficients immediately preceeding them, will produce other co-efficients that will be less than those from which they were derived.

But this series $1 + \frac{1}{n} \times x - \frac{1}{n} \times \frac{n-1}{2n} \times x^2 + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times x^3 - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{3n-1}{4n} \times x^4 + \&c$ is equal to $1 + x^{\frac{1}{n}}$, or to the $\frac{1}{n}$ th power, or the n th root, of the binomial quantity $1 + x$.

Therefore, by extracting the n th root of any binomial quantity, as $1 + x$, we may always obtain a series of quantities, of which the first term shall be 1, and the second, third, fourth, and other following, terms shall involve the several powers of x , to wit, x , x^2 , x^3 , x^4 , x^5 , x^6 , &c *ad infinitum*, combined

with, or multiplied into, a set of co-efficients of which the first will be the fraction $\frac{1}{n}$, and the second, third, fourth, and other following ones will be a set of fractions less than $\frac{1}{n}$, and continually decreasing.

And, after making these reflections on the serieses that are equal to the m th power and to the $\frac{1}{n}$ th power, or the n th root, of the binomial quantity $1 + x$, it seems reasonable to suppose that Dr. Halley would have extended them to the serieses that would be equal to the like powers of a trinomial quantity, (as $1 + bx + cx^2$), or of a quadrinomial quantity, (as $1 + bx + cx^2 + dx^3$), or of a quinquinomial quantity, (as $1 + bx + cx^2 + dx^3 + ex^4$), or, in general, of a multinomial quantity, consisting of any greater number of terms involving the powers of the same quantity x , (as $1 + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$.) and would have concluded that, from the analogy that subsists between a binomial quantity, (such as $1 + x$, or $1 + bx$), and a trinomial quantity, (such as $1 + bx + cx^2$), and a quadrinomial, or quinquinomial, or other higher multinomial, quantity, the observations which he had found to take place concerning the serieses that are equal to the m th and the $\frac{1}{n}$ th powers of the binomial quantity $1 + x$, or $1 + bx$, would likewise be true concerning the serieses that would be equal to the m th and the $\frac{1}{n}$ th powers of the multinomial quantity $1 + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, and consequently that, if we were to find a series that should be equal to the n th root, or the $\frac{1}{n}$ th power, of a multinomial quantity $1 + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, the numeral co-efficients of the powers of x , to wit, $x, x^2, x^3, x^4, x^5, x^6, x^7, \&c$, in the terms of such a series, and more especially in some of the first terms of it, would be a series of decreasing fractions. And he would probably also have supposed that, since Sir Isaac Newton's series for exhibiting the value of $\overline{1+x}^m$, or the integral power of the binomial quantity $1 + x$ denoted by the whole number m , is found to be true also when the whole number m is converted into the fraction $\frac{1}{n}$, or that the series resulting from such conversion is equal to the $\frac{1}{n}$ th power, or the n th root, of the said binomial quantity $1 + x$, it would likewise be true that, if in the complicated series which Mr. De Moivre had shewn to be equal to $\overline{a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c}^m$, or the m th power of the multinomial quantity $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, when m is a whole number, we were to convert the whole number m into the fraction $\frac{1}{n}$, the series resulting from such conversion would be equal to the $\frac{1}{n}$ th power, or the n th root, of the said multinomial quantity $1 + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$.

And,

And, when Dr. Halley had thus extended these reflections to the serieses that are equal to the m th and the $\frac{1}{n}$ th powers of a multinomial quantity, as $a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, or $1 + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + bx^7 + \&c$, he would next be induced to apply them to the multinomial quantity, or series, $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$, continued to t terms, which forms the first, or left-hand, side of the above-mentioned equation which he had obtained in the manner described in art. 10 for the determination of the unknown quantity v ; which application would consist in raising the said series and it's equal, the absolute term, or known quantity, $\frac{z}{at}$, of the said equation, to some power of which the index would be a fraction, or number less than 1, such as $\frac{1}{n}$, or in extracting the n th roots of both sides of the said equation, n being some whole, or mixt, number, or number greater than 1. And, in chusing the said fraction $\frac{1}{n}$, which was to be the index of the power to which both sides of the said equation was to be raised, it was natural to make it equal to $\frac{1}{\frac{t-1}{2}}$, or to make n equal to $\frac{t-1}{2}$, or the co-

efficient of v in the second term of the said series, because then the two first terms of the new series, that would be equal to the $\frac{1}{n}$ th power of the former series, would be 1 and v . For, if the said former series $1 + \frac{t-1}{2} \times v + \frac{t-1}{2} \times \frac{t-2}{3} \times vv + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times v^3 + \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times \frac{t-4}{5} \times v^4 + \&c$ be raised to the $\frac{1}{\frac{t-1}{2}}$ th power by means of Mr. De Moivre's Multinomial

Theorem, it will be found that the five first terms of the series that is equal to the said power of the said former series will be $1 + v + \frac{t+1}{12} \times vv + 0 \times v^3 - \frac{-t^3 + 4t^2 - t - 6}{1440} \times v^4$, or $1 + v + \frac{t+1}{12} \times vv + 0 - \sqrt{\frac{t^3 - 4t^2 + t + 6}{1440}} \times v^4$,

as has been shewn at large above in pages 241, 242, 243, &c - - - 252 of the present Volume. Now in this series it luckily happens that the co-efficient of the fourth term (which involves the cube of v ,) is 0, and consequently that the said fourth term is equal to 0, or is wanting; and therefore we may with the greater safety suppose that the three first terms of this new series are nearly equal to the whole series, and consequently to it's equal, the absolute term of

the new equation, or to $\sqrt[\frac{2}{t-1}]{\frac{z}{at}}$; and then we shall have the quadratic equation

tion $1 + v + \frac{t+1}{12} \times vv$ (nearly) $= \frac{z}{at} \sqrt[t-1]{2}$ for our new and final equation, by the resolution of which the value of the unknown quantity v is to be determined.

When Dr. Halley had thus obtained the quadratic equation $1 + v + \frac{t+1}{12} \times vv = (\text{nearly}) \frac{z}{at} \sqrt[t-1]{2}$, the remaining part of his process in order to obtain his final expression of the value of $1 + v$, or r , would be very easy, and would be as follows.

He would first subtract 1 from both sides of the equation, and thereby obtain the equation $v + \frac{t+1}{12} \times vv = \frac{z}{at} \sqrt[t-1]{2} - 1$; and then, putting y for the known quantity $\frac{z}{at} \sqrt[t-1]{2} - 1$, (in order to shorten and simplify the notation,) he would convert the last equation into the equation $v + \frac{t+1}{12} \times vv = y$. Then, by multiplying all the terms of this equation into the fraction $\frac{12}{t+1}$, (in order to free vv , or the square of the unknown quantity v , from its co-efficient $\frac{t+1}{12}$;) he would obtain the equation $\frac{12}{t+1} \times v + vv = \frac{12}{t+1} \times y$. And this equation he would simplify in the notation of its terms by substituting $2b$ in it instead of the fraction $\frac{12}{t+1}$; whereby it would be converted into the equation $2b \times v + vv = 2by$. Then, by adding bb to both sides, he would obtain the equation $bb + 2bv + vv = bb + 2by$; and, by extracting the square-roots of both sides of this last equation, he would have $b + v = \sqrt{bb + 2by}$; and lastly, by subtracting b from both sides, he would obtain the equation $v = \sqrt{bb + 2by} - b$, or $v = (\sqrt{bb + 2by})^{\frac{1}{2}} - b$; whence $1 + v$, or r , would be $= 1 + \sqrt{bb + 2by} - b$, or $1 + (\sqrt{bb + 2by})^{\frac{1}{2}} - b$; which is the expression he has given us. Q. E. I.

Art. 12. In this manner I think it is probable that Dr. Halley investigated this expression. But it is to be lamented that he did not himself favour the Publick with an account of his investigations of this and the other two expressions given us in his Discourse on Compound Interest, as well as communicate the expressions themselves in the concise and abrupt manner in which they are stated in that Discourse.

End of the Tract intitled, "A Letter from Mr. De Moivre to Dr. Halley."

A LETTER

TO

JAMES WEST, Esq.

PRESIDENT OF THE ROYAL SOCIETY,

CONTAINING

THE INVESTIGATIONS OF TWENTY CASES OF COMPOUND INTEREST,

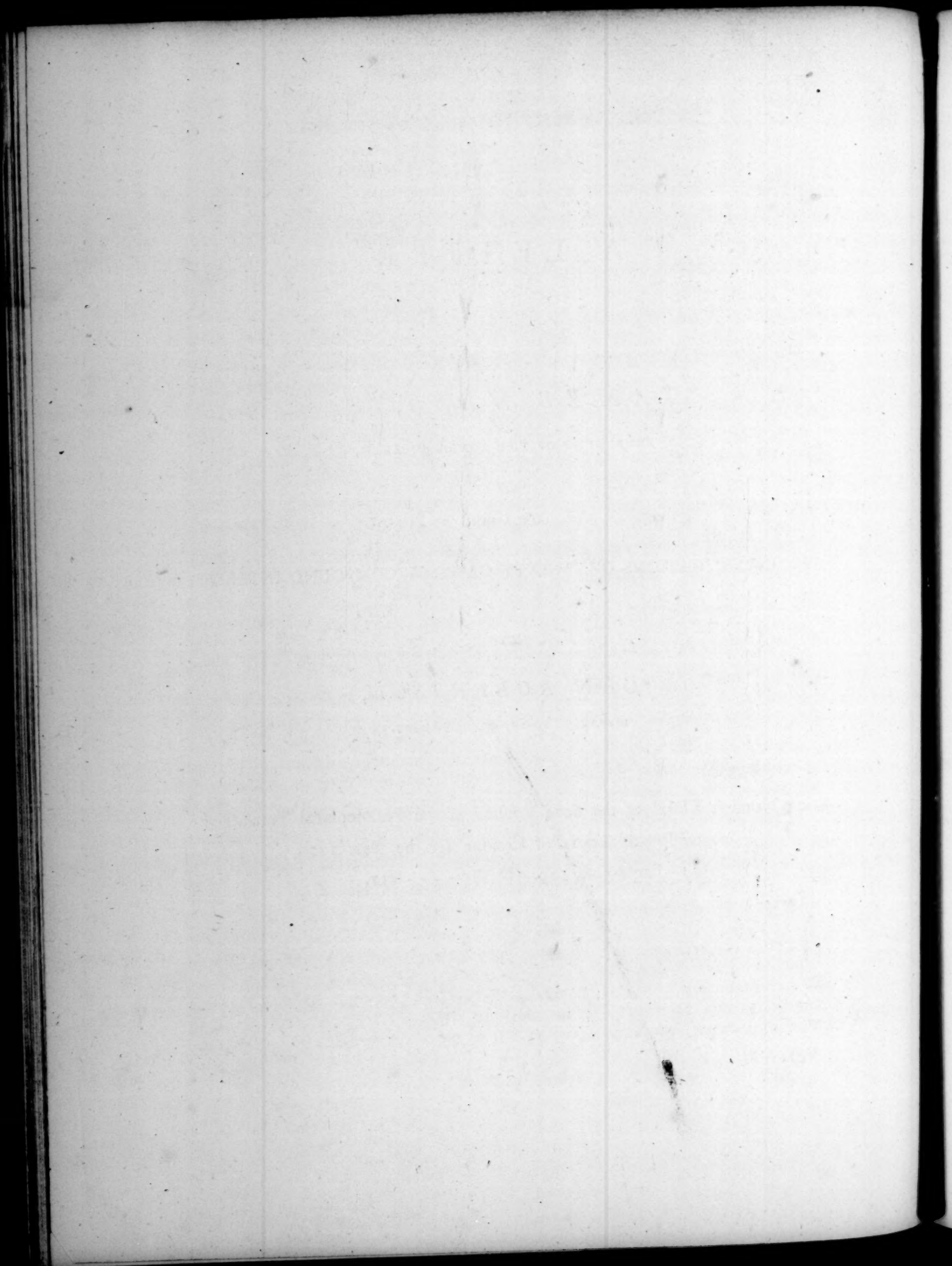
BY

JOHN ROBERTSON,

LIBRARIAN TO THE ROYAL SOCIETY.

Being Number XLIII of the 60th Volume of the Philosophical Transactions
of the Royal Society of London for the Year 1770,

Pages 508, 509, 510, &c - - - 517.



A LETTER

TO

JAMES WEST, Esq.

PRESIDENT OF THE ROYAL SOCIETY,

CONTAINING

THE INVESTIGATIONS OF TWENTY CASES OF COMPOUND INTEREST,

BY

JOHN ROBERTSON,

LIBRARIAN TO THE ROYAL SOCIETY.

Read before the Royal Society of London on the 13th of December, 1770.

SIR,

IT is well known that the consideration of Compound Interest is of great utility in the Computations respecting the values of Pensions, Annuities, Reversions, and other affairs relating to Money concerns; and therefore, many Mathematicians have bestowed some time on this useful subject, and have endeavoured to discover practical methods for solving the various cases that might occur: And in these inquiries, the use of Logarithms has been found of singular service in facilitating the operations.

The late William Jones, Esq. F. R. S. among the variety of Mathematical matters to which he gave attention, considered the business of Compound Interest fully, and did, many years ago, cause to be engraved on a Copper plate more cases in Interest than had been exhibited before that time: Several copies of impressions from that plate were distributed among his friends; to whom it appeared that he had treated this subject in a more extensive manner than had been done by other Mathematicians.

The Theorems, or Rules, for the Cases of Compound Interest, without their investigations, were inserted by Mr. Jones, in the quarto edition of Logarithms, published by Gardiner; and the Rules were also communicated to Mr. Dodson, who published them, by Mr. Jones's leave, with examples to illustrate the use

of his Antilogarithmic Tables: But the investigations of these Theorems not having yet been made public, it is apprehended that Gentlemen curious in these speculations would be pleased to see them: And should the following Essay be approved of by you, (who have been for many years acquainted with the Finances of this Nation, and the importance of a ready knowledge in computing the Interest on the National Loans,) you will be pleased to communicate it to the Royal Society.

I am, SIR,

Your most humble servant,

JOHN ROBERTSON.

In this subject five particulars are taken into consideration.

- 1st, The Annuity, Rent, or Pension.
- 2d, The Times that Annuity, Rent, or Pension is to continue.
- 3d, The Rate of Interest used in the computation.
- 4th, The Amount of those Rents, and their Interest, when they are forborn to be received any times after they are due.
- 5th, The present worth of those Rents, some times before they are due; or, of a Sum to be received before it is due, Discount being allowed.

And the investigations naturally fall under two heads.

First, The Consideration of Amounts.

Secondly, The Consideration of Discounts.

Under the first head an Equation is to be obtained between the Annuity, Time, Rate and Amount, from the known proportion that subsists between sums of money put to interest, during the same length of time, and the amounts of the principal and interest together.

Under the second head another Equation is to be formed between the Annuity, Rate, Time and present Worth, from the known proportion that subsists between the sums discounted, and their present worths, when done for the same time.

As these Equations involve quantities common to both of them, therefore other Equations may be thence deduced, containing all the five terms before specified. And hence, any three of the five terms being given, the other two are to be found; which admits of 20 Cases.

Some of these Cases will produce Affected Equations, where the index of the highest power of the unknown quantity will be the number of times the Rent is to continue, or to be paid: Therefore, the solution of those Cases will be given by a method of Approximation, as no better way has yet been discovered for the solution of Affected Equations, in numbers, above the third or fourth degree.

In the following investigations,

Let a = Annuity, Rent, or Pension.

n = Number of times that interest is to be paid for the annuity, or sum lent.

r = Rate of interest of 1*l.* for 1 time.

m = Amount of the annuity, or sum lent for n times, at r interest.

p = Principal sum used, or present worth of a sum before it is due.

Of Compound Interest.

First, In Amounts,

Let $q = 1 + r$ = Amount of 1*l.* for 1 time.

Now, a = last year's amount.

And $1 : q :: a : aq$ = last but one year's amount.

$1 : q :: aq : aq^2$ = last but two year's amount.

$1 : q :: aq^2 : aq^3$ = last but 3.

And so on to aq^{n-1} = first year's amount.

Therefore $a + aq + aq^2 + aq^3$ &c. $+ aq^{n-1} = m$.

But $a : aq :: m - aq^{n-1} : m - a$.

Euc. 12. v.

Then $m - a = mq - aq^n$.

Or $m - a = m + mr - aq^n$.

Therefore $aq^n - a = mr$.

Put $A = q^n$.

Then $A - 1 \times a = mr$.

Therefore $A = \frac{mr + a}{a}$.

Hence $a = \frac{mr}{A - 1}$,

$$m = \frac{A - 1 \times a}{r},$$

$$r = \frac{A - 1 \times a}{m}.$$

Second, In Discounts.

Since $q : 1 :: a : \frac{a}{q}$ = 1st year's present worth.

$q : 1 :: \frac{a}{q} : \frac{a}{q^2}$ = 2d year's present worth.

$q : 1 :: \frac{a}{q^2} : \frac{a}{q^3}$ = 3d.

And so on to $\frac{a}{q^n}$ = n th year's present worth.

Therefore

Therefore $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3} + \&c.$ to $\frac{a}{q^n} = p.$

But $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}.$

Euc. 12, v.

Therefore $\frac{pq^n - a}{q^n} = pq - a.$

Or $\frac{pq^n - a}{q^n} = p + pr - a.$

Therefore $aq^n - a = prq^n.$

Or $A - 1 \times a = prA.$

Therefore $A = \frac{a}{a - rp}.$

Hence $A - 1 \times a = mr = prA.$

Therefore $A = \frac{mr + a}{a} = \frac{a}{a - rp} = \frac{m}{p} = \overline{1 + r}^n.$

CASE I. Given a, m, p : Required $r.$

Since $A = \frac{m}{p}.$

Therefore $r = \frac{A - 1 \times a}{m}.$

CASE II. Given p, m, n : Required $r.$

Since $\overline{r + 1}^n = \frac{m}{p}.$

Therefore $r = \left(\frac{m}{p} \right)^{\frac{1}{n}} - 1.$

CASE III. Given a, m, n : Required $r.$

Since $A - 1 \times a = mr.$

Therefore $\left(\frac{A - 1}{nr} \right) = \frac{m}{na} = \frac{\overline{1 + r}^n - 1}{nr}.$

Now $\overline{1 + r}^n = 1 + nr + n \times \frac{n-1}{2} r^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} r^3, \&c.$

Therefore $\frac{m}{na} = 1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2, \&c.$

And

And $\left(\frac{m}{na}\right)^{\frac{2}{n-1}} = \left(1 + \frac{n-1}{2}r + \frac{n-1}{2} \times \frac{n-2}{3}r^2\right)^{\frac{2}{n-1}}$; which, by the Binomial Theorem*, will become $= 1 + r + \frac{n+1}{12}r^2$ (nearly).

$$\text{Let } D = \left(\frac{m}{na}\right)^{\frac{2}{n-1}} = 1 + r + \frac{n+1}{12}r^2,$$

$$\text{Then } r^2 + \frac{12}{n+1}r = D - 1 \times \frac{12}{n+1}, \text{ Let } 2E = \frac{12}{n+1},$$

$$\text{Then } r + E = (\sqrt{2 \times D - 1 + E \times E}) = F.$$

$$\text{Therefore } r = F - E.$$

$$\text{In this Solution, 1st, find } D = \left(\frac{m}{na}\right)^{\frac{2}{n-1}},$$

$$2\text{d, find } E = \frac{6}{n+1},$$

$$3\text{d, find } F = \sqrt{2 \times D - 1 + E \times E},$$

$$4\text{th, find } r = F - E.$$

CASE IV. Given a, p, n : Required r .

$$\text{Since } \overline{A-1} \times a = Apr.$$

$$\text{Therefore } \frac{p}{na} = \frac{A-1}{Anr} = \frac{1-A^{-1}}{nr} = \frac{1-\overline{1+1}^{-n}}{nr}.$$

$$\text{Now, } \overline{1+1}^{-n} = 1 - nr + n \times \frac{n+1}{2}r^2 - n \times \frac{n+1}{2} \times \frac{n+2}{3}r^3 + \&c.$$

$$\text{Therefore } \frac{p}{na} = 1 - \frac{n+1}{2}r + \frac{n+1}{2} \times \frac{n+2}{3}r^2, \text{ nearly.}$$

Then $\left(\frac{p}{na}\right)^{\frac{2}{n+1}} = \left(1 - \frac{n+1}{2}r + \frac{n+1}{2} \times \frac{n+2}{3}r^2\right)^{\frac{2}{n+1}}$; which, by the Binomial Theorem†, will become $= 1 + r - \frac{n-1}{12}rr$, nearly.

$$\text{Now, } \left(\frac{p}{na}\right)^{\frac{2}{n+1}} = \left(\frac{na}{p}\right)^{\frac{2}{n+1}}.$$

* Or, rather, by Mr. De Moivre's Multinomial Theorem, extended to the case of fractional powers.

† Or, rather, by Mr. De Moivre's Multinomial Theorem, extended to the case of fractional and negative powers.

Let

$$\text{Let } G = \left(\frac{na}{p} \right)^{\frac{2}{n+1}} = 1 + r - \frac{n-1}{12} r^2.$$

$$\text{Therefore } rr - \frac{12}{n-1} r = -\overline{G-1} \times \frac{12}{n-1}.$$

$$\text{Let } 2H = \frac{12}{n-1}.$$

$$\text{Then } r - H = (\sqrt{H - 2 \cdot \overline{G-1} \times H}) = K.$$

$$\text{Therefore } r = H - K.$$

CASE V. Given a, n, r : Required m .

$$\text{Since } \overline{A-1} \times a = mr. \text{ Therefore } m = \frac{\overline{A-1} \times a}{r}.$$

CASE VI. Given p, n, r : Required m .

$$\text{Since } A = \frac{m}{p}. \text{ Therefore } m = pA.$$

CASE VII. Given a, p, r : Required m .

$$\text{Since } \frac{m}{p} = \frac{a}{a-rp}. \text{ Therefore } m = \frac{ap}{a-rp}.$$

CASE VIII. Given a, p, n : Required m .

$$\text{Find } G = \left(\frac{na}{p} \right)^{\frac{2}{n+1}}; H = \frac{3}{\frac{1}{2}n-1}; K = \sqrt{H - 2\overline{G-1} \times H}.$$

$$\text{Now, } H - K = r \text{ by 4th. But } \overline{r+1}^n = \frac{m}{p}.$$

$$\text{Therefore } m = \overline{r+1}^n \times p.$$

CASE IX. Given a, n, r : Required p .

$$\text{Since } \overline{A-1} \times a = prA. \text{ Therefore } p = \frac{\overline{A-1} \times a}{Ar}.$$

CASE X. Given m, n, r : Required p .

$$\text{Since } A = \frac{m}{p}. \text{ Therefore } p = \frac{m}{A}.$$

CASE XI.

CASE XI. Given a, m, r : Required p .

Since $\frac{mr + a}{a} = \frac{m}{p}$. Therefore $p = \frac{ma}{mr + a}$.

CASE XII. Given a, m, n : Required p .

Find $D = \sqrt[n-1]{\frac{m}{na}^2}$; $E = \frac{3}{\frac{1}{2}n+1}$; $F = \sqrt{2D-1 + E \times E}$.

Now, $F - E = r$, by 3d. But $\sqrt[n]{1+r} = \frac{m}{p}$.

Therefore $p = \frac{m}{1+r^n}$.

CASE XIII. Given p, n, r : Required a .

Since $\overline{A-1} \times a = prA$. Therefore $a = \frac{prA}{A-1}$.

CASE XIV. Given p, m, n : Required a .

Since $A = \sqrt[n]{r+1} = \frac{m}{p}$. Therefore $r+1 = \left(\frac{m}{p}\right)^n$.

Hence $A-1$, and $B-1 (=r)$ are known.

Therefore $a = \frac{m \times \overline{B-1}}{A-1}$.

CASE XV. Given m, n, r : Required a .

Since $\overline{A-1} \times a = mr$. Therefore $a = \frac{mr}{A-1}$.

CASE XVI. Given p, m, r : Required a .

Since $A = \frac{m}{p}$, $A-1$ is given.

Therefore $a = \frac{mr}{A-1}$.

CASE XVII. Given m, p, r : Required n .

Put L , for logarithm; and L' , for arith. comp. of a log.

$$\text{Since } \overline{1+r}^n = \frac{m}{p}. \text{ Therefore } n = \frac{L, m+L, p}{L, r+1}.$$

CASE XVIII. Given a, p, r : Required n .

$$\text{Since } \overline{1+r}^n = \frac{a}{a-pr}. \text{ Therefore } n = \frac{L, a-L, \overline{a-pr}}{L, r+1}.$$

CASE XIX. Given a, p, m : Required n .

$$\text{Since } A = (\overline{r+1})^n = \frac{m}{p},$$

Then $(L, \overline{r+1})^n = L, m - L, p = L, A$;

Hence A , and $A - 1$, are known.

Also $L, \overline{A-1} + L, a + L', m = L, \overline{B-1}$ (by Case 14th):

Hence $r = (B - 1)$, and $r + 1$, are known.

$$\text{Therefore } n = \frac{L, A}{L, r+1}.$$

CASE XX. Given a, m, r : Required n .

$$\text{Since } (A =) \frac{mr+a}{a} = \overline{1+r}^n.$$

$$\text{Therefore } n = \frac{L, \overline{mr+a} - L, a}{L, 1+r}.$$

THEOREMS, OR RULES,

GIVEN BY THE LATE

WILLIAM JONES, Esq. F. R. S.

FOR THE

SOLUTION OF THE SEVERAL QUESTIONS RELATING
TO COMPOUND INTEREST :

Extracted from the Introduction to Gardiner's large Tables of Logarithms,
published in Quarto in the year 1742, intituled *The Explication*
and Use of the Logarithmick Tables, in pages 8
and 9 of the said Introduction.

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II. OF COMPOUND INTEREST.

1. **L**ET p be the principal sum, or present worth of several equal sums due hereafter at several equal times; n the number of those times; a the annuity or sum payable at one of those times; m the amount; the rate of compound interest being that of 1 to $1 + r$, where r is the gain of 1*l.* in one time: Any three of these five terms (p, n, a, m, r) being given, the rest are found by the following Table.

Where the capital letters $A, M, R, \&c.$ represent the logarithms of the numbers expressed by the small letters $a, m, r, \&c.$ $A', M', R', \&c.$ the complements of $A, M, R, \&c.$ Let $b = \frac{m}{p} = \overline{1+r}^n$, $b = a - pr$, $k = a + mr$, $\phi = \frac{1}{2} \overline{n-1}$, $\pi = \frac{1}{2} \overline{n+1}$, $B = n L. \overline{1+r} = M - P$, $C = \frac{1}{n} B$, $D = L. \overline{b-1}$, $E = D + A + M'$, $F = \frac{M+A'+N'}{\phi}$, $s = \frac{6}{n+1}$, $G = \frac{A+N+P'}{\pi}$, $t = \frac{6}{n-1}$, $V = L. \overline{a+p} + P'$, $U = nV' + A + P'$, $x = 2\overline{f-1} + s$, $y = t - 2\overline{g-1}$, $\frac{1}{2} \overline{X+S} = L. z$; $\frac{1}{2} \overline{Y+T} = L. \beta$.

CASE

CASE	Given	Req ^d	SOLUTION.
1	a, n, r	m	$M = D + A + R$
2	p, n, r	m	$M = B + P$
3	a, p, r	m	$M = H' + P + A$
4	a, n, p	m	$M = \begin{cases} nL.t - \beta + 1 + P, \text{ or} \\ nL.v - u + P, \text{ if } n \leq 40. \end{cases}$
5	a, n, r	p	$P = B' + D + A + R'$
6	m, n, r	p	$P = M - B$
7	a, n, m	p	$P = M - nL.z - s + 1$
8	a, r, m	p	$P = K' + M + A$
9	p, n, r	a	$A = B + D' + P + R$
10	p, n, m	a	$A = D' + L.c - 1 + M$
11	m, n, r	a	$A = D' + M + R$
12	p, m, r	a	$A = D' + M + R$
13	p, m, r	n	$n = \frac{M + P'}{L.1 + r}$
14	a, p, r	n	$n = \frac{A + H'}{L.1 + r}$
15	a, p, m	n	$n = \frac{M + P'}{L.c + 1}$
16	a, m, r	n	$n = \frac{K + A'}{L.1 + r}$
17	a, p, m	r	$R = D + A + M'$
18	p, n, m	r	$L.1 + r = \frac{M + P'}{n}$
19	a, n, m	r	$r = z - s.$
20	a, p, n	r	$r = t - \beta, \text{ or } = v - u - 1.$

In Case 20, take the latter, if n exceeds 40.

2. If the annuity commences ν time hence; let n = time of continuance;
 $b = \overline{1+r}^n$; $q = \overline{1+r}^\nu$; $\mathcal{Q} + P + R = L. w.$

1. Given, a, n, r, ν ; required p . $P = D + A + B' + R' + \mathcal{Q}'.$

2. Given, p, n, ν, r ; required a . $A = B + \mathcal{Q} + P + R + D'.$

3. Given, a, p, n, r ; required ν . $\nu = \frac{D + A + B' + R' + P'}{L. \overline{1+r}}.$

4. Given, a, p, r, ν ; required n . $n = \frac{L' \overline{a - w} + A}{L. \overline{1+r}}.$

5. Given, a, p, n, ν ; required r . (Put $\delta = \frac{2\nu + n + 1}{2}$, $u = \frac{12\delta}{nn-1}$,
 $\frac{A+N+P'}{\delta} = L. v$, $z = u - 2\overline{v-1}$, $\frac{1}{2}Z + U = L. \varepsilon$.) $r = u - \varepsilon.$

3. Let p be the present worth of an estate in fee-simple, whose annual rent is a , at the rate, or gain, r on $1l.$ per annum.

Then $A = P + R.$

4. Let n be the number of years' purchase that will buy an estate in fee-simple, at the rate, or gain, r on $1l.$ per annum.

Then $N = R'$; and $R = N'$; as $0 = L. 1.$

5. If an estate in fee-simple, whose annual rent is a , to commence ν years hence, is worth p ready money, at the rate r on $1l.$ per annum. Any three of the terms a, p, r, ν being given, the fourth is determined as follows; putting $q = \overline{1+r}^\nu.$

1. Given, a, r, ν ; required p . Then $P = A + R' + \mathcal{Q}'.$

2. Given, p, r, ν ; required a . Then $A = \mathcal{Q} + P + R.$

3. Given, a, p, r ; required ν . Then $\nu = \frac{A + P' + R'}{L. \overline{1+r}}.$

4. Given, a, p, ν ; required r . (Put $\rho = \frac{a}{p}$, $v = \nu + 1$, $\theta = \rho + 1$,
 $\theta' = \rho' + 1$, $\theta'' = \rho'' + 1$, &c. and $\lambda = v\rho + 1$, $\lambda' = v\rho' + 1$,
 $\lambda'' = v\rho'' + 1$, &c.) Then $r = \rho' = \frac{\lambda\rho}{\theta^v + v\rho}$, or $= \rho'' = \frac{\lambda'\rho}{\theta'^v + v\rho}$, or
 $= \rho''' = \frac{\lambda''\rho}{\theta''^v + v\rho}$, &c.

6. What sum p , at compound interest, in the time N amounts to M , and in the time n amounts to m ?

$$L.p = \frac{NL.m - nL.M}{N - n}.$$

7. If the sum p , at compound interest, in the time n amounts to m ; what will be the amount M of that sum p in the time N ?

$$L.M = \frac{N \times \overline{L.m - L.p} + nL.p}{n}.$$

8. If the sum p , at compound interest, in n years amounts to m ; in what time N will that sum p amount to M ?

$$N = \frac{n \times \overline{L.M - L.p}}{L.m - L.p}.$$

9. If the sums B, C, D, E , are to be paid b, c, d, e years (or spaces of time) hence; to find the time t when the sum $p (= B + C + D + E)$ may be paid, without detriment to either party, at a given rate r of compound interest:

Put $q = 1 + r =$ amount of 1*l.* for 1 time; $p' = \frac{B}{q^b}$, $p'' = \frac{C}{q^c}$, $p''' = \frac{D}{q^d}$,

$$p^{iv} = \frac{E}{q^e}.$$

$$\text{Then } t = \frac{L.p - L.p' + p'' + p''' + p^{iv}}{L.q}.$$

10. If the sum a , is to be paid at each payment, for n several payments; to find the time t , when the whole sum na , may be paid all together, at a given rate r of compound interest: Put $b = \overline{1 + r}^n$.

$$\text{Then } t = \frac{L.b - 1 + L.\overline{1 + r}^n + L.n + L.r}{L.1 + r}.$$

THE SAME
THEOREMS, OR RULES,
OF THE SAID
Mr. WILLIAM JONES,
CONCERNING
THE SEVERAL QUESTIONS RELATING TO COMPOUND INTEREST,
TOGETHER WITH
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Extracted from the Preliminary Discourse of Mr. James Dodson to his
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Folio, in the year 1742, in pages 52, 53, &c,
--- 65 of the said Preliminary Discourse.

THEORY OF ROLLS

WILLIAM JONES

OF THE UNIVERSITY OF CAMBRIDGE

IN TWO VOLUMES

LONDON: PRINTED BY J. JOHNSON, ST. PAUL'S CHURCH-YARD, 1801.

THE SAME
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--- 65 of the said Preliminary Discourse.

TO shorten the Rules and Examples following, observe that when n is put
to represent any number, N represents it's Logarithm, and $\bar{n}$ the Arith-
metical Complement of that Logarithm, and the same of any other Letter.
Also, $L. \overline{a+n}$ signifies the Logarithm of the Sum of a and n , and $L. \overline{a+n}$
= the Arithm. Comp. thereof.

PROPOSITION V.

To solve all Questions relating to Compound Interest, and Annuities com-
puted thereby,

Put a = Annuity,
 n = Time,

r = Rate,

m = Amount,

p = Principal, or present Worth.

Also, let $n \times L. \overline{r+1} = B;$

Likewise $M + \bar{p} = B,$

$$\frac{M + \bar{p}}{n} = E,$$

$$P + R = C,$$

$$M + R = D.$$

Case. Given. Required.

SOLUTION.

1	a, r, r	m	$\left\{ \begin{array}{l} M = L. \overline{b-1} + A + R. \\ M = B + P. \\ M = l. \overline{a-c} + P + A. \\ M = n \times L. \overline{q-s+1} + P. \text{ (See Case 20.)} \end{array} \right.$
2	p, n, r		
3	a, p, r		
4	a, p, n		
5	a, n, r	p	$\left\{ \begin{array}{l} P = L. \overline{b-1} + A + B + R. \\ P = M + B. \\ P = M - n \times L. \overline{i-b+1}. \text{ (See Case 19.)} \\ P = l. \overline{d+a} + M + A. \end{array} \right.$
6	n, m, r		
7	a, m, n		
8	a, m, r		
9	p, n, r	a	$\left\{ \begin{array}{l} A = l. \overline{b-1} + R + B + P. \\ A = L. \overline{e-1} + M + P + l. \overline{m-p}. \\ A = M + R + l. \overline{b-1}. \\ A = M + R + P + l. \overline{m-p}. \end{array} \right.$
10	p, m, n		
11	m, n, r		
12	p, m, r		
13	m, p, r	n	$\left\{ \begin{array}{l} n = \frac{M + P}{L. r + 1}. \\ n = \frac{A + l. \overline{a-c}}{L. r + 1}. \\ \text{Put } L. \overline{m-p} + A + P + P = F. \\ \text{Then } n = \frac{M + P}{L. f + 1}. \\ n = \frac{L. \overline{d+a} + A}{L. r + 1}. \end{array} \right.$
14	a, p, r		
15	a, p, m		
16	a, m, r		
17	a, m, p	r	$\left\{ \begin{array}{l} R = L. \overline{m-p} + A + P + P. \\ L. r + 1 = \frac{M + P}{n}. \\ \text{Put } \frac{M + A + P}{\frac{1}{2}n - 1} = G, \text{ and } \frac{6}{n+1} = b. \\ \frac{L. 2 \times g - 1 + b + H}{2} = I, \text{ then } i - b = r. \\ \text{Put } \frac{A + N + P}{\frac{1}{2}n + 1} = K, \text{ and } \frac{6}{n-1} = q. \\ \frac{L. q - 2 \times k - 1 + Q}{2} = S, \text{ then } q - s = r. \end{array} \right.$
18	p, m, n		
19	a, m, n		
20	a, p, n		

E X.

EXAMPLES.

1. If an Annuity of 1171*l.* 17*s.* 6*d.* per Year, (*a*) remain unpaid 16 Years, (*n*) what will it amount to at 5 $\frac{1}{4}$ per Cent? (*r**) See Case 1.

$$r = .0525 + 1 = 1.0525. \text{ It's Logarithm } 0.0222221.04$$

$$n = \underline{\quad 16 \quad}$$

$$n L. \overline{r+1} = B = 0.3555537, \text{ and } b = 2.2675333$$

$$b - 1 = 1.2675333. \text{ It's Logarithm } 0.1029594$$

$$A = 3.0688813$$

$$R = \underline{1.2798407}$$

$$M = 4.4516814, \text{ and } m = 28293.156*l.*$$

Answer, 28293*l.* 3*s.* 1 $\frac{1}{2}$ *d.*

2. If an Annuity of 1171*l.* 17*s.* 6*d.* per Year, payable Quarterly, remain unpaid 16 Years, what will it amount to at 5 $\frac{1}{4}$ per Cent?

$$\text{Now } a = \frac{1}{4} \times 1171.875 = 292.96875. n = 16 \times 4 = 64. 1 + r = \sqrt[4]{1.0525}$$

$$\frac{1}{4} L. \overline{1.0525} = 0.0055555.26 = L. \overline{1+r} \text{ and } 1 + r = 1.01287424 \text{ and}$$

$$n = \underline{\quad 64 \quad} \quad (r = .01287424)$$

$$B = 0.3555537, \text{ and } b = 2.2675333$$

$$L. \overline{b-1} = 0.1029594$$

$$A = 2.4668213$$

$$R = \underline{1.8902784}$$

$$M = 4.4600591, \text{ and } m = 28844.241 = 28844*l.* 4*s.* 10*d.*$$

3. What is the Amount of 12477*l.* 10*s.* 0 $\frac{3}{4}$ *d.* (*p*) put out to Interest for 16 Years, (*n*) at 5 $\frac{1}{4}$ per Cent. (*r*)? Case 2.

$$r = .0525 + 1 = 1.0525. \text{ It's Log. } 0.0222221.04$$

$$n = \underline{\quad 16 \quad}$$

$$B = 0.3555537$$

$$P = \underline{4.0961277}$$

$$M = 4.4516814, \text{ and } m = 28293.156*l.*$$

Answer, 28293*l.* 3*s.* 1 $\frac{1}{2}$ *d.*

4. Bought an Annuity of 1171*l.* 17*s.* 6*d.* (*a*) to continue a certain Time for 12477*l.* 10*s.* 0 $\frac{3}{4}$ *d.* (*p*) at 5 $\frac{1}{4}$ per Cent. (*r*) What will it amount to, if forborn to the End of that Time? Case 3.

* *r* signifies the Interest of 1*l.* viz. $\frac{5.25}{100} = .0525.$

$$P = 4,0961277$$

$$R = -2,7201593$$

$$C = 2,8162870, \text{ and } a = 1171,875$$

$$c = 655,06893$$

$$a - c = 516,80607 \quad \text{It's Ar. Co.} = -3,2866724$$

$$P = 4,0961277$$

$$A = 3,0688813$$

$$M = 4,4516814$$

Therefore $m = 28293,156 = 28293l. 3s. 1\frac{1}{2}d.$

To perform Case 4. find r by Case 20. and then m by Case 2.

5. What is the present Worth of an Annuity of 1171*l.* 7*s.* 6*d.* per Year, (a) to continue 16 Years, (n) Interest being allowed at 5*l.* per Cent. (r)? Case 5.

$r + 1 = 1,0525$. It's logarithm 0,0222221.04

$$n = 16$$

$$B = 0,3555537, \text{ and } b = 2,2675333.$$

$$b - 1 = 1,2675333. \text{ It's Log.} = 0,1029594$$

$$A = 3,0688813$$

$$B = -1,6444463$$

$$R = 1,2798407$$

$$P = 4,0961277, \text{ and } p = 12477,503.$$

Answer, 12477*l.* 10*s.* 0*l.*

6. What is the present Worth of an Annuity of 1171*l.* 17*s.* 6*d.* per Year, payable Quarterly, to continue 16 Years, Interest at 5*l.* per Cent?

Now $a = \frac{1}{4} \times 1171,875 = 292,96875. n = 64. r + 1 = \sqrt[4]{1,0525},$

$$\frac{1}{4} L. \overline{1,0525} = 0,0055555.26 = L. \overline{1+r}, \text{ and } 1+r = 1,01287424 \therefore r$$

$$n = 64$$

$$(= ,01287424$$

$$B = 0,3555537, \text{ and } b = 2,2675333$$

$$L. \overline{b-1} = 0,1029594$$

$$A = 2,4668213$$

$$B = -1,6444463$$

$$R = 1,8902784$$

$$P = 4,1045054, \text{ and } p = 12720,535 = 12720l. 10s. 8\frac{1}{2}d.$$

7. What

7. What present Money will pay a Debt of 28293*l.* 3*s.* 1½*d.* (*m*) due at the End of 16 Years, (*n*) Discount being allowed at 5¼*l.* per Cent. (*r*)? Case 6.

$$\begin{array}{rcl}
 r + 1 = 1,0525. & \text{It's Log. } 0,0222221.04 & \\
 n & = & 16 \\
 B & = & 0,3555537, \text{ and } \mathfrak{B} = -1,6444463 \\
 & & M = 4,4516814 \\
 P & = & 4,0961277, \text{ and } p = 12477,503
 \end{array}$$

Answer, 12477*l.* 10*s.* 0¾*d.*

To perform Case 7, find *r* by Case 19, and then *p* by Case 6.

8. An Annuity of 1171*l.* 17*s.* 6*d.* (*a*) to continue some limited Time, will amount, if forborn to the End thereof, to 28293*l.* 3*s.* 1½*d.* (*m*) Interest being computed at 5¼*l.* per Cent. (*r*). What is the present Worth thereof? Case 8.

$$\begin{array}{rcl}
 M = 4,4516814 & & \\
 R = -2,7201593 & & \\
 D = 3,1718407, \text{ and } a = 1171,875 & & \\
 & d = 1485,3907 & \\
 d + a = 2657,2657 \text{ it's Ar. Co.} = -4,5755650 & & \\
 & M = 4,4516814 & \\
 & A = 3,0688813 & \\
 & P = 4,0961277 &
 \end{array}$$

Therefore $p = 12477,503 = 12477*l.* 10*s.* 0¾*d.*$

9. What Annuity to continue 16 Years, (*n*) can be purchased for 12477*l.* 10*s.* 0¾*d.* (*p*) Interest being allowed at 5¼*l.* per Cent. (*r*)? Case 9.

$$\begin{array}{rcl}
 r + 1 = 1,0525. & \text{It's Logarithm } 0,0222221.04, & \\
 n = & 16 & \\
 B = 0,3555537 \text{ and } b = 2,2675333 & & \\
 R = -2,7201593 & & \\
 P = 4,0961277 & & \\
 b - 1 = 1,2675333; \text{ it's Ar. Co.} = -1,8970406 & & \\
 A = 3,0688813, \text{ and } a = 1171,875. & &
 \end{array}$$

Answer, 1171*l.* 17*s.* 6*d.*

10. What

10. What Annuity payable Quarterly, and to continue 16 Years, can be purchased for 12720*l.* 10*s.* 8½*d.* Interest being allowed at 5¼*l.* per Cent?

Now $n = 64$. $r + 1 = \sqrt[4]{1.0525}$, then it's Log. = 0,0055555,26
 $L. \frac{r}{r+1} = 0,0055555$, then $r + 1 = 1,01287424$. $n = 64$

$$\begin{aligned} B &= 0,3555537, \text{ and } b = \\ r &= ,01287424. \text{ It's Logarithm} = -2,1097216 \quad (2,2675333) \\ P &= 4,1045054 \\ L. \frac{b}{b-1} &= -1,8970406 \\ A &= 2,4668213, \text{ and } a = \\ & \quad (292,9687) \end{aligned}$$

Answer, The Quarterly Payment is 292*l.* 19*s.* 4½*d.*

11. What Annuity to continue 16 Years, (n) is now worth 12477*l.* 10*s.* 0¼*d.* (p) and at the Expiration thereof, will amount to 28293*l.* 3*s.* 1½*d.* (m)? Case 10.

$$\begin{aligned} M &= 4,4516814 \\ P &= -5,9038773 \end{aligned}$$

$n = 16$ 0,3555537 (0,0222221 = E and $e = 1,0525$, and $e - 1 = ,0525$

$$\begin{aligned} L. \frac{e-1}{e} &= -2,7201593 & m &= 28293,156 \\ M &= 4,4516814 & p &= 12477,503 \\ P &= 4,0961277 \\ L. \frac{m-p}{m} &= -5,8009129 & m-p &= 15815,653 \\ A &= 3,0688813, \text{ and } a = 1171,875 = 1171*l.* 17*s.* 6*d.* \end{aligned}$$

12. What Annuity being forborn 16 Years, (n) will amount to 28293*l.* 3*s.* 1½*d.* (m) if Interest be allowed at 5¼*l.* per Cent. (r)? Case 11.

$r + 1 = 1,0525$. It's Logarithm 0,0222221,04
 $n = 16$

$B = 0,3555537$, and $b = 2,2675333$.

$$\begin{aligned} b - 1 &= 1,2675333 \text{ it's Ar. Co.} = -1,8970406 \\ M &= 4,4516814 \\ R &= -2,7201593 \end{aligned}$$

$A = 3,0688813$ and $a = 1171,875$.

Answer, 1171*l.* 17*s.* 6*d.*

13. What

13. What Annuity payable Quarterly, being forborn 16 Years, will amount to 28844*l.* 4*s.* 10*d.* Interest being allowed at 5½*l.* per Cent?

Now $n = 64$, and $r + 1 = \sqrt[4]{1,0525}$. It's Log. 0,0055555.26
 $n = 64$

$$B = 0,3555537, \text{ and } b = 2,2675333.$$

$$l. \overline{b-1} = -1,8970406$$

$$M = 4,4600591$$

$$R = -2,1097216$$

$$A = 2,4668213, \text{ and } a = 292,9687.$$

Answer, The Quarterly Payment will be 292*l.* 19*s.* 4½*d.*

14. What Annuity is now worth 12477*l.* 10*s.* 0¼*d.* (*p*) and will amount to 28293*l.* 3*s.* 1½*d.* (*m*) if Interest be computed at 5½*l.* per Cent? (*r*) Case 12.

$$m = 28293,156$$

$$p = 12477,503$$

$$m - p = 15815,653, \text{ and } l. \overline{m-p}$$

$$M = 4,4516814$$

$$R = -2,7201593$$

$$P = 4,0961277$$

$$= -5,8009129$$

$$A = 3,0688813, \text{ and } a = 1171,875.$$

15. In what Time will 12477*l.* 10*s.* 0¼*d.* (*p*) being put to Interest at 5½ per Cent. (*r*) amount to 28293*l.* 3*s.* 1½*d.* (*m*)? Case 13.

$$M = 4,45168$$

$$P = -5,90387$$

$$r + 1 = 1,0525. \text{ It's Logarithm } 0,02222 \quad 0,35555 \quad (16 = n)$$

$$13333$$

$$13333$$

$$0000$$

16. If an Annuity of 1171*l.* 17*s.* 6*d.* per Year, (*a*) be sold for 12477*l.* 10*s.* 0¼*d.* (*p*) allowing Interest at 5½*l.* per Cent. (*r*). How long is it to continue? Case 14.

562 MR. JONES'S THEOREMS, OR RULES, WITH EXAMPLES, CONCERNING

$$P = 4,09613$$

$$R = -2,72016$$

$$C = 2,81629, \text{ and } a = 1171,875$$

$$c = 655,073$$

$$a - c = 516,802. \text{ l. } \overline{a - c} = -3,28667$$

$$A = 3,06888$$

$$r + 1 = 1,0525. \text{ It's Logarithm } 0,02222) 0,35555 (16 = n.$$

To perform Case 15, find r by Case 17; and then n by Case 13.

17. How long must an Annuity of 1171*l.* 17*s.* 6*d.* (a) be forborn, to amount to 28293*l.* 3*s.* 1½*d.* (m) Interest being computed at 5½*l.* per Cent. (r)? Case 16.

$$M = 4,45168$$

$$R = -2,72016$$

$$D = 3,17184, \text{ and } a = 1171,87$$

$$d = 1485,39$$

$$a + d = 2657,26. \text{ L. } \overline{a + d} = 3,42443$$

$$Q = -4,93112$$

$$r + 1 = 1,0525. \text{ It's Logarithm } 0,02222) 0,35555 (16 = n.$$

18. At what Rate per Cent. will an Annuity of 1171*l.* 17*s.* 6*d.* (a) which is now worth 12477*l.* 10*s.* 0¼*d.* (p) amount to 28293*l.* 3*s.* 1½*d.* (m)? Case 17.

$$m = 28293,156$$

$$p = 12477,503$$

$$m - p = 15815,653. \text{ It's Logarithm } = 4,1990871$$

$$A = 3,0688813$$

$$Q = -5,5483186$$

$$P = -5,9038723$$

$$R = -2,7201593, \text{ and } r = ,0525\%$$

19. At what Rate of Interest, will 12477*l.* 10*s.* 0¼*d.* (p) put to Interest for 16 Years, (n) amount at the End thereof to 28293*l.* 3*s.* 1½*d.* (m)? Case 18.

$$M = 4,45168$$

$$P = -5,90387$$

$$n = 16) 0,35555 (0,02222 = L. \overline{1,0525} = 1 + r.$$

$$\text{Then } 1,0525 - 1 = ,0525 \times 100 = 5,25, \text{ the Answer.}$$

20. At what Rate of Interest, will an Annuity of 117*l.* 17*s.* 6*d.* (*a*) in 16 Years, (*n*) amount to 28293*l.* 3*s.* 1½*d.* (*m*)? Case 19.

$$\overline{n-1} = 15, \text{ and } \frac{1}{2}\overline{n-1} = 7,5, \text{ and } \overline{n+1} = 17.$$

$$M = 4,45168$$

$$A = -4,93112$$

$$N = -2,79588$$

$$7,5) 0,17868 \text{ (} 0,02382 = G, \text{ and } g = 1,05638 \because g - 1 = ,05638.$$

$$17) 6,00000 \text{ (} ,35294 = b, \text{ and } H = -1,54770$$

$$2 \times \overline{g-1} = ,11276$$

$$0,46570 = 2 \times \overline{g-1} + b.$$

$$L. \overline{0,4657} = -1,66811$$

$$H = -1,54770$$

$$2) -1,21581 \text{ (} -1,60791 = I, \text{ and } i = ,40542$$

$$b = ,35294$$

$$i - b = ,05248 = r \text{ nearly.}$$

21. If an Annuity of 117*l.* 17*s.* 6*d.* (*a*) to continue 16 Years, (*n*) be fold for 12477*l.* 10*s.* 0¾*d.* (*p*) What Rate of Interest was allowed? Case 20.

$$\overline{n-1} = 15, \text{ and } \overline{n+1} = 17, \text{ and } \frac{1}{2}\overline{n+1} = 8,5.$$

$$A = 3,06888$$

$$N = 1,20412$$

$$P = -5,90387$$

$$8,5) 0,17687 \text{ (} 0,02081 = K, \text{ and } k = 1,04908, \text{ and } k - 1 = ,04908.$$

$$15) 6,00000 \text{ (} ,40000 = q, \text{ and } Q = -1,60206$$

$$2 \times \overline{k-1} = ,09816$$

$$,30184 = q - 2 \times \overline{k-1}.$$

$$L. \overline{0,30184} = -1,47978$$

$$Q = -1,60206$$

$$2) -1,08184 \text{ (} -1,54092 = S, \text{ and } s = ,347472$$

$$q = ,400000$$

$$q - s = ,052528 = r.$$

Annuities in Reversion, computed by Compound Interest.

Put a = Annuity,
 n = Time of it's Duration,
 r = Rate,
 m = Amount,
 p = Present Worth,
 x = Time before the Annuity commences.

} Also, $n \times L. \overline{r+1} = B.$

Cafe.	Given.	Required.	SOLUTION.
1	a, r, n, x	p	$P = L. \overline{b-1} + A + R + \overline{x+n} \times L. \overline{r+1}.$
2	p, r, n, x	a	$A = R + P + l. \overline{b-1} + \overline{x+n} \times L. \overline{r+1}.$
3	a, r, n, p	x	$x = \frac{L. \overline{b-1} + A + B + P + R}{L. \overline{r+1}}.$
4	a, r, x, p	n	Put $P + R + x \times L. \overline{r+1} = C.$ Then $n = \frac{A + l. \overline{a-c}}{L. \overline{r+1}}.$
5	a, n, x, p	r	Put $\frac{2x + \overline{n+1}}{2} = b,$ and $u = \frac{12b}{nn-1}.$ $\frac{A + N + P}{b} = V,$ and $z = u - 2 \times \overline{v-1}.$ Then $\frac{Z + U}{2} = E,$ and $u - e = r.$

EXAMPLES.

1. If an Annuity of 117*l.* 17*s.* 6*d.* (a) to continue 16 Years, (n) will commence at the end of 5 Years. (x) What is it's present Worth, Interest at $5\frac{1}{4}\%$ per Cent. (r)? Cafe 1.

$$\overline{r+1} = 1,0525. \text{ It's Logarithm} = 0,0222221,04$$

$$n = \underline{\quad 16 \quad}$$

$$B = 0,3555537, \text{ and } b = 2,2675333.$$

$$b - 1 = 1,2675333. \text{ It's Logarithm} = 0,1029594$$

$$A = 3,0688813$$

$$R = 1,2798407$$

$$L. \overline{r+1} = -1,977777896 \times (n + x =) 21 = -1,5333358$$

$$P = 3,9850172, \text{ and } p =$$

Answer, 966*ol.* 17*s.* 10*d.*

(9660,8913.
2. What

2. What Annuity to commence at the End of 5 Years, (x) and continue 16 Years, (n) can be purchased for 9660*l.* 17*s.* 10*d.* (p) Interest at 5*½**l.* per Cent. (r)? Case 2.

$$\begin{aligned} R &= -2,7201593 \\ P &= 3,9850172 \\ L. \frac{1}{1+r} &= 0,0222221.04 \times (n+x=) 21 = 0,4666642 \\ n &= 16 \end{aligned}$$

$$B = 0,3555537, \text{ and } b = 2,2675333$$

$$L. \frac{b}{b-1} = -1,8970406$$

$$A = 3,0688813, \text{ and } a = 1171,875.$$

3. An Annuity of 1171*l.* 17*s.* 6*d.* per Year, (a) to continue 16 Years, (n) is now sold for 9660*l.* 17*s.* 10*d.* (p) Interest at 5*½**l.* per Cent. (r) How long will it be before the Annuity commences? Case 3.

$$\begin{aligned} L. \frac{1}{1+r} &= 0,02222.21 \\ n &= 16 \end{aligned}$$

$$B = 0,35555.36, \text{ and } b = 2,26753.$$

$$\begin{aligned} L. \frac{b}{b-1} &= 0,10296 \\ A &= 3,06888 \\ B &= -1,64445 \\ P &= -4,01498 \\ R &= 1,27984 \end{aligned}$$

$$L. 1+r = 0,02222) 0,11111 (5 = x.$$

4. An Annuity of 1171*l.* 17*s.* 6*d.* (a) to commence at the End of 5 Years, (x) is now sold for 9660*l.* 17*s.* 10*d.* (p) at 5*½**l.* per Cent. (r). How long is it to continue? Case 4.

$$\begin{aligned} L. \frac{1}{1+r} &= 0,02222.21 \times (x=) 5 = 0,11111 \\ P &= 3,98502 \\ R &= -2,72016 \end{aligned}$$

$$C = 2,81629, \text{ and } c = 655,073.$$

$$\begin{aligned} a &= 1171,875 \\ c &= 655,073 \end{aligned}$$

$$\begin{aligned} a - c &= 516,802 \text{ It's Arith. Comp.} = -3,28667 \\ A &= 3,06888 \end{aligned}$$

$$L. \frac{1}{1+r} = 0,02222) 0,35555 (16 = n.$$

5. At what rate *per Cent.* will an Annuity of 117*l.* 17*s.* 6*d.* *per Year* (*a*), to commence at the End of 5 Years (*x*), and to continue 16 Years (*n*), be now fold for 966*ol.* 17*s.* 10*d.* (*p*)? Case 5.

$$2x = 10 + 16 + 1 = 27, \text{ and } \frac{27}{2} = 13,5 = b; 13,5 \times 12 = 162$$

$$16 \times 16 = 256 - 1 = 255) 162 (= ,635294 = u.$$

$$A = 3,06888$$

$$N = 1,20412$$

$$P = -4,01498$$

$$b = 13,5) 0,28798 (0,02133 = V, \text{ and } v = 1,05034 \therefore v - 1 = ,05034$$

$$(\times 2 = ,10068.$$

$$u = ,635294$$

$$2 \times \frac{u}{v - 1} = ,10068$$

$$z = 0,534614, \text{ and } Z = -1,72804$$

$$U = -1,80297$$

$$2) -1,53101$$

$$E = -1,76550, \text{ and } e = 0,582774 : \text{ Then } u =$$

$$,635294 - (e =) ,582774 = ,05252 = r.$$

The Value of Estates in Fee-Simple computed by Compound Interest.

Put *a* = the Yearly Income,
r = the Rate of Interest,
p = the present Value,
n = the Number of Years' Purchase.

Case. Given. Required. SOLUTION.

1	<i>a</i> , <i>r</i> ,	<i>p</i>	$P = A + R.$
2	<i>p</i> , <i>r</i> ,	<i>a</i>	$A = P + R.$
3	<i>a</i> , <i>p</i> ,	<i>r</i>	$R = A + P.$
4	<i>r</i> ,	<i>n</i>	$N = R.$
5	<i>n</i> ,	<i>r</i>	$R = P.$

EX-

EXAMPLES.

1. What is an Estate of 1171*l.* 17*s.* 6*d.* *per Ann.* (*a*) in Fee-Simple, worth at 5½*l.* *per Cent.* (*r*) ?

$$A = 3,0688813$$

$$R = 1,2798407$$

$$P = 4,3487220, \text{ and } p = 22321,429 = 22321*l.* 8*s.* 7*d.*$$

2. What Yearly Rent in Fee-Simple can be purchased for 22321*l.* 8*s.* 7*d.* (*p*) at 5½*l.* *per Cent.* (*r*) ?

$$P = 4,3487220$$

$$R = -2,701593$$

$$A = 3,0688813, \text{ and } a = 1171,875 = 1171*l.* 17*s.* 6*d.*$$

3. If an Estate of 1171*l.* 17*s.* 6*d.* *per Ann.* (*a*) in Fee-Simple, be sold for 22321*l.* 8*s.* 7*d.* (*p*) at what Rate *per Cent.* was the Interest computed ?

$$A = 3,0688813$$

$$P = -5,6512700$$

$$R = -2,7201593, \text{ and } r = ,0525 = 5\frac{1}{4} \text{ per Cent.}$$

4. If an Estate in Fee-Simple be sold at 5½*l.* *per Cent.* (*r*). How many years' Purchase was given ?

$$R = N = 1,27984, \text{ and } n = 19,0476 = 19 \text{ Years, } 17 \text{ Days.}$$

5. If an Estate in Fee-Simple be sold for 19 Years, 17 Days, Purchase (*n*), at what Rate *per Cent.* is the Interest allowed to the Purchaser ?

$$R = R = -2,7201593, \text{ and } r = ,0525 = 5\frac{1}{4} \text{ per Cent.}$$

The Value of Estates in Reversion computed by Compound Interest.

Put a = the Yearly Income,

r = the Rate of Interest,

x = Time before the Payments commence,

p = the present Worth.

Case.	Given.	Required.	SOLUTION.
1	a, r, x	p	$P = A + R + x \times L. \overline{r+1}.$
2	p, r, x	a	$A = P + R + x \times L. \overline{r+1}.$
3	p, a, r	x	$x = \frac{A + P + R}{L. \overline{r+1}}.$

EXAMPLES.

1. What is the present Worth of an Estate of 1171*l.* 17*s.* 6*d.* per Ann. (a) in Fee-Simple, to commence at the End of 5 Years (x), Interest being allowed at 5*½* *l.* per Cent. (r)?

$$L. \overline{1+r} = 0,0222221.04$$

$$x = \underline{\quad 5 \quad}$$

$$0,1111105; \text{ it's Ar. Co. } = -1,8888895$$

$$R = 1,2798407$$

$$A = \underline{3,0688813}$$

$$P = 4,2376115, \text{ and } p = 17282,697 = (17282*l.* 13*s.* 11*¼**d.*)$$

2. What Yearly Rent in Fee-Simple, to commence at the End of 5 Years (x), can be purchased for 17282*l.* 13*s.* 11*¼**d.* (p) allowing Interest at 5*½* *l.* per Cent. (r)?

$$L. \overline{1+r}$$

$$L. \overline{1+r} = 0,0222221,04$$

$$x = 5$$

$$0,1111105$$

$$R = -2,7201593$$

$$P = 4,2376115$$

$$A = 3,0688813, \text{ and } a = 1171,875 = 1171l. 17s. 6d.$$

3. An Estate in Reversion of 1171l. 17s. 6d. *per Ann.* (*a*) Fee-Simple, was at 5 $\frac{1}{4}$ l. *per Cent.* (*r*), bought for 17282l. 13s. 11 $\frac{1}{4}$ d. (*p*) at the End of what Time doth it commence?

$$A = 3,06888$$

$$R = -5,76239$$

$$R = 1,27984$$

$$L. \overline{1+r} = 0,02222) 0,11111 (5 = x.$$

What Sum is that (*p*) which put to Compound Interest for 7 Years (*n*), will amount to 2996,6l. (*m*), and in 9 Years (*x*), to 3366,9812l. (*z*)?

SOLUTION.

$$P = \frac{xM - nZ}{x - n}.$$

$$\text{Example. } M = 3,4766288 \times 9 (= x) = 31,2896592$$

$$Z = 3,5272407 \times 7 (= n) = 24,6906849$$

$$x - n = 2) 6,5989743$$

$$P = 3,2994871, \text{ and } p = 1992,9074l.$$

If the Sum 1992,907l. (*p*) in 7 Years (*n*), amounts to 2996,6l. (*m*); What Sum (*z*) will it amount to in 9 Years (*x*), at Compound Interest?

SOLUTION.

$$Z = \frac{xM - x - n \times P}{n}.$$

$$\text{Example. } M = 3,4766288 \times 9 (= x) = 31,2896592$$

$$P = 3,2994870 \times 2 (= x - n) = 6,5989740$$

$$n = 7) \underline{24,6906852}$$

$$Z = 3,5272407, \text{ and } z = 3366,9812.$$

If the Sum 1992,907*l.* (*p*) amounts to 2996,6*l.* (*m*) in 7 Years (*n*), in what Time (*x*) will it amount to 3366,9812*l.* (*z*), at Compound Interest?

SOLUTION.

$$x = \frac{Z - P}{M - P} \times n.$$

$$\text{Example. } M = 3,4766288 \text{ and } Z = 3,5272407$$

$$P = 3,2994870 \quad = 3,2994870$$

$$\underline{0,1771418}$$

$$\underline{0,2277537} \text{ (1,285714,}$$

$$\text{And } 1,285714 \times 7 = x = 9 \text{ nearly.}$$

Equation of Payments.

If the Sums 20*l.* (*b*); 45*l.* (*c*); 56*l.* (*d*), &c. are to be paid 2 (*n*), 3 (*x*), 5 (*y*), &c. Years, or other Spaces of Time, hence; in what Time may the whole Sum 121*l.* (*p* = *b* + *c* + *d*, &c.) be paid, without any Detriment to the Parties, at 5*l. per Cent.* (*r*) Compound Interest?

SOLUTION.

(Put $q = 1 + r$ = amount of 1*l.* in 1 Time; also, let $B - nQ = K'$; $C - xQ = K''$; $D - yQ = K'''$, &c. And $s = k' + k'' + k'''$, &c.)

Then $\frac{P - s}{Q}$ = the Time required.

$$\text{Example. } r + 1 = 1,05, \text{ It's Logarithm} = Q = 0,0211893.$$

$$B = 1,3010300 - 0,0423786 (= Q \times 2) = K' = 1,2586514 \text{ \& } k' = 18,1406$$

$$C = 1,6532125 - 0,0635679 (= Q \times 3) = K'' = 1,5896446 \text{ \& } k'' = 38,8727$$

$$D = 1,7481880 - 0,1059465 (= Q \times 5) = K''' = 1,6422415 \text{ \& } k''' = 43,8775$$

$$s = 100,8908$$

$$P =$$

$$P = 2,082,7854$$

$$S = 2,003,8515$$

$$Q = 0,0211893) 0,0789339 \text{ (3,72518 Years, the Answer.)}$$

If the Sum 1175,875*l.* (*a*) is to be paid at each payment for 16 (*n*) several Payments; in what Time (*x*) should the whole Sum (*na*) be paid altogether, at 5*l.* $\frac{1}{4}$ (*r*) per Cent. Compound Interest?

SOLUTION.

(Put $B = n \times L. \overline{r+1}$. Then,

$$x = \frac{N + R + B + L. \overline{b-1}}{L. \overline{r+1}}.$$

Example. $r + 1 = 1,0525$. It's Log. 0,0222221.04
 $n = 16$

$$B = 0,3555537 \text{ and } b = 2,2675333.$$

$$N = 1,2041200.$$

$$R = -2,7201593$$

$$b - 1 = 1,2675333; \text{ it's Ar. Co. } = -1,8970406$$

$$L. \overline{r+1} = 0,0222221) 0,1768736 \text{ (7,95935 = } x.$$

Answer, 7,95935 Years.

THE SEVERAL QUESTIONS RELATING TO THE FOLLOWING PROBLEM

$$P = 1,000,000$$

$$i = 0.05$$

$$Q = 0.01803 \text{ (3.75\% Yearly, the Answer)}$$

If the sum 1,758,874 (a) is to be paid at each payment for 10 years.
 Payments in what time (x) should the whole sum (a) be paid altogether
 at 5% (r) per Cent Compound Interest?

SOLUTION

Put $B = a \times \frac{1}{1+i}$. Then,

$$x = \frac{V + B + B + B + \dots + B}{1+i}$$

Example: $r + i = 1.05$, its log. 0.0212185

$B = 0.9555537$ and $a = 1,000,000$
 $V = 1,000,000$
 $R = -0.05$
 $b - 1 = 1.9555537$ its Ar. Co. = -1.8970408

$$L + r = 0.0212185 \text{ (0.0212185)} \quad 0.168736 \quad 7.92222 = n$$

Answer, 7.9222 Years.

SEVERAL QUESTIONS,

WITH THEIR SOLUTIONS,

RELATING TO

COMPOUND INTEREST

AND

ANNUITIES:

Extracted from the First Volume of Dodson's Mathematical Repository,
published in the year 1748, in pages 298,
299, 300, &c - - - 312.

SEVEN

WITH THIS SOLUTION

COMPOUND INTEREST

ANNUITIES

Published by the Author, 10, North Street, New York.

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SEVERAL QUESTIONS,

WITH THEIR SOLUTIONS,

RELATING TO

COMPOUND INTEREST AND ANNUITIES:

Extracted from the First Volume of Dodson's Mathematical Repository,
published in the year 1748, in pages 298,
299, 300, &c - - - 312.

IN computations relating to Compound Interest and Annuities,

Let $\begin{Bmatrix} p \\ a \\ r \\ n \\ m \end{Bmatrix}$ represent $\begin{cases} \text{the principal or present worth,} \\ \text{the annuity,} \\ \text{the interest of 1*l.* in 1 time,} \\ \text{the number of times,} \\ \text{the amount;} \end{cases}$

QUESTION CXLI.

Then, p, r, n , are given; to find m ?

Now, because $\begin{Bmatrix} p \\ p \times \overline{1+r} \\ p \times \overline{1+r}^2 \\ \&c. \\ p \times \overline{1+r}^{n-1} \end{Bmatrix}$ will amount to $\begin{Bmatrix} p \times \overline{1+r} \\ p \times \overline{1+r}^2 \\ p \times \overline{1+r}^3 \\ \&c. \\ p \times \overline{1+r}^n \end{Bmatrix}$ = the amount of p at the end of $\begin{Bmatrix} 1 \\ 2 \\ 3 \\ \&c. \\ n \end{Bmatrix}$ times.

Therefore $p \times \overline{1+r}^n = m$;

In Logarithms, $P + n \times L. \overline{1+r} = M$.

QUESTION CXLII.

When p, r , and m , are given; to find n ?

Then $n = \frac{M - P}{L. \overline{1+r}}$.

QUES.

QUESTION CXLIII.

When p , n , m , are given; to find r ?

$$\text{Then } L. \frac{1}{1+r} = \frac{M-P}{n}.$$

QUESTION CXLIV.

When r , n , and m , are given; to find p ?

$$\text{Then } P = M - n \times L. \frac{1}{1+r}.$$

Concerning Annuities.

QUESTION CXLV.

In annuities computed by Compound Interest, a , r , and n , are given; to find m ?

$$\text{Then } \left\{ \begin{array}{l} a \\ a \times \frac{1}{1+r} \\ a \times \frac{1}{(1+r)^2} \\ a \times \frac{1}{(1+r)^3} \\ \&c. \\ a \times \frac{1}{(1+r)^{n-1}} \end{array} \right\} \text{ becomes due in the } \left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \\ 3^{\text{d}} \\ 4^{\text{th}} \\ \&c. \\ n^{\text{th}} \end{array} \right\} \text{ time.}$$

Therefore their sum $\frac{a \times \frac{1}{(1+r)^n} - a}{r} = m$ by Quest. 83.

In Logarithms, let $n \times L. \frac{1}{1+r} = B$; then $A + L. \frac{1}{b-1} - R = M$.

QUESTION CXLVI.

When m , r , and n , are given; to find a ?

$$\text{Then } a = \frac{rm}{\frac{1}{(1+r)^n} - 1}:$$

Or (if $n \times L. \frac{1}{1+r} = B$) $A = M + R - L. \frac{1}{b-1}$.

QUESTION CXLVII.

When a , m , and r , are given; to find n ?

$$\text{Then } a \times \frac{1}{(1+r)^n} = mr + a;$$

$$\text{Therefore } \frac{1}{(1+r)^n} = \frac{mr+a}{a};$$

$$\text{And } n = \frac{L. \frac{mr+a}{a} - A}{L. \frac{1}{1+r}}.$$

QUES-

QUESTION CXLVIII.

When a , n , and m , are given; to find r ?

$$\text{Then } \frac{a \times \overline{1+r}^n - a}{\overline{1+r} - 1} = m,$$

$$\text{Or } a \times \overline{1+r}^n - a = m \times \overline{1+r} - m;$$

$$\text{Therefore } a \times \overline{1+r}^n - m \times \overline{1+r} + m - a = 0:$$

If $\overline{1+r}$ could reasonably be expected to be a whole number, or rational fraction, it might be found in the same manner as r in Quest. 92; but as that will seldom or never happen, use the following approximation*.

$$\text{Let } D = \frac{M - A - N}{\frac{1}{2} \times n - 1}; \text{ and } e = \frac{6}{n + 1};$$

$$\text{Also, } F = \frac{L \cdot 2 \times \overline{d-1} + e + E}{2}. \text{ Then will } r = f - e.$$

For example; if an annuity of 50*l.* forborn 18 years amounts to 1342*l.* 15*s.* what rate of interest was allowed?

$$\begin{array}{ll} \text{Here } 50 = a, & M = 3,12800, \\ 18 = n, & A = 1,69897, \\ 1342,75 = m, & N = 1,25527; \end{array}$$

$$\begin{array}{ll} 8,5 = \frac{1}{2} \times n - 1, & 8,5) 0,17375 \text{ (} 0,02044 = D, \\ 0,31579 = \frac{6}{n+1} = e; \text{ and } & 1,04819 = d. \end{array}$$

$$\text{Then } 2 \times 0,04819 = 0,09638, \quad E = 1,49940,$$

$$e = 0,31579, \quad L \cdot 2 \times \overline{d-1} + e = 1,61508,$$

$$\text{Therefore } 2 \times \overline{d-1} + e = 0,41217; \quad 2) \overline{1,11448},$$

$$\overline{1,55724} = F.$$

$$\text{Therefore } (f =) 0,36078 - 0,31579 = (0,04499 =) 0,045 = r.$$

Answer, 4½*l.* per cent.

QUESTION CXLIX.

In annuities computed by Compound Interest, a , r , and n , are given; to find p ?

* The most excellent mathematician William Jones, Esq. hath given leave for the publication of this approximation in two books relating to Logarithms. See above, page 556, Case 19.

$$\text{Then } \left\{ \begin{array}{l} \frac{a}{1+r} \\ \frac{a}{1+r^2} \\ \frac{a}{1+r^3} \\ \&c. \\ \frac{a}{1+r^n} \end{array} \right\} \text{ is the present value of the sum payable at the end of the } \left\{ \begin{array}{l} 1^{\text{st}} \\ 2^{\text{d}} \\ 3^{\text{d}} \\ \&c. \\ n^{\text{th}} \end{array} \right\} \text{ time ;}$$

Therefore their sum $\frac{1+r^n - 1}{r \times 1+r} \times a = p$, by Quest. 95.

In Logarithms, let $n \times L. \overline{1+r} = B$;

Then $A + L. \overline{b-1} - R - B = P$.

QUESTION CL.

When p , r , and n , are given ; to find a ?

Then $a = \frac{pr \times \overline{1+r}^n}{\overline{1+r}^n - 1}$;

Or (if $n \times L. \overline{1+r} = B$; $A = \frac{P + R + B}{L. \overline{b-1}}$).

QUESTION CLI.

When a , p , and r , are given ; to find n ?

Then $\overline{1+r}^n \times a = pr \times \overline{1+r}^n + a$.

Therefore $\overline{1+r}^n = \frac{a}{a - pr}$:

And $n = \frac{A - L. \overline{a - pr}}{L. \overline{1+r}}$.

QUESTION CLII.

When a , n , and p , are given ; to find r ?

Then $\frac{a \times \overline{1+r}^n - a}{\overline{1+r} - 1 \times \overline{1+r}^n} = p$.

Therefore $p \times \overline{1+r}^{n+1} - \overline{a + p} \times \overline{1+r}^n + a = 0$.

And

And here, as in Quest. 148. (since the value of $\overline{1+r}$ is not to be expected in a whole number or rational fraction,) the following approximation (quoted from the same gentleman)* may be used:

$$\text{Let } G = \frac{A + N - P}{\frac{1}{2} \times n + 1}; \quad b = \frac{6}{n-1};$$

$$K = \frac{L \cdot b - 2 \times \overline{g-1} + H}{2}; \quad \text{then } r = b - k.$$

For example; suppose a bookseller purchases a work for 40*l.* and pays for printing a thousand copies thereof 15*l.*; for paper, 20*l.*; and for advertising and other incident charges, 10*l.* Now, if he sells the edition at 3*s.* each copy, in 10 years (*i. e.* 100 copies every year,) what does he gain per cent.?

Here the bookseller lays out 85*l.* to purchase an annuity of 15*l.* per year, to continue 10 years.

$$\text{Therefore } 15 = a,$$

$$10 = n,$$

$$85 = p,$$

$$\frac{2}{3} = \frac{6}{n-1} = b,$$

$$5,5 = \frac{1}{2} \times n + 1;$$

$$b = 0,66667,$$

$$1 \times 0,10879 = 0,21758,$$

$$b - 2 \times \overline{g-1} = 0,44909;$$

$$A = 1,17609,$$

$$N = 1,00000,$$

$$2,17609,$$

$$P = 1,92942,$$

$$5,5) 0,24667 (= 0,04485 = G;$$

$$\text{Therefore } 1,10879 = g.$$

$$H = 1,82391,$$

$$L \cdot \overline{b-2 \times \overline{g-1}} = \overline{1,65233},$$

$$2) \overline{1,47624},$$

$$K = \overline{1,73812};$$

$$\text{Then } 0,66667 - (k =) 0,54717 = 0,11950 = r.$$

Answer, 11*l.* 19*s.* per cent.

QUESTION CLIII.

In annuities computed by Compound Interest, a , p , and r , are given; to find m ?

$$\text{By Quest. 151, } \overline{1+r}^n = \frac{a}{a-pr}.$$

$$\text{By Quest. 141, } \overline{1+r}^n = \frac{m}{p}.$$

$$\text{Therefore } - \frac{m}{p} = \frac{a}{a-pr}, \text{ and } m = \frac{pa}{a-pr}.$$

* See above, page 556, Case 20.

QUESTION CLIV.

When a , r , and m , are given; to find p ?

$$\text{Then } \frac{ma}{a + mr} = p.$$

QUESTION CLV.

When a , p , and m , are given; to find r ?

$$\text{Then } \frac{m - p \times a}{mp} = r.$$

QUESTION CLVI.

When p , r , and m , are given; to find a ?

$$\text{Then } a = \frac{mrp}{m - p}.$$

QUESTION CLVII.

In computing annuities by Compound Interest, p , m , and n , are given; to find a ?

$$\text{By Quest. 141, } 1 + r = \sqrt[n]{\frac{m}{p}}.$$

$$\text{Therefore } r = \sqrt[n]{\frac{m}{p}} - 1.$$

$$\text{By Quest. 146, } a = \frac{rm}{1 + r^n - 1}.$$

$$\text{If } M - P = B, \text{ and } \frac{M - P}{n} = C;$$

$$\text{Then } A = \frac{L \cdot \overline{c - 1} + M}{L \cdot \overline{b - 1}}.$$

QUESTION CLVIII.

When a , p , and m , are given; to find n ?

$$\text{By Quest. 155, } r = \frac{m - p \times a}{mp}.$$

$$\text{Then, by Quest. 142, } n = \frac{M - P}{L \cdot 1 + r}.$$

QUESTION CLIX.

When p , n , and a , are given; to find m ?

First find r by Quest. 152; and then m by Quest. 153.

QUES-

QUESTION CLX.

When m , n , and a , are given; to find p ?

First find r by Quest. 148; and then p by Quest. 154.

In computing the values of annuities in reversion by Compound Interest;

Let $\begin{Bmatrix} a \\ p \\ t \\ x \\ r \end{Bmatrix}$ represent the $\begin{cases} \text{annuity,} \\ \text{present worth thereof,} \\ \text{time of it's duration,} \\ \text{time before it commences,} \\ \text{interest of 1} \text{ in 1 time.} \end{cases}$

QUESTION CLXI.

Then a , t , x , and r , are given; to find p ?

By Quest. 149, the present worth of a to continue $\begin{Bmatrix} x+t \\ x \end{Bmatrix}$ years is $\begin{cases} \frac{\overline{1+r}^{x+t} - 1 \times a}{r \times \overline{1+r}^{x+t}}; \\ = \text{to } \frac{\overline{1+r}^x - 1 \times a}{r \times \overline{1+r}^x}. \end{cases}$

$$\text{Then } \frac{\overline{1+r}^{x+t} \times a - a}{r \times \overline{1+r}^{x+t}} - \frac{\overline{1+r}^x \times a - a}{r \times \overline{1+r}^x} = p,$$

$$\text{Therefore } - \quad - \quad \frac{\overline{1+r}^t \times a - a}{r \times \overline{1+r}^{x+t}} = p.$$

In Logarithms, (if $t \times L. \overline{1+r} = B$.)

$$\text{Then } A + L. \overline{b-1} - R - \overline{x+t} \times L. \overline{1+r} = P.$$

QUESTION CLXII.

When p , t , x , and r , are given; to find a ?

$$\text{Then } A = P + R + \overline{x+t} \times L. \overline{1+r} - L. \overline{b-1}.$$

QUESTION CLXIII.

When p , a , t , and r , are given; to find x ?

$$\text{Then } \frac{A + L. \overline{b-1} - R - P - B}{L. \overline{1+r}} = x.$$

QUES.

QUESTION CLXIV.

When p , a , x , and r , are given; to find t ?

If $P + R + x \times L. \overline{1+r} = C$;

Then $t = \frac{A - L. \overline{a-c}}{L. \overline{1+r}}$.

QUESTION CLXV.

When p , a , x , and t , are given; to find r ?

The following approximation (by the gentleman mentioned in Quest. 148,) may be used.

Let $\frac{2x + x + 1}{2} = b$; $u = \frac{12b}{u-1}$;

And $\frac{A + T - P}{b} = V$; $z = u - \overline{v-1} \times 2$;

Then $\frac{Z + U}{2} = E$; and $u - e = r$.

QUESTION CLXVI.

A gentleman who had 10 different annuities of 100*l.* each, the longest being to continue 60 years, the second 59, the third 58 years, &c. fold them all at 5*l.* per cent. compound interest; how much money did he receive?

If $a = 100$, and $r = 0.05$; then the present worth of 100*l.* a year, to continue

$$\left. \begin{array}{l} 60 \\ 59 \\ 58 \\ \text{\&c.} \end{array} \right\} \text{ years will be } \left\{ \begin{array}{l} \frac{\overline{1+r}^{60} - 1}{r \times \overline{1+r}^{60}} \times a = 1 - \frac{1}{\overline{1+r}^{60}} \times \frac{a}{r} \\ \frac{\overline{1+r}^{59} - 1}{r \times \overline{1+r}^{59}} \times a = 1 - \frac{1}{\overline{1+r}^{59}} \times \frac{a}{r} \\ \frac{\overline{1+r}^{58} - 1}{r \times \overline{1+r}^{58}} \times a = 1 - \frac{1}{\overline{1+r}^{58}} \times \frac{a}{r} \\ \text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{array} \right\} \text{ by Quest. 149.}$$

Therefore the value of 10 such annuities will be

$$10 - \frac{\overline{1+r}^{10} - 1}{r \times \overline{1+r}^{60}} \times \frac{a}{r} \text{ by Quest. 83.}$$

Now

$$\text{Now } (1 + 0,05)^{10} = 1,05^{10} = 1,628895;$$

$$\text{Therefore } - \quad 1,05^{10} - 1 = 0,628895;$$

$$\text{Also, } - \quad 1,05^{60} = 18,679186;$$

$$\text{Therefore } \frac{0,628895}{0,05 \times 18,679186} = 0,673368;$$

$$\text{And } - \quad 10 - 0,673368 \times \frac{100}{0,05} = 18653,26\%.$$

QUESTION CLXVII.

If to enjoy the benefit of an estate for 23 years, after the expiration of 8 years, be worth 400*l.* present money, what will the same estate be worth for 21 years, after the expiration of 10 years; allowing compound interest at 5 per cent.?

Suppose the estate was *aL* per year, and *x* = the number required;

$$\left. \begin{array}{l} \text{Then } \frac{1,05^{23} - 1 \times a}{0,05 \times 1,05^{31}} = 400 \\ \text{And } \frac{1,05^{21} - 1 \times a}{0,05 \times 1,05^{31}} = x. \end{array} \right\} \text{per Quest.}$$

$$\text{That is, } - \quad a = \frac{400 \times 0,05 \times 1,05^{31}}{1,05^{23} - 1},$$

$$\text{And } - \quad a = \frac{x \times 0,05 \times 1,05^{31}}{1,05^{21} - 1};$$

$$\text{Then } \frac{x \times 0,05 \times 1,05^{31}}{1,05^{21} - 1} = \frac{400 \times 0,05 \times 1,05^{31}}{1,05^{23} - 1}.$$

$$\text{Therefore } - \quad x = \frac{400 \times 1,05^{21} - 1}{1,05^{23} - 1};$$

$$\text{That is, } - \quad x = \left(\frac{400 \times 1,785963}{2,071524} \right) = 344,8597.$$

QUESTION CLXVIII.

If 150*l.* be lent, on condition that 12*l.* per annum be paid, until the principal and it's interest at 5 per cent. be satisfied; and that, at every payment, the interest then due shall be discharged, and the remainder, by which such payment exceeds that interest, applied to reduce the principal: How many years must the said payment continue?

If $p = 150$; $a = 12$; $r = .05$; and $n =$ the time required.

$$\text{Then } \left\{ \begin{array}{l} p + pr - - - = \text{sum due before} \\ -\frac{p}{a} + pr - - - = \text{sum due after} \end{array} \right\} \text{1st payment.}$$

$$\text{And } \left\{ \begin{array}{l} \overline{p-a} \times r + pr^2 = \text{interest due at} \\ -\frac{p}{2a} + \overline{2p-a} \times r + pr^2 = \text{sum due after} \end{array} \right\} \text{2d payment.}$$

$$\text{Also, } \left\{ \begin{array}{l} \overline{p-2a} \times r + \overline{2p-a} \times r^2 + pr^3 = \text{interest due at} \\ -\frac{p}{3a} + \overline{3p-3a} \times r + \overline{3p-a} \times r^2 + pr^3 = \text{sum due after} \end{array} \right\} \text{3d payment.}$$

$$\text{Again, } \left\{ \begin{array}{l} \overline{p-3a} \times r + \overline{3p-3a} \times r^2 + \overline{3p-a} \times r^3 + pr^4 = \text{interest due at} \\ \left\{ -\frac{p}{4a} \right\} + \overline{4p-6a} \times r + \overline{6p-4a} \times r^2 + \overline{4p-a} \times r^3 + pr^4 = \text{sum due after} \end{array} \right\} \text{4th paym.}$$

$$\text{Therefore } \left\{ \begin{array}{l} p + npr + \frac{n \cdot n-1}{1 \cdot 2} pr^2 + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} pr^3, \&c. \\ -na - \frac{n \cdot n-1}{2} ar - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} ar^2 - \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} ar^3, \&c. \end{array} \right.$$

will be the sum due after the n th payment; which, by the question, is nothing.

$$\text{Therefore } p \times 1 + nr + \frac{n \cdot n-1}{1 \cdot 2} r^2, \&c. = a \times n + \frac{n \cdot n-1}{1 \cdot 2} r + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} r^2, \&c.$$

$$\text{Or } - - - p \times \overline{1+r}^n = a \times \frac{\overline{1+r}^n - 1}{r},$$

$$\text{Or } - - - \frac{pr \times \overline{1+r}^n}{a} = \overline{1+r}^n - 1.$$

$$\text{Or } - 1 = 1 - \frac{pr}{a} \times \overline{1+r}^n;$$

$$\text{Or } - 1 = \frac{a - pr}{a} \times \overline{1+r}^n.$$

$$\text{Therefore } \frac{a}{a - pr} = \overline{1+r}^n.$$

$$\text{And } \frac{A - L \cdot \overline{a - pr}}{L \cdot \overline{1+r}} = n.$$

Scholium. By comparing this result with Quest. 151, it appears, that, where a debt is discharged by many equal payments, and the interest due at the time of each payment is cleared before any part of the principal, compound interest is allowed to the lender.

But it is both legal and customary, when money is paid in part of a debt, to deduct the interest then due out of such payment; and to apply, only, the remaining part to the discharge of the principal.

Therefore, in computing the present values of annuities, &c. the rules founded on the principles of Compound Interest will give their legal value.

QUESTION CLXIX.

In a geometrical progression infinitely decreasing, are given the greatest term z , and the ratio r ; to find the sum s ?

By Quest. 85, $z + \frac{z-a}{r-1} = s$.

Where a , in a finite progression, signifies the least term; but, in an infinite progression, the least term is inconsiderable;

Therefore $z + \frac{z}{r-1} = s$. Or $\frac{rz}{r-1} = s$.

QUESTION CLXX.

What is the present value p of an estate of a per year, in fee-simple, allowing compound interest at r per pound per annum?

Now $p = \frac{a}{1+r} + \frac{a}{(1+r)^2} + \frac{a}{(1+r)^3}$, &c. ad infinitum.

Therefore $p = \left(\frac{\frac{a}{1+r} \times \overline{1+r}}{1+r-1} \right) = \frac{a}{r}$, by Quest. last.

QUESTION CLXXI.

What is the present value, p , of the reversion of an estate of a per annum, in fee simple, to commence at the end of x years; allowing compound interest at r per pound per annum?

By Quest. 149, an annuity } $= \frac{a \times \overline{1+r}^x - a}{r \times \overline{1+r}^x}$.
of a , to continue x years, }

By Quest. 170, a perpetuity, } $= \frac{a}{r}$.
 a , per annum, }

Therefore $\frac{a}{r} - \frac{a \times \overline{1+r}^x - a}{r \times \overline{1+r}^x} = p$,

Or $\frac{a \times \overline{1+r}^x - a \times \overline{1+r}^x + a}{r \times \overline{1+r}^x} = p$;

Therefore $\frac{a}{r \times \overline{1+r}^x} = p$;

Or $A - R - x \times L \cdot \overline{1+r} = P$.

Corol. 1. $A = P + R + x \times L \cdot \overline{1+r}$.

Corol. 2. $x = \frac{A - P - R}{L \cdot \overline{1+r}}$.



Johannes Elard
De Civitat Cestriæ
Ætat: suæ 58 An^o Dom: 1706

E X T R A C T S

FROM

MR. JOHN WARD'S
YOUNG MATHEMATICIAN'S GUIDE,

From Page 233 to Page 283,

Being the Xth, XIth, and XIIth, Chapters of the Second Part of that
useful Work.

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EXTRACTS

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CHAP. X.

The Solution of Adfectèd Equations in Numbers.

BEFORE we proceed to the Solution of Adfectèd Equations, it may not be amiss to shew the Investigation (or Invention) of those Theorems or Rules for extracting the Roots of Simple Powers, made use of in Chapter XI, Part I. I shall here make choice of the same Letters to represent the Numbers both given and sought, as in my Compendium of *Algebra*.

Viz. Let $\left\{ \begin{array}{l} G, \text{ always denote the given Resolvend.} \\ r = \left\{ \begin{array}{l} \text{any Number taken as near the true Root as may be,} \\ \text{whether it be greater or less.} \end{array} \right. \\ e = \left\{ \begin{array}{l} \text{the unknown Part of the Root sought by which } r \text{ is to} \\ \text{be either increased or decreased.} \end{array} \right. \end{array} \right.$

Then, if r be any Number less than the true Root, it will be $r + e =$ the Root sought. But, if r be taken greater than the true Root, it will then be $r - e =$ the Root sought. And put D for the Dividend that is produced from G , after it is lessened and divided by r , &c. (into the Co-efficients of Adfectèd Equations,) according as the Nature of the Root requires. These Things being premised, we may proceed to raising the Theorems.

SECT.

S E C T. I.

I. For the Square Root, viz. $aa = G$. Quære a .

Let	1	$r + e = a,$
$1 \text{ } \textcircled{+}^2$	2	$rr + 2re + ee = aa = G,$
$2 - rr$	3	$2re + ee = G - rr.$ Call it D , viz. $D = G - rr.$
Then	4	$\left\{ \frac{D}{2r + e} = e \right\}$ This shews the 1st Method of extracting the Square Root, Sect. 5. Chap. 11. Part 1.
$3 \div 2$	5	$re + \frac{1}{2}ee = \frac{G - rr}{2} = D.$

Which gives this Theorem, $\left\{ \frac{D}{r + \frac{1}{2}e} = e. \right.$

The Arithmetical Operations of both these Theorems, you have in the Examples of Section 2, Page 126, to which I refer the Learner, supposing him by this Time to understand them without any more words than what is there expressed.

II. To extract the Cube Root; viz. $aaa = G$. Quære a .

Let	1	$r + e = a$, supposing r less than the true Root.
$1 \text{ } \textcircled{+}^3$	2	$rrr + 3rre + 3ree + eee = aaa = G,$
$2 - rrr$	3	$3rre + 3ree + eee = G - rrr,$
$3 \div 3r$	4	$re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D.$

Let $\frac{eee}{3r}$ be rejected or cast off, as being of small Value; then it will be $re + ee = D$, which gives this following Theorem, $\frac{D}{r + e} = e.$

By this Theorem or Rule, the 1st and 2d Examples in Case 1, Page 132, are performed; the which being compared with this Theorem may be easily understood.

Again, Suppose $aaa = G$, as before, and let r be taken greater than the true Root.

Then	1	$r - e = a,$	$\left\{ \begin{array}{l} eee \text{ being rejected as be-} \\ \text{fore.} \end{array} \right.$
$1 \text{ } \textcircled{-}^3$	2	$rrr - 3rre + 3ree = a^3 = G,$	
$2 \pm$	3	$3rre - 3ree = rrr - G,$	
$3 \div 3r$	4	$re - ee = \frac{rrr - G}{3r} = D.$	

Which gives this Theorem, $\frac{D}{r - e} = e.$

By this Theorem the third Example in Case 2, Page 133, is performed.

III. To

III. To extract the *Biquadrate* Root; viz. $a^4 = G$. Quære a .

Let	1	$r - e = a$, supposing r less than just.	
$1 \ominus r^4$	2	$r^4 + 4rrre + 6rree = a^4 = G$,	{ rejecting all the powers of e above ee .
$2 - r^4$	3	$4rrre + 6rree = G - r^4$,	
$3 \div 2rr$	4	$2re + 3ee = \frac{G - r^4}{2rr} = D$.	

Which gives this *Theorem*, $\frac{D}{2r + 3e} = e$.

By this *Theorem* the *Biquadrate* Root of any Number may be extracted. But, as I have already said, Page 134, those Extractions may be very well performed by two Extractions of the Square Root. *Vide Example, Page 135.*

IV. To extract the *Sur-solid* Root, viz. $a^5 = G$. Quære a .

If r be taken less than just, then $r + e = a$, as before, and $\frac{G - r^5}{5r^3} = D$, which gives this *Theorem*, $\frac{D}{r + 2e} = e$. By this *Theorem* the *Sur-solid* Root, Example 1, Page 136, is extracted. But if r be taken greater than just; then $r - e = a$, and $\frac{r^5 - G}{5r^3} = D$, which gives this *Theorem*, $\frac{D}{r - 2e} = e$. By this last *Theorem* the Example in Page 137 is performed.

I presume it needless to pursue the raising of those *Theorems* for extracting the Roots of Simple Powers, any further; because the Method of doing it is general, how high soever they are; and therefore it may be easily understood by what is already done.

S E C T. II.

NOTWITHSTANDING I have already shewed the Solution of Quadratick Equations, two several Ways, viz. by casting off the lowest Term, and by compleating the Square, *vide* Section 2, Page 195, &c.; yet it may not be amiss to shew how those Equations may be resolved into Numbers by this Universal Method of continued Series; wherein, if the first r be taken equal to the first true Root, or single Side of the Resolvend; and every single Value of e (as it becomes found) be still added to it, for a new r , then those Roots may be extracted without repeating a second Operation, as before in the single Powers.

Case

Case 1. Let $aa + 2ba = G$. It is required to find the Value of a .

Put	1	$r + e = a,$
$1 \odot^2$	2	$rr + 2re + ee = aa,$
$1 \times 2b$	3	$2br + 2be = 2ba,$
$2 + 3$	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G,$
$4 - rr, \&c.$	5	$2re + 2be + ee = G - rr - 2br,$
$5 \div 2$	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D.$

Which gives this *Theorem*, $\frac{D}{r + b + \frac{1}{2}e} = e.$

Suppose $b = 364$, and $G = 38692865$: If $r = 6000$, then $rr = 36000000$, and $2br = 4368000$. But $36000000 + 4368000 = 40368000 > 38692865 = G$. Therefore the first $r < 6000$. Let $r = 5000$, then

1st $r = 5000$	$19346432,5 = \frac{1}{2}G,$	
$b = 364$	$- 1432000, = \frac{1}{2}rr + br.$	
1st $r + b = 5364$	$5026432,5 = D$ ($800 = e.$	
$+ \frac{1}{2}e = 400$	46112	
1 Divisor 5764)	41523	(60 = e.
2d $r + b = 6164$	37164	
$+ \frac{1}{2}e = 30$		
	4359	(7 = e.
2 Divisor 6194)	43592,5	
3d $r + b = 6224$		867 = e.
$+ \frac{1}{2}e = 3,5$	(0)	
3 Divisor 6227,5		
First $r = 5000$		
$+ e = 867$		
		$\} = 5867 = a, \text{ as was required.}$

Case 2. If $aa - 2ba = G$, then, proceeding as above, there will arise this *Theorem*, $\frac{D}{r - b + \frac{1}{2}e} = e, \&c.$ And in Case 3, viz. $2ba - aa = G$, you will have this *Theorem*, $\frac{D}{b - r - \frac{1}{2}e} = e, \&c.$ as above.

I think it needless to trouble the Reader with the Work of these two Theorems in Numbers; because, if the last Example of Case 1. be understood, the other will be easy. Not but that the Method of compleating the Square is very ready and easy, as you may observe by the Work in several Questions of this Chapter.

S E C T. III.

IN the Solution of all Adfected Equations, that are above (or higher than) Quadratics, it will be the best way to take r = the next nearest Root of the Equation: and then it will be $r + e = a$, if r be less than just; or $r - e = a$, if r be greater than just (as at the beginning of this Chapter). And all the Powers of the unknown Part of the Root, (*viz.* e) above it's Square (ee) are to be rejected or cast off, as before in raising the Theorems for the Simple Powers. And therefore it is, that, to supply the want of those Powers (above ee in the Theorem), the Operation must be repeated; as in the Example of extracting the Cube Root, Page 133, *viz.* when the Figures in the Root consist of more than three Places. (*Vide* Page 140, and 141.)

Suppose $aaa + ba = G$. Quære a .

Let	1	1	$r + e = a$; <i>viz.</i> let r be supposed less than just.
$1 \ominus^3$	2	2	$rrr + 3rre + 3ree = aaa$,
$1 \times b$	3	3	$br + be = ba$,
$2 + 3$	4	4	$rrr + br + 3rre + be + 3ree = a^3 + ba = G$,
$4 \div 3r$	5	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$,
$5 - \&c.$	6	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$.

Which gives this Theorem, $\frac{D}{r + \frac{b}{3r} + e} = e$.

But if r be taken greater than just, then it will be $re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D$, which produces this Theorem, $\frac{D}{r + \frac{b}{3r} - e} = e$.

By either of these two Theorems the Value of a may be easily found. Or rather otherwise, as in the following Example.

Let $aaa + 24a = 587914$. Here $b = 24$. Suppose the first $r = 90$; then $r^3 = 729000 > 587914$, without the 24×90 being added to it: therefore $r < 90$. Again, suppose $r = 80$; then $r^3 = 512000$, and $24r = 1920$. But $512000 + 1920 = 513920 < 587914$; hence r is > 80 , but nearer to it than 90. Therefore

it must be	1	$r + e = a$, less than just.
$1 \odot^3$	2	$rrr + 3rre + 3ree = aaa$,
1×24	3	$24r + 24e = 24a$,
2 in Numb.	4	$512000 + 19200e + 240ee = aaa$,
3 in Numb.	5	$1920 + 24e = 24a$,
$4 + 5$	6	$513920 + 19224e + 240ee = 587914$,
$5 - 513920$	7	$19224e + 240ee = 73994$,
$7 \div 240$	8	$80,1e + ee = 308,31 = D$,
$8 \div$	9	$e = \frac{D}{80,1 + e}$.

Operation 80,1

$$+ e = 3,$$

1 Divisor 83,1)

$$+ e = 3,6$$

2 Divisor 86,7)

$$+ e = ,67$$

87,37)

$$\begin{array}{r} 308,31 \quad (80, = r, \\ 249,3 \quad 3,68 \text{ \&c.} = e, \\ \hline 83,68 \text{ \&c.} = r + e. \\ 59,01 \\ 52,02 \\ \hline 6,99 \text{ \&c.} \end{array}$$

Or rather, new $r = 83,7$ for a second Operation, which, being involved and tried (as above), will be found greater than just: therefore

it must be	1	$r - e = a$,
$1 \odot^3$	2	$rrr - 3rre + 3ree = aaa$,
1×24	3	$24r - 24e = 24a$,
2 in Numb.	4	$586376,253 - 21017,07e + 251,1ee = aaa$,
3 in Numb.	5	$2008,8 - 24e = 24a$,
$4 + 5$	6	$588385,053 - 21041,07e + 251,1ee = 587914$,
$6 +$	7	$21041,07e - 251,1ee = 471,053$,
$7 \div 251,1$	8	$83,7955e - ee = 1,87595778 = D$,
$8 \div$	9	$e = \frac{D}{83,7955 - e}$.

2d Ope.

2d Operation 83,7955

$$-e = ,02$$

1st Divisor 83,7755)

$$-e = ,022$$

2d Divisor 83,7535)

$$-e = ,0023$$

3d Divisor 83,7512)

$$-e = 3 \text{ \&c.}$$

83,751

...

Here the new Divisors are rejected, as insignificant.

$$\begin{array}{r}
 1,87595778 \quad (83,70000000 = r, \\
 1,675510 \quad 00,02239331 = e, \\
 \hline
 83,67760669 = a = r - e. \\
 2004477 \\
 1675070 \\
 \hline
 03294078 \\
 02512536 \\
 \hline
 00781542 \\
 00753760 \\
 \hline
 27782 \\
 25125 \\
 \hline
 2657 \\
 2512 \\
 \hline
 145 \\
 83 \\
 \hline
 \end{array}$$

All the remaining Examples of extracting Roots (except Page 260) are left in the Author's own Method; which by this Time, it is presumed, the Learner will easily know how to correct of himself, if he takes due Notice of what has been delivered Page 131, 132, &c.

But if more Exactness be required, you may make the new $r = 83,6776067$, and proceed with it to a third Operation; which will afford twenty-seven Places of Figures for the Value of a ; that is, every Operation will produce triple the Places of Figures to those of the Precedent r . And this, tripling the Places of Figures in the Root at every Operation, holds good, and is to be observed in the Solution of all Adfected Equations (how high soever they are), according to this Method of resolving them. See Page 141.

Example 2. Suppose $aaa - ba = G$. Quære a . If $r + e = a$, then $re - \frac{1}{3}be + ee = \frac{1}{3}G + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem, $\frac{D}{r - \frac{1}{3}b + e} = e$. But if $r - e = a$, then $re + \frac{1}{3}be + ee = \frac{1}{3}G + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem, $\frac{D}{r + \frac{1}{3}b + e} = e$.

Or you may proceed otherwise, as in the last Example. Let $aaa = 6438a$ $= 104785688$; here $b = 6438$. Suppose the first $r = 500$; rrr will be $= 125000000$, and $br = 3219000$, then $125000000 - 3219000 = 121781000$. But $121781000 > 104785688$; therefore $r < 500$. Again, suppose $r = 400$, $rrr = 64000000$, and $br = 2575200$; then will $64000000 - 2575200 = 61424800$. But $61424800 < 104785688$; hence $r > 400$; consequently r is betwixt 400 and 500. But 500 is the next nearest; therefore let $r = 500$, being greater than just.

Then	1	$r - e = a,$
10^3	2	$rrr - 3rre + 3ree = aaa,$
$1 \times b$	3	$br - be = ba,$
2 in Numb.	4	$125000000 - 750000e + 1500ee = aaa,$
3 in Numb.	5	$3219000 - 6438e = 6438a,$
$4 - 5$	6	$121781000 - 743562e + 1500ee = 104785688,$
$6 +$	7	$743562e - 1500ee = 16995312,$
$7 \div 1500$	8	$495e - ee = 11330 = D,$
$8 \div$	9	$e = \frac{D}{495 - e}.$

Operation 495

$$- e = 20$$

1 Divisor 475)

$$- e = 3$$

472)

$$\begin{array}{r}
 11330 \quad (500,0 = r, \\
 950 \quad 23,8 = e, \\
 \hline
 1830 \quad 476,2 = r - e = a. \\
 1416 \\
 \hline
 414,0 \\
 377,6 \\
 \hline
 \end{array}$$

Let new $r = 476$ for a 2d Operation; then $r^3 = 107850176$, and $br = 3064488$; but $107850176 - 3064488 = 104785688$, the same with the Refolvend. Consequently $a = 476$ just.

Example 3. Let $ba - aaa = G$. Quære a . If $r + e = a$, then $\frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this *Theorem*, $\frac{D}{\frac{\frac{1}{3}b}{r} - r - e} = e$. But

if $r - e = a$, then $re - \frac{\frac{1}{3}be}{r} - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this *Theorem*, $\frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e$.

Or

Or otherwise, as before in the two last Examples. Thus, let $123456a - aaa = 12272861$. Here $b = 123456$. Suppose the first $r = 200$; then rrr will be $= 8000000$, and $br = 24691200$; then $24691200 - 8000000 = 16691200$: but $16691200 > 12272861$; therefore r is here less than just, because the highest Power is $-$, or Negative. Again, suppose $r = 300$; then r^3 will be $= 27000000$, and $br = 37036800$; then $37036800 - 27000000 = 10036800 < 12272861$. Consequently $r < 300$, and $r > 200$. Let $r = 300$, being the next nearest, but more than just.

Then	1	$r - e = a,$
10^3	2	$rrr - 3rre + 3ree = aaa,$
$1 \times b$	3	$br - be = ba,$
2 in Numb.	4	$27000000 - 2700000e + 9000ee,$
3 in Numb.	5	$37036800 - 123456e,$
$5 - 4$	6	$10036800 + 146544e - 9000ee = 12272861,$
$6 -$	7	$146544e - 9000ee = 2236061,$
$7 \div 900$	8	$162e - ee = 2484 = D,$
$1 \div \&c.$	9	$e = \frac{D}{162 - e}.$

Operation 162

$$- e = 10$$

1st Divisor 152)

$$- e = 6$$

2d Divisor 146

$$\begin{array}{r} 2484 \left(\begin{array}{l} 300,0 = r, \\ 16,6 = e, \end{array} \right. \\ \underline{152} \\ 283,4 = r - e = a. \\ \underline{964} \\ 876 \\ \underline{} \\ 88,0 \\ \underline{86,6} \\ \end{array}$$

Or new $r = 283$, which, being involved, $\&c.$ will appear to be the true Root, that is, $a = 283$ just.

Note, These are usually called the three Forms of Cubick Equations; and in the Solution of the third or last Form, viz. $ba - aaa = G$, you may meet with some seeming Difficulties; especially in making Choice of the first r , because this Equation is an ambiguous Equation, and hath two Affirmative Roots, viz. a greater and lesser Root. But having once found either of them, the other may be easily obtained by Division only; as in the Quadratick Equations. *Vide* Chap. 8. As for instance, in the last Example, $a = 283$, and $123456a - aaa = 12272861$. Make these two Equations $= 0$, to wit, let $a - 283 = 0$, and $-aaa + 123456a - 12272861 = 0$.

Then,

$$\begin{array}{rcl}
 \text{Then } a - 283) & -aaa & * \quad + 123456a - 12272861 \quad (-aa \\
 & -aaa + 283aa & \\
 \hline
 & * & -283aa + 123456a \quad (-283a \\
 & & -283aa + 80089a \\
 \hline
 & * & + 43367a - 12272861 \quad (+ 43367 \\
 & & + 43367a - 12272861 \\
 \hline
 & * & *
 \end{array}$$

Hence it appears that $-aa - 283a + 43367 = 0$. Consequently $aa + 283a = 43367$. This Equation being solved, $a = 110,2722$, &c. which is the lesser Root of the aforefaid Equation $ba - aaa = G$, &c. After this Manner all the possible and impossible Roots of any Equation may be easily discovered, any one of it's Roots being once found. I shall therefore omit inserting more Examples of that kind.

Suppose $aaa + baa + ca = G$. Quære a . Let $b = 74$, $c = 8729$, and $G = 560783$. By Trial (as before) it will be found that the next nearest $r = 40$, being something less than just.

Therefore	1	$r + e = a,$
$1 \times c$	2	$cr + ce = ca,$
$1 \otimes^2 : \times b$	3	$brr + 2bre + bee = baa,$
$1 \otimes^3$	4	$rrr + 3rre + 3ree = aaa,$
2 in Numb.	5	$349160 + 8729e,$
3 in Numb.	6	$118400 + 5920e + 74ee,$
4 in Numb.	7	$64000 + 4800e + 120ee,$
$5 + 6 + 7$	8	$531560 + 19449e + 194ee = 560783,$
$8 - 531560$	9	$19449e + 194ee = 29223,$
$9 \div 194$	10	$100,2e + ee = 153,06 = D,$
$10 \div$	11	$e = \frac{D}{100,2 + e}.$

$$\begin{array}{r}
 \text{Operation } 100,2 \\
 + e = 1
 \end{array}$$

$$\begin{array}{r}
 \text{1st Divisor } 101,2 \\
 + e = .5
 \end{array}$$

$$\begin{array}{r}
 \text{2d Divisor } 101,7)
 \end{array}$$

$$\begin{array}{r}
 153,06 \quad (40,0 = r, \\
 101,2 \quad 1,5 = e, \\
 \hline
 51,86 \quad 41,5 = r + e = a. \\
 50,85 \\
 \hline
 1,01
 \end{array}$$

Or

Or new $r = 41,5$ for a second Operation, which, being duly involved, $\&c.$ will be found more than just.

$$\begin{array}{l|l|l} \text{Therefore} & 1 & r - e = a, \\ & 2 & cr - ce = ca, \\ \text{Then } \left\{ & 3 & brr - 2bre + bee = baa, \\ & 4 & rrr - 3rre + 3ree = aaa. \end{array}$$

These being turned into Numbers, $\&c.$ as above, they will be $20037,75^e - 198,5ee = 390,375$, which, being divided by $198,5$, the Co-efficient of ee , will become $100,946e - ee = 1,956624$, $\&c. = D$.

$$\begin{array}{r} \text{Operation } 100,946 \\ - e = \quad 01 \end{array}$$

$$\begin{array}{r} \text{1st Divisor } 100,936 \\ - e = \quad ,009 \\ \hline \text{2d Divisor } 100,927 \end{array} \quad \begin{array}{r} 1,966624 \quad (41,5000000 = r, \\ 1,00936 \quad ,0194847 = e, \\ \hline 41,4805153 = r - e = a. \\ 957264 \\ 908343 \\ \hline \end{array}$$

* Here I proceed by plain Division, without forming new Divisors.

$$\begin{array}{r} * 489210 \\ 403708 \\ \hline 855020 \\ 807416 \\ \hline 476040 \\ 403708 \\ \hline 72332 \text{ \&c.} \end{array}$$

Let the last Equation in the Enigma, Chap. 9. be here proposed for a Solution. *Viz.* $aaaa + baaa - caa - da = G$; $b = 2$, $c = 288$, $d = 506$, and $G = 1513$, Quære a . By Trials it will be found, that the next nearest $r = 20$, being something more than just.

$$\begin{array}{l|l|l} \text{Therefore} & 1 & r - e = a, \\ 1 \times d & 2 & dr - de = da, \\ 1 \textcircled{c}^3 \times c & 3 & crr - 2cre + cee = caa, \\ 1 \textcircled{c}^3 \times b & 4 & brrr - 3brre + 3bree = baaa, \\ 1 \textcircled{c}^4 & 5 & r^4 - 4rrre + 6rree = aaaa. \end{array}$$

These being turned into Numbers, and those duly collected, according as the Signs of the Equation direct, they will become $50680 - 22374e + 2232ee = 1513$, which, being all divided by 2232 , the Co-efficient of ee , will be $10e - ee = 22 = D$.

$$\text{Then } \frac{D}{10 - e} = e.$$

Operation

Operation 10

$$-e = \frac{3}{7}$$

Divisor 7)

$$\begin{array}{r} 22 \\ 21 \\ \hline 1 \end{array} \left(\begin{array}{l} 20 = r, \\ 3 = e, \end{array} \right.$$

$$\frac{21}{1} \quad 17 = r - e = a \text{ just.}$$

See the End of Chap. 9.

By what hath been already done about the Solution of these few Equations (being carefully observed), I presume the Learner will easily conceive how to proceed in the Solution of all Kinds of Equations, be they never so high, or adfectet; therefore I shall not here propose many various Examples, but only take them as they fall in Course, when I come to the next Part, wherein you will (perhaps) find such Equations, with their Solutions, as are not common.

C H A P. XI.

Of Simple Interest, Annuities, or Pensions, &c.

INTEREST, or the Use paid for the Loan of Money, is either Simple; or Compound.

SECT. I. *Of Simple Interest.*

SIMPLE Interest, is that which is paid for the Loan of any Principal or Sum of Money, lent out for some Time, at any Rate *per Cent.* agreed on between the Borrower and the Lender; which, according to the late Laws of *England*, ought to be six Pounds for the Use of 100*l.* for one Year, and twelve Pounds for the Use of 100*l.* for two Years: and so on for a greater, or lesser, Sum, proportionable to the Time proposed.

There are several Ways of computing (or answering Questions about) Simple Interest; as, by the single and double Rule of Three (See Page 96, &c.). Others make use of Tables composet at several Rates *per Cent.*; as Sir *Samuel Moreland*, in his *Doctrine of Interest*, both simple and compound, which is all performed by Tables; wherein he hath detected several material Errors committed by Sir *Isaac Newton*, Mr. *Kersey* upon *Wingate*, and Mr. *Clavil*, &c. in the Business of computing Interest, &c. by their Tables, too tedious to be here repeated.

But

But I shall in this Tract take other Methods, and shew that all Computations relating to Simple Interest are grounded upon Arithmetical Progression; and from thence raise such general Theorems, as will suit with all Cases. In order to that,

Let $\begin{cases} P = \text{any Principal or Sum put to Interest,} \\ R = \text{the Ratio of the Rate, per Cent. per Annum,} \\ t = \text{the Time of the Principal's Continuance at Interest,} \\ A = \text{the Amount of the Principal, and it's Interest.} \end{cases}$

Note, The Ratio of the Rate, is only the Simple Interest of 1*l.* for one Year, at any given Rate; and is thus found.

Viz. $100 : 6 :: 1 : 0,06$ = the Ratio at 6 *per Cent. per Annum.*

Or $100 : 7 :: 1 : 0,07$ = the Ratio at 7 *per Cent. &c.*

Again, $100 : 7,5 :: 1 : 0,075$ = the Ratio at 7 and $\frac{1}{2}$ *per Cent.*

And if the given Time be whole Years; then t = the Number of whole Years: but if the Time given be either pure Parts of a Year, or Parts of a Year mixed with Years, those Parts must be turned into Decimals; and then t = those Decimals, or the Years together with those Decimals. Now the common Parts of a Year may be easily turned, or converted, into Decimal Parts, if it be considered

That one $\begin{cases} \text{Day is the } \frac{1}{365} \text{ Part of a Year} = 0,00274 \text{ ferè.} \\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = 0,083333, \text{ \&c.} \\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = 0,25. \end{cases}$

These Things being premised, we may proceed to raising the *Theorems.*

Let R = the Interest of 1*l.* for one Year, as before.

Then $2R$ = the Interest of 1*l.* for two Years.

And $3R$ = the Interest of 1*l.* for three Years.

$4R$ = the Interest of 1*l.* for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the Simple Interest of one Pound is a series of Terms in Arithmetical Progression increasing; whose first Term and common Difference is R , and the Number of all the Terms is t . Therefore the last Term will always be tR = the Interest of 1*l.* for any given Term signified by t .

Then, $\begin{cases} \text{As one Pound : is to the Interest of 1*l.* :: so is any Principal or given} \\ \text{Sum : to it's Interest.} \end{cases}$

That is, $1*l.* : tR :: P : tRP$ = the Interest of P . Then the Principal being added to it's Interest, their Sum will be = A , the Amount required; which gives this general Theorem.

$$\text{Theorem 1. } tRP + P = A.$$

From whence the three following *Theorems* are easily deduced.

$$\text{Theorem 2. } \frac{A}{tR+1} = P. \quad \text{Theorem 3. } \frac{A-P}{tP} = R. \quad \text{Theorem 4. } \frac{A-P}{RP} = t.$$

These four *Theorems* resolve all Questions about Simple Interest.

Question 1. What will 256l. 10s. amount to in 3 Years, one Quarter, 2 Months, and 18 Days, at 6 per Cent. per Annum?

Here is given $P = 256,5$; $R = 0,06$; and $t = 3,46599$.

For 3 Years = 3,

Quære A . per *Theorem 1*.

one Quarter = 0,25,

2 Months = 0,16667 = $0,08333 \times 2$,

18 Days = 0,04932 = $0,00274 \times 18$,

$$\text{Hence } t = 3,46599 : \times 0,06 = 0,2079594 = tR.$$

$$\text{Then } 0,2079594 \times 256,5 = 53,341586 = tRP.$$

And $53,341586 + 256,5 = 309,841586 = tRP + P = A$. That is, $309,841586 = 309\text{l. } 16\text{s. } 10\text{d.}$ being the Answer required.

Question 2. What Principal or Sum, being put to Interest, will raise a Stock of 309l. 16s. 10d. in three Years, one Quarter, two Months, and 18 Days, at 6 per Cent. per Annum?

Or the same Question otherwise stated thus.

What is 309l. 16s. 10d. due 3 Years, one Quarter, 2 Months and 18 Days hence, worth in ready Money; abating or discounting 6 per Cent, &c.?

Here is given $A = 309,841586$; $R = 0,06$; $t = 3,46599$ (found as before); thence to find P . Per *Theorem 2*. First, $3,46599 \times 0,06 = 0,2079594 = tR$. Then $tR + 1 = 1,2079594$ $309,841586 = A$ ($256,5 = P$; that is, $256,5 = 256\text{l. } 10\text{s.}$ the Answer required.

Question 3. At what Rate of Interest, per Cent, &c. will 256l. 10s. amount to 309l. 16s. 10d. in three Years, one Quarter, two Months, and 18 Days?

Here is given, $P = 256,5$; $A = 309,841586$; and $t = 3,46599$; to find R . Per *Theorem 3*. First, $309,841586 - 256,5 = 53,341586 = A - P$. Next, $3,46599 \times 256,5 = 889,026435 = tP$. And $tP = 889,026435$ $53,341586$ ($00,06 =$ the Ratio. Then $1\text{l.} : 0,06 :: 100 : 6 =$ the Rate required.

Question 4. In what Time will 256l. 10s. raise a Stock of (or amount to) 309l. 16s. 10d. at 6 per Cent. &c.?

Here is given, $P = 256,5$; $A = 309,841586$, and $R = 0,06$, to find t . Per *Theorem 4*. First, $309,841586 - 256,5 = 53,341586 = A - P$. And $256,5$

$256,5 \times 0,06 = 15,39 = PR$. Then $15,39) 53,341586 (3,46599 = t$; that is, $t = 3$ Years and $,46599$ Decimal Parts of a Year; which may be brought into common Parts of a Year, thus:

$0,46599$
 $0,25 =$ one Quarter.

And $0,08333) 0,21599$ (2 Months.
 $,16666$

$0,21599$

$0,00274) ,04933$. (18 Days.

Hence $t = 3$ Years, one Quarter, 2 Months, and 18 Days; the Answer required.

It must needs be easy to conceive, that what is here done at 6 *per Cent.* may be done at any other Rate of Interest, by forming the Ratio (*viz.* R) accordingly.

SCHOLIUM.

Although it be according to the Laws and Custom of *England*, to compute Interest at the Proportion of 6 *per Cent.* (as above), yet he that takes up Money at Interest for any Time less than even or compleat Years, pays more Interest than seems reasonably due, according to the Rules of Art. As, for instance; if 100*l.* be forborn at Interest one whole Year, it amounts to 106*l.* But (I say) if it be paid at the half Year's End, it should not amount to 103; as appears from this following Proportion.

Let $a =$ the Amounts due at the half Year's End; then it will be $100 : a :: a : 106$, the Amount at the Year's End. *Ergo* $aa = 10600$, and $a = \sqrt{10600} = 102,9563 = 102*l.* 19*s.* 1½*d.* which is less than 103*l.* by 10½*d.* And if it be paid in less than half a Year's Time, the Error must needs be the greater.$

SECT. II. Of Annuities, or Pensions, in Arrears, computed at Simple Interest.

ANNUITIES, or Pensions, &c. are said to be in Arrears, when they are payable or due, either Yearly, or Half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became due: Now, in order to that,

Put $\begin{cases} u = \text{the Annuity, Pension, or Yearly Rent, \&c.} \\ t = \text{the Time of it's Continuance, or being unpaid.} \\ R = \text{the Ratio, or Interest of 1\% for 1 Year, as before.} \\ A = \text{the Amount of the Annuity and it's Interest.} \end{cases}$

Then, if u = the first Year's Rent, due without Interest,

$\begin{matrix} Ru = \text{the Interest} \\ 2u = \text{the Rent.} \end{matrix} \left. \vphantom{\begin{matrix} Ru \\ 2u \end{matrix}} \right\} \text{due at the End of the second Year.}$
 $\begin{matrix} 2Ru = \text{the Interest} \\ 3u = \text{the Rent.} \end{matrix} \left. \vphantom{\begin{matrix} 2Ru \\ 3u \end{matrix}} \right\} \text{due at the End of the third Year.}$
 $\begin{matrix} 3Ru = \text{the Interest} \\ 4u = \text{the Rent.} \end{matrix} \left. \vphantom{\begin{matrix} 3Ru \\ 4u \end{matrix}} \right\} \text{due at the End of the fourth Year.}$
 $\begin{matrix} 4Ru = \text{the Interest} \\ 5u = \text{the Rent.} \end{matrix} \left. \vphantom{\begin{matrix} 4Ru \\ 5u \end{matrix}} \right\} \text{due at the End of the fifth Year.}$

And so on for any Number of Years. Hence it is evident, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$, the Sum of all the Rents and their Interest, being forborn 5 Years.

From whence it follows, that $Ru + 2Ru + 3Ru + 4Ru = A - tu$. Here $t = 5$. Divide by u , then $R + 2R + 3R + 4R = \frac{A - tu}{u}$.

Next, to find the Sum of this Progression (See Page 185.) thus: Let $R + 2R + 3R + 4R$ &c. = z , then $1 + 2 + 3 + 4$ &c. = $\frac{z}{R}$. Here the Sum of the first and last Terms are $4 + 1 = 5 = t$, and the Number of all the Terms is $4 = t - 1$. Therefore $\frac{t-1}{2} \times t =$ the Sum of all the Terms; that is, $\frac{u-t}{2} = \frac{z}{R}$: hence $\frac{uR - tR}{2} = z$. Consequently $\frac{uR - tR}{2} = \frac{A - tu}{u}$. Now, from this Equation it will be easy to deduce the following *Theorems*.

Theorem 1. $\frac{uRu - tRu + 2tu}{2} = A$, or $\frac{tu - tu}{2} R + tu = A$.

Theorem 2. $\frac{2A}{uR - tR + 2t} = u$. *Theorem 3.* $\frac{2A - 2tu}{tu - tu} = R$.

Let $\frac{2}{R} - 1 = x$, then $t = \sqrt{\frac{2A}{Ru} + \frac{xx}{4}} - \frac{1}{2}x$, *Theorem 4* *.

* For, since $\frac{2A - 2tu}{tu - tu}$ is = R , we shall have $2A - 2tu = tuR - tuR$, and $2A = tuR - tuR + 2tu$, and $\frac{2A}{Ru} (= tt - t + \frac{2t}{R} = tt - 1 \times t + \frac{2}{R} \times t = tt + \left(\frac{2}{R} - 1\right) \times t) = u + x \times t$. Therefore $u + x \times t + \frac{xx}{4}$ will be = $\frac{2A}{Ru} + \frac{xx}{4}$, and $t + \frac{x}{2}$ will be = $\sqrt{\frac{2A}{Ru} + \frac{xx}{4}}$, and t will be = $\sqrt{\frac{2A}{Ru} + \frac{xx}{4}} - \frac{x}{2}$. Q. E. D.

Question

Question 1. If 250l. yearly Rent (or Pension, &c.) be forborn or unpaid seven Years; what will it amount to in that Time, at 6 per Cent. for each Payment, as it becomes due?

Here is given $u = 250$, $t = 7$, and $R = 0,06$; to find A . *Per Th. 1.* First, $250 \times 7 = 1750 = tu$, $1750 \times 7 = 12250 = ttu$. Again, $12250 - 1750 = 10500 = ttu - tu$, and $\frac{10500}{2} \times 0,06 = 315$. Lastly, $315 + 1750 = 2065 = A$; *Viz.* 2065*l.* is the *Ans.* required.

But if the Annuity, Rent or Pension, is to be paid by Quarterly or Half Yearly Payments, &c. Then $\frac{0,06}{2} = 0,03 = R$ for half yearly Payments: and $\frac{0,06}{4} = 0,015 = R$ for quarterly; or $0,045 = R$ for three quarterly Payments. Example of half-yearly Payments.

Suppose 250l. per Annum, to be paid by half-yearly Payments, were in Arrears, or unpaid for seven Years; what would it amount to, allowing 6 per Cent. per Annum for each Payment, as it becomes due?

In this Example there is given $u = 125 = \frac{250}{2}$; $t = 14$, the Number of payments; and $R = 0,03 = \frac{0,06}{2}$; thence to find A .

First, $125 \times 14 = 1750 = tu$; $1750 \times 14 = 24500 = ttu$. Again, $24500 - 1750 = 22750 = ttu - tu$; then $\frac{22750}{2} = 11375$, and $11375 \times 0,03 = 341,25$. Lastly, $341,25 + 1750 = 2091,25$; that is, $A = 2091*l.* 5*s.*$ the Answer required.

N. B. Hence it may be observed, that half-yearly Payments are more advantageous than yearly. For 2091*l.* 5*s.* > 2065*l.* by 26*l.* 5*s.*; consequently, quarterly Payments are more advantageous than half-yearly Payments.

*Question 2. What yearly Rent, Pension, &c. being forborn or unpaid seven Years, will raise a Stock of 2065*l.* allowing 6 per Cent. per Annum for each Payment, as it becomes due?*

Here is given $A = 2065$, $t = 7$, and $R = 0,06$; to find u . *Per Theorem 2.* First, $7 \times 0,06 = 0,42 = tR$, and $0,42 \times 7 = 2,94 = ttR$. Then $ttR - tR = 2,52$. Lastly, $ttR - tR + 2t = 16,52$ $4130 = 2A$ ($250 = u$; that is, 250*l.* per Annum, &c. will raise 2065*l.* the Stock required.

*Question 3. In what Time will 250l. yearly Rent raise a Stock of 2065*l.* allowing 6 per Cent, &c. for the Forbearance of the Payments as they become due?*

Here is given $u = 250$, $A = 2065$, and $R = 0,06$; to find t . *Per Theorem 4.* First, $\frac{2}{R} = \frac{2}{0,06} = 33,3333$; and $33,3333 - 1 = 32,3333 = x = \frac{2}{R} - 1$. Then $16,16666 \text{ \&c.} = \frac{1}{2}x$; $261,3605 \text{ \&c.} = \frac{1}{4}xx$. Again, $\frac{4130}{15} = 275,333$

$275,333 = 2A \div Ru$, and $275,333 + 261,3605 = 536,6938 = \frac{2A}{Ru} + 2u$.
Then $\sqrt{536,6938} = 23,1666$. Lastly, $23,1666 - 16,1666 = 7 = t$, the Time required.

Question 4. If 250l. yearly Rent, being forborn seven Years, will amount to 2065l. allowing Simple Interest for every Payment as it becomes due; what must the Rate of the Interest be per Cent, &c.?

Here is given $u = 250$, $A = 2065$, and $t = 7$; to find R . Per Theorem 3.

$$\text{Thus, } \begin{cases} tu = 12250 \\ tu = 1750 \end{cases} \quad \begin{cases} 4130 = 2A \\ 3500 = 2tu \end{cases}$$

$$tu - tu = 10500 \quad 630 = 2A - 2tu \quad (0,06 = R)$$

Then $1 : 0,06 :: 100 : 6$, the Rate required.

SECT. III. *The Present Worth of Annuities, or Pensions, &c. computed at Simple Interest.*

THE Business of purchasing Annuities, or taking of Leafes, &c. for any assigned Time, depends upon the true equating of the Principal or Money laid out on the Purchase, with the Annuity or Yearly Rent, by allowing (or discounting) the same Rate of Interest to both Parties. Which may be easily performed by duly applying the respective Theorems of the two last Sections together; as will fully appear by the following Question.

Question 1. What is 75l. yearly Rent, to continue nine Years, worth in ready Money, at 6 per Cent. per Annum Simple Interest?

1. Per Theorem 1. of the last Section, find what the proposed yearly Rent would amount to, if it were forborn 9 Years, at 6 per Cent.

Thus, $u = 75$, $t = 9$, and $R = 0,06$:

$$\begin{array}{r} tu = 6075 \\ tu = 675 \\ \hline \end{array}$$

$$tu - tu = 5400$$

$$\begin{array}{r} \text{Quære } A. \\ \text{Then 2) } 5400 \text{ (2700)} \\ R = 0,06 \end{array} \left. \vphantom{\begin{array}{r} 5400 \\ 2700 \end{array}} \right\} \text{Multiply}$$

$$\begin{array}{r} 162, \\ + tu = 675, \end{array} \left. \vphantom{\begin{array}{r} 162, \\ 675, \end{array}} \right\} = 837 = A.$$

2. Then, by Theorem 2, Section 1, find what Principal, being put to Interest for the same Time, and at the same Rate, will amount to $837l. = A$. Thus, $tR = 0,54 = 9 \times 0,06$; $tR + 1 = 1,54$ $837 (543,5064 = P$; that is, $P = 543l. 10s. 1\frac{1}{2}d.$, which is the Worth of 75l. a Year, as was required.

From

From the Work of these two Operations (duly considered) it must needs be easy to conceive, how the two *Theorems* by which they were performed, may be combined in one.

For, 1. $\frac{tRu - tRu + 2tu}{2} = A$; and, 2. $PtR + P = A$. Consequently $PtR + P = \frac{tRu - tRu + 2tu}{2}$. And from this Equation may be deduced the following *Theorems*.

$$\text{Theorem 1. } \frac{tRu - tRu + 2tu}{2tR + 2} = P, \text{ or } \frac{tR - tR + 2t}{2tR + 2} \times u = P.$$

By this *Theorem* all Questions of the same Kind with the last (*viz.* that above) may be easily and readily answered at one Operation.

$$\text{Theorem 2. } \frac{2PtR + 2P}{tR - tR + 2t} = u, \text{ or } \frac{tR + 1}{tR - tR + 2t} : \times 2P = u.$$

$$\text{Theorem 3. } \frac{2P - 2tu}{tu - tu - 2Pt} = R^*.$$

Let $\frac{2}{R} - \frac{2P}{u} - 1 = x$, then will $tt \pm xt = \frac{2P}{Ru}$. Which gives this *The-*

$$\text{orem 4. } \sqrt{\frac{2P}{Ru} + \frac{xx}{4}} : \pm \frac{x}{2} = t \dagger.$$

By the second and fourth *Theorems*, two very useful Questions may be easily answered.

* For, since $PtR + P$ is $= \frac{tRu - tRu + 2tu}{2}$, we shall have $2PtR + 2P = tRu - tRu + 2tu$, and $2P = tRu - tRu + 2tu - 2PtR$, and $2P - 2tu = tRu - tRu - 2PtR = tu - tu - 2Pt$ $\times R$, and consequently $R = \frac{2P - 2tu}{tu - tu - 2Pt}$. Q. E. D.

† For, since $PtR + P$ is $= \frac{tRu - tRu + 2tu}{2}$, we shall have $2PtR + 2P = tRu - tRu + 2tu$, and consequently $2P = tRu - tRu + 2tu - 2PtR$, and $\frac{2P}{Ru} = t - t + \frac{2t}{R} - \frac{2Pt}{u} = t - 1 \times t + \frac{2}{R} \times t - \frac{2P}{u} \times t = t + \left(\frac{2}{R} - \frac{2P}{u} - 1 \right) \times t = t + x \times t$. Therefore $t + x \times t + \frac{xx}{4}$ will be $= \frac{2P}{Ru} + \frac{xx}{4}$, and $t + \frac{x}{2}$ will be $= \sqrt{\frac{2P}{Ru} + \frac{xx}{4}}$, and consequently t will be $= \sqrt{\frac{2P}{Ru} + \frac{xx}{4}} - \frac{x}{2}$. Q. E. D.

1. As,

1. *As, for Instance: If it be required to find what Annuity, or yearly Rent, &c. may be purchased, for any proposed Sum, to continue any assigned Time, allowing any Rate of Interest?*

This Question may be answered by *Theorem 2.*

2. *Again: If it be required to find how long any yearly Rent, Pension, or Annuity, &c. may be purchased (or enjoyed) for any proposed Sum, at any given Rate of Interest?*

All Questions of this Kind are easily answered by *Theorem 4.*

In these Questions it is supposed, that the Purchase, or yearly Rent, is to commence or be immediately entered-upon. But if it be required to find the Value or Purchase of an Annuity or yearly Rent, &c. in Reversion; that is, when it is not to be entered-upon until after some Time, or Number of Years are past; then you must first find what the Sum proposed to be laid out in the Purchase would amount to, if it were put out to Interest, during the Time the Annuity, &c. is not to be put in present Possession; and make that Amount the Sum for the Purchase, proceeding with it as in either of the two last Questions, &c.

Note, *From the first Question of this Section it will be easy to conceive how to perform the Equation of Payments, between Debtor or Creditor, at any Rate of Interest, without doing any Damage to either Party.*

That is, when several Sums of Money are to be paid, at several different Times, to find the Time when all the Payments may be truly discharged at once: as if one Sum were to be paid at the End of two Months, another at six Months, and perhaps a third Sum at eight Months' End, &c. And if it were required to find the Time when all those Sums may be truly discharged at one Payment without Loss, &c.

CHAP. XII.

Of Compound Interest, and Annuities, &c.

COMPOUND Interest is that which arises from any Principal and it's Interest put together, as the Interest so becomes due; so that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reason it is called Interest upon Interest, or Compound Interest.

As,

As, for Instance; Suppose 100*l.* were lent out for two Years, at 6 *per Cent.* *per Annum*, Compound Interest: then, at the End of the first Year, it will only amount to 106*l.* as in Simple Interest. But for the second Year this 106*l.* becomes Principal, which will make the whole Sum of Principal and Interest amount to 112*l.* 7*s.* 2½*d.* at the second Year's End, whereas by Simple Interest it would have amounted to but 112*l.*

And, although it be not lawful to let out Money at Compound Interest, yet in purchasing of Annuities or Pensions, &c. and taking Leases in Reversion, it is very usual to allow Compound Interest to the Purchaser for his ready Money; and therefore it is very requisite to understand it.

SECT. I. Of Compound Interest.

Let $\left\{ \begin{array}{l} P = \text{the Principal put to Interest,} \\ t = \text{the Time of it's Continuance,} \\ A = \text{the Amount of the Principal and Interest,} \\ R = \left\{ \begin{array}{l} \text{the Amount of 1*l.* and it's Interest for 1 Year, at any given} \\ \text{Rate, which may be thus found.} \end{array} \right. \end{array} \right\}$ as before.

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1*l.* at 6 *per Cent.*

Or 100 : 105 :: 1 : 1,05 = the Amount of 1*l.* at 5 *per Cent.*

And so on for any other assigned Rate of Interest.

Then, if $R =$ the Amount of 1*l.* for one Year, at any Rate,

$R^2 =$ the Amount of 1*l.* for two Years,

$R^3 =$ the Amount of 1*l.* for three Years,

$R^4 =$ the Amount of 1*l.* for four Years,

$R^5 =$ the Amount of 1*l.* for five Years.

Here $t = 5$.

For 1 : $R :: R : RR :: RR : RRR :: RRR : R^4 : R^4 : R^5 : \&c.$ in $\div$.

That is, $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound at one Year's} \\ \text{End :: so is that Amount : to the Amount of one Pound at} \\ \text{two Years' End, \&c.} \end{array} \right.$

Whence it is plain, that Compound Interest is grounded upon a Series of Terms, increasing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years) does always assign the Index of the last and highest Term:

Viz. the Power of R , which is R^t .

Again, As 1 : $R^t :: P : PR^t = A$, the Amount of P for the Time, that $R^t =$ the Amount of 1*l.*

That is, $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound for any given Time} \\ \text{: : so is any proposed Principal (or Sum) to it's Amount for the} \\ \text{same Time.} \end{array} \right.$

From the Premises (I presume) the Reason of the following *Theorems* may be very easily understood.

Theorem 1. $PR' = A$, as above.

From hence the two following *Theorems* are easily deduced.

Theorem 2. $\frac{A}{R'} = P$. *Theorem 3.* $\frac{A}{P} = R'$.

By these three *Theorems*, all Questions about Compound Interest may be truly resolved by the Pen only, viz. without Tables; though not so readily as by the Help of Tables, calculated on Purpose; as will appear further on.

Question 1. What will 256l. 10s. amount to in seven Years, at 6 per Cent. per Annum Compound Interest?

Here is given $P = 256,5$; $t = 7$; and $R = 1,06$; which, being involved until it's Index = t (viz. 7), will become $R' = 1,50363$. Then $1,50363 \times 256,5 = 385,6811 = A = 385l. 13s. 7\frac{1}{2}d.$ which is the Answer required.

Question 2. What Principal or Sum of Money must be put (or let) out to raise a Stock of 385l. 13s. 7 $\frac{1}{2}$ d. in seven Years, at 6 per Cent. per Annum Compound Interest?

Here is given $A = 385,6811$; $R = 1,06$; and $t = 7$; to find P , by *Theorem 2*. Thus, $R' = 1,50363$ $385,6811 = A$ ($256,5 = P$). That is, $P = 256l. 10s.$ which is the Principal or Sum, as was required.

Question 3. In what Time will 256l. 10s. raise a Stock of (or amount to) 385l. 13s. 7 $\frac{1}{2}$ d. allowing 6 per Cent. per Annum Compound Interest?

Here is given $P = 256,5$; $A = 385,6811$; $R = 1,06$; to find t by the third *Theorem*, $R' = \frac{A}{P} = \frac{385,6811}{256,5} = 1,50363$, which being continually divided by $R = 1,06$, until nothing remain, the Number of those Divisions will be $7 = t$. Thus, $1,06$ $1,50363$ ($1,41852$). And $1,06$ $1,41852$ ($1,338225$). Again, $1,06$ $1,338225$ ($1,262477$). And so on until it become $1,06$ $1,06$ (1). which will be at the seventh Division. Therefore it will be $t = 7$, the Number of Years required by the Question.

Question

Question 4. If 256l. 10s. will amount to (or raise a Stock of) 385l. 13s. 7½d. in seven Years' Time; what must the Rate of Interest be, per Cent. per Annum?

Here is given $P = 256,5$; $A = 385,6811$; and $t = 7$. Quære R . By Theorem 3. $\frac{A}{P} = R^t = 1,50363$; as before in the last Question. And if $R^t = R^7 = 1,50363$, then $R = \sqrt[7]{1,50363}$, which may be thus extracted.

Put	1	$r + e = R$, then
$1 \text{ } \textcircled{7}$	2	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$,
$2 - r^7$	3	$7r^6e + 21r^5ee = G - r^7$,
$3 \div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$,
$4 \div$	5	$e = \frac{D}{r + 3e}$; let $r = 1$, then D will be $(= \frac{G - r^7}{7r^5} = \frac{1,50363 - 1}{7 \times 1}$ $= \frac{0,50363}{7}) = 0,0719$;

and e (being $= \frac{D}{r + 3e}$) will be less than $\frac{D}{r}$, or than $\frac{0,0719}{1}$, or than $0,0719$, and therefore nearly $= 0,06$.

$$\begin{array}{r} \text{Operation } r = 1,00 \\ + 3e = 0,18 \\ \hline \end{array}$$

$$\begin{array}{r} \text{Divisor } 1,18) \quad 0,0719 \quad (1,00 = r, \\ \quad \quad \quad 708 \quad \quad \quad 0,06 = e, \\ \hline \quad \quad \quad 1,06 = r + e = R. \\ \quad \quad \quad \text{11 to be rejected.} \end{array}$$

Then $1 : 0,06 :: 100 : 6$, the Rate *per Cent.* required.

The first three Questions may be much more easily performed by the following Table, which is only the Amounts of one Pound for thirty-nine Years.

That is, of $R \cdot RR \cdot RRR \cdot R^4 \cdot R^5$. and so on to R^{39} .

Years = t .	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Years = t .	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.	Years = t .	The Amounts of 1l. at 6 per Cent. &c. Compound Interest.
1	$1.06 = R$.	14	2.260,903,955,7	27	4.822,345,940,7
2	$1.1236 = RR$.	15	2.396,558,193,1	28	5.111,686,697,1
3	$1.191,016 = R^3$.			29	5.418,387,899,0
4	1.262,476,96	16	2.540,351,684,7	30	5.743,491,172,9
5	1.338,225,577,6	17	2.692,772,785,7		
		18	2.854,339,152,9	31	6.082,100,643,2
6	1.418,519,112,2	19	3.025,599,502,1	32	6.453,386,681,8
7	1.503,630,259,0	20	3.207,135,472,2	33	6.840,589,882,8
8	1.593,848,074,5			34	7.251,025,275,7
9	1.689,478,959,0	21	3.399,563,600,5	35	7.686,086,792,3
10	1.790,847,696,5	22	3.603,537,416,6		
		23	3.819,749,661,6	36	8.147,251,999,8
11	1.898,298,558,3	24	4.048,934,641,3	37	8.636,087,119,8
12	2.012,196,471,8	25	4.291,870,719,7	38	9.154,252,347,0
13	2.132,928,260,1	26	4.549,382,962,9	39	9.703,507,487,8

The Title of this Table shews it's Construction, and it's Use will easily appear by an Example or two.

EXAMPLE I.

What will 375l. 10s. amount to in nine Years, at 6 per Cent. per Annum, &c.?

The tabular Number against 9 Years is 1,689479, which, being multiplied with the Principal 375,5, will produce 634,3993, &c. viz. 634l. 8s. *ferè*, being the Amount or Answer required.

EXAMPLE II.

What Principal (or Sum) must be put to Interest to raise a Stock of 634l. 8s. in nine Years' Time, at 6 per Cent. per Annum, &c.?

If the proposed Stock (*viz.* 634,4) be divided by the tabular Number that is against the given Number of Years (*viz.* 9), the Quotient will be the Principal (or Sum) required. *Viz.* against 9 is 1,689479. Then 1,689479) 634,4 (375,5 = 375l. 10s. the Principal (or Sum) required.

EXAMPLE III.

In what Time will 375l. 10s. raise a Stock of (or amount to) 634l. 8s. at 6 per Cent. &c.?

Divide the proposed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5), and the Quotient will shew the tabular Number that stands over against the Time sought. Thus, 375,5) 634,4 (1,689479, &c. This Number, being sought in the Table, will be found to stand against 9 Years, which therefore is the Time required.

But, if the Quotient cannot be truly found in the Table of Amounts for Years, as above; then take out of that Table the nearest Number that is less, and make it a Divisor, by which you must divide the first Quotient; and then seek the second Quotient in the Table of Amounts for Days (which is inserted a little further on), and it will assign the Number of Days: as in this Example.

In what Time will 563l. amount to 860l. at 6 per Cent. per Annum Compound Interest?

Answer. In 7 Years and 99 Days.

Thus, 563) 860 (1,52753, which shews the Time to be more (or above) seven Years; for over against 7 Years is 1,50363, which being made the new Divisor; *viz.* 1,50363) 1,52753 (1,01589, &c. this Number is the nearest Amount to 99 Days.

Note, If the Stock, Principal, and Time, be given; the Rate of Interest will be best found by extracting the Root, &c. as before in the fourth Question.

The next Thing that I shall here propose, is to make this Table (which is only calculated for the Rate of 6 *per Cent.*) universally useful for all the Rates of Compound Interest, *which, I may presume to say, is a new Improvement of my own*, being well satisfied it never was published before; and not only so, but I have heard several very good Artists affirm it was impossible to be done.

The Method of performing it is briefly thus: Let x = the Difference between $1.06 = R$, the Amount of 1*l.* for one Year (in the Table), and any other proposed Amount of 1*l.* for one Year; which admits of two Cases.

Case 1. If the proposed Rate be greater than the $1.06 = R$, then will $R + x$ = the true Amount of 1*l.* for one Year at that Rate.

Case 2. But if the proposed Rate be less than $1.06 = R$, then it will be $R - x$ = the Amount of 1*l.* &c.

Make $\begin{cases} t - 1 = b, & t - 2 = c, & t - 3 = d, & t - 4 = f, & \&c. \\ \frac{1}{1}tb = g, & \frac{1}{2}cg = m, & \frac{1}{3}dm = n, & \frac{1}{4}fn = s, & \&c. \end{cases}$

Then will $R^t + tR^b x + gR^c x^2 + mR^d x^3 + nR^f x^4 \&c.$ = the Amount of 1*l.* at the given Rate, for any Time denoted by t , in Case 1. And $R^t - tR^b x + gR^c x^2 - mR^d x^3 + nR^f x^4 \&c.$ will be = the Amount of 1*l.* for the same Time in Case 2.

Which is no more but this: Let $R + x$, or $R - x$ (which soever it is), be involved (as directed in *Señ. 5. Chap. 2.*) to the same Power or Height as the Index t , the given Time in the Question, denotes: rejecting all the Powers of x above xxx , or $xxxx$ at most, as useless*. Then multiply that Power of $R + x$, or $R - x$, into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

EXAMPLE I.

Suppose it were required to find what 256*l.* would amount to in fifteen Years, at 8*l.* per Cent. per Annum Compound Interest?

Here $t = 15$.

* For, by Newton's Binomial Theorem, $\overline{R + x}^t$ is = the Series $R^t + t \times R^{t-1} \times x + t \times \frac{t-1}{2} \times R^{t-2} \times x^2 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times R^{t-3} \times x^3 + t \times \frac{t-1}{2} \times \frac{t-2}{3} \times \frac{t-3}{4} \times R^{t-4} \times x^4 + \&c. = R^t + tR^b \times x + t \times \frac{b}{2} \times R^c \times x^2 + t \times \frac{b}{2} \times \frac{c}{3} \times R^d \times x^3 + t \times \frac{b}{2} \times \frac{c}{3} \times \frac{d}{4} \times R^f \times x^4 + \&c. = R^t + tR^b \times x + gR^c \times x^2 + mR^d \times x^3 + nR^f \times x^4 + \&c.$

a. e. d.

First,

First, $100 : 108 :: 1 : 1,08$, the Amount of 1*l.* at 8 *per Cent.* Next, $1,08 - 1,06 = 0,02 = x$. And $R + x = 1,08$, as in Case 1. Then $R^{15} + 15R^{14}x + 105R^{13}xx + 455R^{12}xxx$, &c. = the Amount of 1*l.* for 15 Years, at 8 *per Cent.*

Here $x = 0,02$. $xx = 0,0004$. and $xxx = ,000008$

By the Table, $R^{15} = 2,396558$

$$\text{And } \begin{cases} 15R^{14}x = 2,260904 \times 15 \times ,02 = 0,678271 \\ 105R^{13}xx = 2,132928 \times 105 \times ,0004 = 0,089583 \\ 455R^{12}xxx = 2,012196 \times 455 \times ,000008 = 0,007324 \end{cases}$$

Sum = 3,171736

Then $3,171736 \times 256 = 811,964416 = A$.

That is, 811*l.* 19*s.* 3 $\frac{1}{2}$ *d.* *ferè.* Which is the Answer required.

EXAMPLE II.

What will 365*l.* amount to in seven Years at four and a half *per Cent.*, &c.

First, $100 : 104,5 :: 1 : 1,045$, the Amount of 1*l.* at 4 $\frac{1}{2}$ *per Cent.*

Next, $1,06 - 1,045 = 0,015 = x$. Consequently $R - x = 1,045$, as in Case 2.

Then $R^7 - 7R^6x + 21R^5xx - 35R^4xxx$, &c. = the Amount of 1*l.* for 7 Years, at 4 $\frac{1}{2}$ *per Cent.*

Here $x = ,015$; $xx = ,000225$; and $xxx = 000003375$.

By the Table, $R^7 = + 1,503630$

$$\text{And } \begin{cases} - 7R^6x = - 0,148944 \\ + 21R^5xx = + 0,006323 \\ - 35R^4xxx = - 0,000141 \end{cases}$$

$$R^7 - 7R^6x + 21R^5xx - 35R^4xxx = 1,360868$$

Then $1,360868 \times 365 = 496,71682 = A$.

That is, 496*l.* 14*s.* 3 $\frac{1}{2}$ *d.* is the Answer required.

If the Reason of these two Operations be but well understood, it will be very easy to conceive how to find *P*, the Principal, by having *A*, *t*, and *x* given (because *R* and it's Powers are always given by the Table).

For $R^t \pm tR^bx + gR^cxx \pm mR^dxxx \times P = A$ (as above).

Therefore $\frac{A}{R^t \pm tR^bx + gR^cxx \pm mR^dxxx} = P$.

Or, if *A*, *P*, and *t*, be given, *x* may be found.

For

For $R^t \pm tR^bx + gR^c_{xx} \pm mR^d_{xxx} = \frac{A}{P}$. This Equation being solved (as in Chap. 10.), the Value of x will be found; and then either $R + x$, or $R - x$, will shew the Rate of Interest, &c.

But I shall leave the numerical Operations to the Learner's Practice, supposing enough done to shew how all Questions of this Kind that are limited by whole Years may be computed.

And if the Time given or sought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-Years, &c. for resolving such Questions, the best Way will be to reduce those Parts of a Year into Days; that done, find an Answer, according to the demand of the Question (and agreeing to *1l.* as before), for that Number of Days; and in order to that, it will be requisite to find the Amount of *1l.* for one Day (as in my *Compendium of Algebra*, Page 110), which I shall here insert.

Put a = the Amount sought, then it will be

$$1 : a :: a : aa :: aa : aaa :: aaa : aaaa \div \text{to } a^{365}.$$

That is, $\left\{ \begin{array}{l} \text{As one Pound is to it's Amount for one Day} :: \text{so is that Amount : to} \\ \text{the Amount of two Days} :: \text{and so is that of two Days : to that of} \\ \text{three Days. And so on in } \div \text{ to 365 Days.} \end{array} \right.$

Then the last of the Terms will be $a^{365} = 1,06$.

Put	1	$r + e = a$. And let $r = 1$,
$1 \text{ } \textcircled{365}$	2	$r^{365} + 365r^{364}e + 66430r^{363}ee = a^{365} = 1,06$,
2 in Numb.	3	$1 + 365e + 66430ee = 1,06$,
$3 - \bar{1}$	4	$365e + 66430ee = 0,06$,
$4 \div 66430$	5	$,00549e + ee = 0,0000009032 = D$,
$5 \div$	6	$e = \frac{D}{,00549 + e}$.

Operation	$,00549$	
$+ e =$	$,0001$	
1st Divisor	$,00559$	$0,0000009032$
$+ e =$	$,00015$	559
2d Divisor	$,00574$	3442
$+ e =$	$,000059$	2870
3d Divisor	$,005799$	57200
&c.		&c.

$(1,00000000 = r,$
 $0,0001598 = e,$
 $1,0001598 = r + e = a$, true to the
 7th Figure, and only too much by
 2 in the 8th, at one Operation.

Now $r = 1,00016$ for a second Operation. Then

2 in Numb.	7	$1,06013401407 + 386,887e + 70402,172ee = 1,06.$
		Hence it appears that $r - e = a.$
Therefore	8	$1,06013401407 - 386,887e + 70402,172ee = 1,06,$
$8 \pm$	9	$386,887e - 70402,172ee = 0,00013401407,$
$9 \div$	10	$,0054953e - ee = ,0000000019035503,$
$10 \div$	11	$e = \frac{,0000000019035503}{,0054953 - e}.$

Operation	$,0054953$		
$- e =$	3		
1st Divisor	$,0054950$	$0,0000000019035503$	$(1,00016 = r,$ $0,0000000346417 = e,$
	34	164850	$1,000159653583 = r - e.$
	$,00549466$	$)2550503$	
$- e =$	46	2197864	
2d Divisor	$,005494614$	$)35263900$	
$- e =$	64	32967684	
3d Divisor	$,00549460$	$)2296216$	
		2197840	
		98376	
		54946	
		$\&c.$	

Which, being further pursued to a third Operation, will give $a = 1,000159653587453, \&c.$

This Value of a is the Amount of 1% for one Day, from which, if 1% be subtracted, the Remainder $= ,000159653587, \&c.$ will be the Interest of 1% for one Day. Consequently, if any proposed Principal be multiplied into either of these, the respective Product will be the Amount or Interest of that Principal for one Day, at 6 per Cent, $\&c.$

And that the Amount (or Interest) of any Principal or Sum may be easily computed for any Number of Days less than a Year; I have here inserted the following Table, which, with a great deal of Care (and I believe Exactness), is calculated from the last-found ($1,000159653587453$) Amount of 1% for one Day. To which also is annexed a Table of the Amounts of 1% for Months.

Days.

Days.	Amounts of 1/. &c.	Days.	Amounts of 1/. &c.	Days.	Amounts of 1/. &c.
1	1.000,159,653,6	46	1.007,370,508,2	91	1.014,633,351,1
2	1.000,319,332,6	47	1.007,531,338,5	92	1.014,795,347,8
3	1.000,479,037,2	48	1.007,692,194,5	93	1.014,957,456,5
4	1.000,638,767,3	49	1.007,853,076,2	94	1.015,119,398,1
5	1.000,798,522,9	50	1.008,013,982,5	95	1.015,281,465,5
6	1.000,958,303,9	51	1.008,174,916,6	96	1.015,443,558,9
7	1.001,118,110,5	52	1.008,335,875,3	97	1.015,605,678,1
8	1.001,277,942,6	53	1.008,496,859,7	98	1.015,767,823,2
9	1.001,437,800,2	54	1.008,657,869,9	99	1.015,929,994,1
10	1.001,597,682,4	55	1.008,818,905,7	100	1.016,092,191,0
11	1.001,757,592,0	56	1.008,979,967,3	101	1.016,254,413,8
12	1.001,917,526,2	57	1.009,141,054,5	102	1.016,416,662,4
13	1.002,077,485,9	58	1.009,302,167,5	103	1.016,578,937,0
14	1.002,237,471,2	59	1.009,463,306,2	104	1.016,741,237,5
15	1.002,397,482,0	60	1.009,624,470,7	105	1.016,903,563,8
16	1.002,557,518,4	61	1.009,785,660,8	106	1.017,065,916,1
17	1.002,717,580,3	62	1.009,946,876,7	107	1.017,228,294,4
18	1.002,877,667,7	63	1.010,108,118,4	108	1.017,390,798,5
19	1.003,037,780,8	64	1.010,269,385,8	109	1.017,551,308,6
20	1.003,197,919,3	65	1.010,430,678,9	110	1.017,715,584,6
21	1.003,358,081,0	66	1.010,591,997,8	111	1.017,878,066,5
22	1.003,518,273,2	67	1.010,753,342,4	112	1.018,040,574,4
23	1.003,678,488,5	68	1.010,914,712,8	113	1.018,203,103,3
24	1.003,838,729,4	69	1.011,076,109,0	114	1.018,365,568,0
25	1.003,998,995,8	70	1.011,237,530,9	115	1.018,528,257,8
26	1.004,159,287,9	71	1.011,398,978,6	116	1.018,690,865,5
27	1.004,319,605,5	72	1.011,560,452,1	117	1.018,853,003,1
28	1.004,479,948,7	73	1.011,721,951,3	118	1.019,016,166,7
29	1.004,640,317,5	74	1.011,883,476,4	119	1.019,178,856,3
30	1.004,800,712,0	75	1.012,045,027,2	120	1.019,341,571,9
31	1.004,961,132,0	76	1.012,206,603,8	121	1.019,504,313,4
32	1.005,121,577,6	77	1.012,368,206,2	122	1.019,667,081,9
33	1.005,281,048,8	78	1.012,530,834,4	123	1.019,829,874,5
34	1.005,441,545,7	79	1.012,691,488,5	124	1.019,992,693,4
35	1.005,603,068,2	80	1.012,853,168,3	125	1.020,155,533,9
36	1.005,763,616,4	81	1.013,014,873,9	126	1.020,318,111,0
37	1.005,924,191,1	82	1.013,176,605,4	127	1.020,481,308,4
38	1.006,084,789,5	83	1.013,338,362,7	128	1.020,644,231,2
39	1.006,245,414,6	84	1.013,500,145,8	129	1.020,807,181,4
40	1.006,406,065,3	85	1.013,661,954,7	130	1.020,970,156,9
41	1.006,566,741,6	86	1.013,822,789,5	131	1.021,133,158,5
42	1.006,727,443,6	87	1.013,985,650,1	132	1.021,296,186,1
43	1.007,888,171,2	88	1.014,147,536,5	133	1.021,459,239,7
44	1.007,048,924,5	89	1.014,309,448,8	134	1.021,622,319,3
45	1.007,209,703,5	90	1.014,471,386,9	135	1.021,785,421,0

Days.	Amounts of <i>l.</i> &c.	Days.	Amounts of <i>l.</i> &c.	Days.	Amounts of <i>l.</i> &c.
136	1.021,948,556,7	181	1.029,316,023,1	226	1.036,737,570,1
137	1.022,111,714,4	182	1.029,190,837,2	227	1.036,603,088,9
138	1.022,274,898,2	183	1.029,645,197,5	228	1.037,068,634,2
139	1.022,438,108,1	184	1.029,829,584,1	229	1.037,234,205,9
140	1.022,601,344,0	185	1.029,973,996,9	230	1.037,399,404,1
141	1.022,764,606,0	186	1.030,138,435,9	231	1.037,563,428,7
142	1.022,927,894,0	187	1.030,302,901,2	232	1.037,731,279,8
143	1.023,090,208,1	188	1.030,467,392,8	233	1.037,896,757,3
144	1.023,254,548,3	189	1.030,631,920,6	234	1.038,062,401,2
145	1.023,417,914,6	190	1.030,796,455,7	235	1.038,224,191,6
146	1.023,581,300,9	191	1.030,961,025,1	236	1.038,393,948,4
147	1.023,744,725,3	192	1.031,125,621,6	237	1.038,559,731,8
148	1.023,908,169,9	193	1.031,290,244,5	238	1.038,725,541,5
149	1.024,071,640,5	194	1.031,454,893,7	239	1.038,891,377,8
150	1.024,235,137,2	195	1.031,619,569,2	240	1.039,057,405,5
151	1.024,398,660,0	196	1.031,784,270,9	241	1.039,223,129,8
152	1.024,562,208,9	197	1.031,948,999,0	242	1.039,389,045,4
153	1.024,725,783,0	198	1.032,113,753,4	243	1.039,554,987,6
154	1.024,889,385,1	199	1.032,278,534,1	244	1.039,720,956,3
155	1.025,053,012,4	200	1.032,443,341,0	245	1.039,886,951,5
156	1.025,216,665,8	201	1.032,608,174,2	246	1.040,052,973,2
157	1.025,380,345,3	202	1.032,773,033,9	247	1.040,219,021,4
158	1.025,544,050,9	203	1.032,937,919,8	248	1.040,385,096,1
159	1.025,707,782,7	204	1.033,102,832,1	249	1.040,551,197,3
160	1.025,871,540,6	205	1.033,267,770,6	250	1.040,717,325,0
161	1.026,035,324,7	206	1.033,432,735,5	251	1.040,883,479,3
162	1.026,199,144,9	207	1.033,597,726,8	252	1.041,049,660,1
163	1.026,362,971,3	208	1.033,762,744,4	253	1.041,215,867,4
164	1.026,526,833,8	209	1.033,927,788,3	254	1.041,382,101,2
165	1.026,690,722,5	210	1.034,092,858,6	255	1.041,548,361,6
166	1.026,854,637,4	211	1.034,257,955,2	256	1.041,714,648,5
167	1.027,018,578,4	212	1.034,423,078,2	257	1.041,880,962,0
168	1.027,182,545,6	213	1.034,588,227,5	258	1.042,047,302,1
169	1.027,346,538,9	214	1.034,753,403,3	259	1.042,213,668,7
170	1.027,510,558,5	215	1.034,918,605,4	260	1.042,380,051,8
171	1.027,674,604,0	216	1.035,083,833,8	261	1.042,546,481,5
172	1.027,838,676,4	217	1.035,249,088,7	262	1.042,712,927,8
173	1.028,002,774,6	218	1.035,414,369,9	263	1.042,879,400,7
174	1.028,166,898,9	219	1.035,579,677,5	264	1.043,045,900,1
175	1.028,331,240,1	220	1.035,745,011,5	265	1.043,212,426,1
176	1.028,495,220,2	221	1.035,910,371,9	266	1.043,378,978,7
177	1.028,659,429,1	222	1.036,075,758,7	267	1.043,545,557,9
178	1.028,823,658,3	223	1.036,241,171,9	268	1.043,712,163,7
179	1.028,987,913,7	224	1.036,406,611,6	269	1.043,878,796,1
180	1.029,152,195,3	225	1.036,571,077,6	270	1.044,045,455,1

Days.

Days.	Amounts of <i>l.</i> &c.	Days.	Amounts of <i>l.</i> &c.	Days.	Amounts of <i>l.</i> &c.
271	1.044,212,140,7	308	1.050,308,252,1	345	1.056,621,011,2
272	1.044,378,852,9	309	1.050,565,951,9	346	1.056,789,704,5
273	1.044,545,591,8	310	1.050,733,678,6	347	1.056,958,424,8
274	1.044,712,357,2	311	1.050,901,432,0	348	1.057,127,172,0
275	1.044,879,149,3	312	1.051,069,212,1	349	1.057,295,950,4
276	1.045,045,968,0	313	1.051,237,019,1	350	1.057,464,747,2
277	1.045,212,813,3	314	1.051,404,852,9	351	1.057,633,575,3
278	1.045,379,685,3	315	1.051,572,713,4	352	1.057,802,430,3
279	1.045,544,658,4	316	1.051,740,600,8	353	1.057,971,312,2
280	1.045,713,509,2	317	1.051,908,515,0	354	1.058,140,221,1
281	1.045,880,461,1	318	1.052,076,455,9	355	1.058,309,157,0
282	1.046,047,439,7	319	1.052,244,423,7	356	1.058,478,119,9
283	1.046,214,444,9	320	1.052,412,418,3	357	1.058,647,109,7
284	1.046,381,476,8	321	1.052,580,439,7	358	1.058,816,126,5
285	1.046,548,435,3	322	1.052,748,483,0	359	1.058,985,170,3
286	1.046,715,620,6	323	1.052,916,563,1	360	1.059,154,241,1
287	1.046,882,732,5	324	1.053,084,665,0	361	1.059,323,338,9
288	1.047,049,871,1	325	1.053,252,793,7	362	1.059,492,463,6
289	1.047,217,036,3	326	1.053,420,949,3	363	1.059,661,515,4
290	1.047,384,228,3	327	1.053,589,131,7	364	1.059,830,794,2
291	1.047,551,446,9	328	1.053,757,341,0	365	1.06
292	1.047,718,692,3	329	1.053,925,577,1	Months.	The Amounts of <i>l.</i> at 6 per Cent, For Months.
293	1.047,885,964,3	330	1.054,093,840,1		
294	1.048,053,263,1	331	1.054,262,130,0		
295	1.048,220,588,5	332	1.054,430,446,7		
296	1.048,387,940,7	333	1.054,598,790,3		
297	1.048,555,319,6	334	1.054,767,160,8		
298	1.048,722,725,2	335	1.054,935,558,2		
299	1.048,890,157,6	336	1.055,103,982,4		
300	1.049,057,616,6	337	1.055,272,433,6		
301	1.049,225,102,5	338	1.055,440,911,6		
302	1.049,392,615,0	339	1.055,609,416,5		
303	1.049,560,154,3	340	1.055,777,948,4		
304	1.049,727,720,4	341	1.055,946,507,1		
305	1.049,895,313,2	342	1.056,115,092,7		
306	1.050,062,932,7	343	1.056,283,705,3		
307	1.050,230,579,0	344	1.056,452,344,8		
				1	1.004,867,550,5
				2	1.009,758,794,2
				3	1.014,673,846,2
				4	1.019,612,822,4
				5	1.024,575,839,4
				6	1.029,563,014,1
				7	1.034,574,164,1
				8	1.039,610,307,6
				9	1.044,670,663,4
				10	1.049,755,650,7
				11	1.054,865,389,4
				12	1.06

The use of this Table is in all respects like that of whole Years, in finding the Amount of any given Sum for any proposed Number of Days less than a Year.

EXAMPLE I.

*Suppose it were required to find the Amount of 375*l.* for 210 Days, at 6 per Cent.*

The Amount of 1*l.* for 210 Days is 1,0340928, &c. per Table. Then $1,0340928 \times 375 = 387,7848$, &c. = 387*l.* 15*s.* 8 $\frac{1}{4}$ *d.* which is the Amount required. And the rest of the Variations may be performed just as in the Examples of whole Years.

But if the Time given consists of Years, and Parts of a Year; as Quarters, Months, &c. Then reduce the odd Time or Parts of the Year into Days; and the Answer may then be found at two Operations; as in the following Example.

EXAMPLE II.

*Suppose it were required to find what 265*l.* would amount to in five Years and 135 Days at 6 per Cent. &c.*

First, the Amount of 1*l.* for $\begin{cases} 5 \text{ Years is } 1,338225, \text{ \&c.} \\ 135 \text{ Days is } 1,021785, \text{ \&c.} \end{cases}$

Then $1,338225 \times 1,021785 \times 265*l.* = 362,355232*l.*$, &c. = 362*l.* 7*s.* 1 $\frac{1}{4}$ *d.*, being the Amount or Answer required.

Or, if the Amount and Time are given, to find the Principal; then multiply the Amount of 1*l.* for the Years, and the Amount of 1*l.* for the odd Days, together; and by their Product divide the given Amount, the Quotient will be the Principal required.

EXAMPLE III.

*What Principal will raise a Stock of 362*l.* 7*s.* 1 $\frac{1}{4}$ *d.* or 362,355232*l.* in 5 Years and 135 Days, at 6 per Cent. &c.*

The Amount of 1*l.* for $\begin{cases} 5 \text{ Years is } 1,338225*l.*, \&c. \\ 135 \text{ Days is } 1,021785*l.*, \&c. \end{cases}$

Then $1,338225 \times 1,021785 = 1,367378$, &c. the Divisor. Next, $1,367378 \mid 362,355232 = A$ (265*l.* the Principal required.

Again, if the Principal and it's Amount are given, to find the Time, at 6 per Cent. &c. you must divide the Amount by it's Principal, and then proceed as in the Third Example, Page 612, for the Answer required.

But

But if the *Amount* and it's *Principal*, with the *Time* of it's being at *Interest*, are given, to find the *Rate of Interest*; then proceed as in the Fourth *Question*, Page 611, &c.

Now, in order to make this *Table of Amounts* for Days, useful for all *Rates* of *Interest* (as before in that for *Years*), you must first find the *Simple Interest* of 1*l.* for one Day, both at the given *Rate*, and also at 6 *per Cent.* And call their *Difference* *x*.

Thus, suppose the given *Ratio* were 8 *per Cent. per Annum.* First, $100 : 8 :: 1 : 0,08$, and $100 : 6 :: 1 : 0,06$, the two *Simple Interests* for one Year.

Then $365) 0,08 (0,00021917$, &c. the *Simple Interest* of 1*l.* for one Day, at 8 *per Cent.*

And $365) 0,06 (0,00016438$, &c. the *Simple Interest* of 1*l.* for one Day, at 6 *per Cent.*

Their *Difference* $0,00005479 = x$, which may do indifferently well for ordinary small *Questions*; but where *Exactness* is required, it will be convenient to make Use of this *Proportion*.

Viz. $\left\{ \begin{array}{l} \text{As the Simple Interest of 1*l.* for one Day, at 6 *per Cent.* : is to the} \\ \text{Tabular Interest of 1*l.* for one Day :: so is the Simple Interest of} \\ \text{1*l.* for one Day, at any given Rate : to a Fourth Number.} \end{array} \right.$

That is, $0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286$.

Then $0,00021286 - 0,00015965 = 0,00005321 = x$.

This *x* being *involved* with the respective *Amounts* for Days, in the same Manner as was done with those for *Years* (vide Page 613), the Result will be the *Answer* to the *Question*.

SECT. II. *Annuities, or Pensions in Arrear, computed at Compound Interest.*

WHEN *Annuities*, &c. are said to be in *Arrear*, see Page 603. And I shall here make use of the same Letters to represent the same Things as before in that Page, save only that *R* is here equal to the *Amount* of 1*l.* as in Section I. of this Chapter.

Suppose *u* = the First Year's Rent of any Annuity without *Interest*.

Then will $Ru + u = \left\{ \begin{array}{l} \text{the Amount of the First Year's Rent, and it's Interest;} \\ \text{more the 2d Year's Rent.} \end{array} \right.$

And $RRu + Ru + u = \left\{ \begin{array}{l} \text{the Amount of the 1st and 2d Year's Rents, with} \\ \text{their Interests; more the 3d Year's Rent, \&c.} \end{array} \right.$

Here

Here $RRu + Ru + u = A$, the *Amount* of any *Yearly Rent* or *Annuity*, being forborn *Three Years*. And from hence may be deduced these *Proportions*.

Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$; and so on in $\div$ for any *Number* of *Terms* or *Years* denoted by t , wherein the last Term will always be uR^{t-1} .

Consequently $A - uR^{t-1}$ will be = the Sum of all the Antecedents; and $A - u$ will be = the Sum of all the Consequents in the Series.

And therefore it would be $u : uR :: A - uR^{t-1} : A - u$. Vide *Page 188* of the *Young Mathematician's Guide* in octavo.

Ergo $Au - uu = RuA - uuR^t$, which, being divided all by u , will become $A - u = RA - uR^t$.

From this last *Equation* it will be easy to raise the following *Theorems*.

$$\text{Theorem 1. } \left\{ \frac{uR^t - u}{R - 1} = A. \quad \text{Theorem 2. } \left\{ \frac{RA - A}{R^t - 1} = u. \right. \right.$$

Theorem 3. $\left\{ \frac{RA + u - A}{u} = R^t. \right.$ If this *Equation* be continually divided by R , until nothing remain, the *Number* of those *Divisions* will be t . See *Page 611*.

Theorem 4. $\left\{ \frac{A}{u}R - R^t = \frac{A - u}{u}. \right.$ If this *Equation* be resolved into *Numbers*, according to the Method proposed in *Secl. III. Chap. 10.* the *Root* will show the *Value* of R .

Question 1. If 30*l.* *Yearly Rent* or *Annuity*, &c. be forborn (i. e. remain unpaid) *Nine Years*; what will it amount to, at 6 per Cent. per Annum Compound Interest?

Here is given $u = 30$, $t = 9$, and $R = 1.06$; to find A , per *Theorem 1*.

$$R^9 = 1.689479, \text{ by the Table of Amounts for Years.}$$

$$30 = u$$

$$\begin{array}{r} R^9 u = 50.684370 \\ - u = 30, \end{array}$$

$R - 1 = 0.06$) 20,684370 (344,7395 = 344*l.* 14*s.* 9½*d.* = A , the Amount required.

Question

Question 2. *What Yearly Rent or Annuity, &c. being forborn or unpaid Nine Years, will raise a Stock of 344*l.* 14*s.* 9½*d.* = 344,7395, at 6 per Cent. &c.?*

Here is given $A = 344,7395$, $t = 9$, and $R = 1,06$; to find u , per Theorem 2.

$$\begin{array}{r} AR = 344,7395 \times 1,06 = 365,42387 \\ - A = 344,7395 \\ \hline \end{array}$$

$$R^t - 1 = 1,689479 - 1 = 0,689479 \quad 20,68437 \quad (30 = u).$$

Question 3. *In what Time will 30*l.* Yearly Rent raise a Stock of, or amount to, 344*l.* 14*s.* 9½*d.* allowing 6 per Cent. for the Forbearance of Payments?*

Here is given $u = 30$, $A = 344,7395$, and $R = 1,06$; to find t , per Theorem 3.

First, $AR + u - A = (344,7395 \times 1,06 + 30 - 344,7395 = 365,42387 + 30 - 344,7395 = 395,42387 - 344,7395) = 50,68437$. And $u = 30$) 50,68437 ($1,689479 = R^t$. Then

$R = 1,06$) 1,689479 ($1,593848$. And $1,06$) 1,593848 ($1,50363$; and so on until it become $1,06$) 1,06 (1 , which will be at the Ninth Division; therefore $t = 9$.

Or $R^t = 1,689479$, being sought in the Table of Amounts for Years, will be found to stand overagainst 9 Years, which is the Time required:

Question 4. *If 30*l.* per Annum, being unpaid Nine Years, will amount to 344*l.* 14*s.* 9½*d.* allowing Compound Interest for every Payment as it becomes due, What must the Rate of Interest be per Cent. &c.?*

Here is given $u = 30$, $A = 344,7395$, and $t = 9$; to find R by the last of the Four Equations, *Viz.* $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$.

First, $\frac{A}{u} = \frac{344,7395}{30} = 11,491317$. And $\frac{A - u}{u}$ is ($= \frac{A}{u} - \frac{u}{u} = \frac{A}{u} - 1$ $= 11,491317 - 1) = 10,491317$.

Hence there is this Equation, $11,491317R - R^9 = 10,491317$.

Let

Let	1	$r + e = R$, and suppose $r = 1$.
10^9	2	$r^9 + 9r^8e + 36r^7ee = R^9$,
$1 \times \frac{A}{u}$ in Numb.	3	$11,491317 + 11,491317e = 11,491317R$,
2 in Numb.	4	$1,000000 + 9,000000e + 36ee = R^9$,
3 — 4	5	$10,491317 + 2,491317e - 36ee = (11,491317R - R^9) = 10,491317$,
+ 36ee	6	$10,491317 + 2,491317e = 10,491317 + 36ee$,
— 10,491317	7	$2,491317e = 36ee$,
$7 \div e$	8	$2,491317 = 36e$,
$8 \div 36$	9	$e = 0,06$, &c.

First, $r = 1$
 $+ e = 0,06$ } $= 1,06 = R$, { *As may be easily tried by involving it, and ordering it, as the Equation above directs.*

SECT. III. To find the Present Worth of Annuities, Pensions, or Leases, &c. at Compound Interest.

LET P = the present Worth of any Annuity, or Lease, &c. and the rest of the Letters, as before.

Then, from what has been said in Section III. Chap. 11. about Purchasing of Annuities, &c. at Simple Interest, it will be easy to form the like Theorems here at Compound Interest, viz. by combining Theorem 1, Page 622, and Theorem 1, Page 610, into one Theorem.

For $\left\{ \frac{uR^t - u}{R - 1} = A, \right\}$ { *The Amount of any Yearly Rent being unpaid any Number of Years. Per Theorem 1. of the last Section, Page 622.*

And $PR^t = A$, { *The Amount of any Principal or Sum being put to Interest, for the same Number of Years. Per Theorem 1, Page 610.*

Hence it follows, that $PR^t = \frac{uR^t - u}{R - 1}$:

Viz. $PR^{t+1} - PR^t = uR^t - u$, being the very same Equation with that in my Compendium of Algebra, Page 112, which is there raised from the Consideration of purchasing annuities, or taking of Leases, &c. to be grounded upon a Rank or Series of Geometrical Proportionals continually decreasing. Thus, $\frac{u}{R}$ is the First and Greatest Term; R the common Ratio of all the Terms; and P is the Sum of all the Series.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$, &c. in $\div$ until the last Term $= \frac{u}{R^t}$. Then will $P - \frac{u}{R^t}$ be the Sum of all the Antecedents, and $P - \frac{u}{R}$, the Sum of all the Consequents.

Therefore

Therefore it will be

$\frac{u}{R} : \frac{u}{RR}$, or (in the same Ratio) $u : \frac{u}{R} :: P - \frac{u}{R} : P - \frac{u}{R}$, which produces $PR^{t+1} - uR^t = PR^t - u$, as above *.

From this Equation may be deduced the following Theorems.

Theorem 1. $\left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P. \right.$ Theorem 2. $\left\{ \frac{PR^t \times R - PR^t}{R^t - 1} = u. \right.$

Theorem 3. $\left\{ \frac{u}{P + u - PR} = R^t, \right.$ {Which, being continually divided by R, will give t.

Theorem 4. $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1}; \right.$ the resolving of which Equation will discover the Value of R.

Question 1. What is 30l. Yearly Rent, to continue Seven Years, worth in ready Money, allowing 6 per Cent. Compound Interest to the Purchaser?

Here is given $u = 30$, $t = 7$, and $R = 1.06$; to find P , per Theorem 1.

Viz. $\frac{u}{R^t} = \frac{30}{1.50363} = 19.9517$.

And $30 - 19.9517 = 10.0483 = u - \frac{u}{R^t}$.

Then $R - 1 = 0.06$ 10.0483 ($167.4716 = P = 167l. 9s. 5d.$, being the Answer required.

Question 2. What Annuity or Yearly Rent, to continue Seven Years, may be purchased for 167l. 9s. 5d. allowing 6 per Cent. Compound Interest to the Purchaser?

In this Question there is given $P = 167.4716$, $t = 7$, and $R = 1.06$; to find u , by the Second Theorem.

* For u is to $\frac{u}{R}$ as $P - \frac{u}{R^t}$ to $P - \frac{u}{R} :: \frac{PR^t - u}{R^t} : \frac{PR - u}{R} :: \frac{PR^t - u}{R^t} : \frac{PR^t - uR^{t-1}}{R^t}$
 $:: PR^t - u : PR^t - uR^{t-1}$; and consequently $uPR^t - uuR^{t-1}$ will be $= uPR^{t-1} - \frac{uu}{R}$,
 and $uPR^{t+1} - uuR^t$ will be $= uPR^t - uu$, and $PR^{t+1} - uR^t$ will be $= PR^t - u$. Q. E. D.

$$\text{First, } PR^t \times R = 251,8153 \times 1,06 = 266,9242$$

$$\text{And } - PR^t = 167,4716 \times 1,50363 = 251,8153$$

$$\text{Then } R^t - 1 = 0,50363 \quad 15,1089 \quad (30 = u.$$

That is, $u = 30$ l. the Answer required.

Question 3. *How long may One have a Lease of 30l. Yearly Rent, for 167l. 9s. 5d. allowing 6 per Cent. Compound Interest to the Purchaser?*

Here is given $P = 167,4716$, $u = 30$, and $R = 1,06$; to find t , by the Third Theorem.

$$\text{First, } P + u = 167,4716 + 30 = 197,4716$$

$$\text{And } - PR = 177,5199$$

$$\text{Then } 19,9517) 30 = u \quad (1,50363 = R^t.$$

If this $1,50363 = R^t$ be either continually divided by $1,06 = R$, until nothing remain (as before in Page 610); or if it be sought in the Table of Amounts for Years, &c., it will discover $t = 7$, which is the true Answer required.

Question 4. *Suppose One should give 167l. 9s. 5d. for the Purchase of a Pension, or Annuity of 30l. per Annum, to continue Seven Years; at what Rate of Interest, per Cent. would that Purchase be made, allowing Compound Interest to the Purchaser?*

In this Question there is given, $P = 167,4716$, $u = 30$, and $t = 7$; to find R . Per Theorem 4, in this Equation $\left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1} \right\}$, which, being brought into Numbers, and it's Root extracted, as in the fourth Question of the last Section; the Value of R will be found 1,06, and then it will be $1 : 0,06 :: 100 : 6$, the Rate per Cent. as was required.

These Four Questions include all the Varieties that can be proposed about purchasing Annuities or Leases, &c. which are to be either immediately entered upon, or in Possession at the Time when the Purchase is made.

But such Questions as relate to Annuities, or a taking of Leases, &c. in Reversion, must be parted or divided into two distinct Questions, each to be separately considered by itself (See Page 608). As in the following Examples.

Example 1. *Suppose it were required to compute the present Worth of 75l. Yearly Rent, which is not to commence or be entered upon, until Ten Years hence; and then to continue Seven Years after that Time; at 6 per Cent. &c. Compound Interest?*

The First Work in this Question is, to find what 75l. per Annum, to continue Seven Years, is worth in ready Money; as if it were to be immediately entered

upon. And to perform that, there is given $u = 75$, $R = 1,06$, and $t = 7$; to find P , as in the First *Question* of this *Section*.

Thus, $\frac{u}{R^t} = \frac{75}{1,50363} = 49,8793$, and $75 - 49,8793 = 25,1207 = u - \frac{u}{R}$:

Then, $R - 1 = 0,06$) $25,1207 = 418,6783 = 418\text{ l. } 14\text{ s. } 6\frac{1}{2}\text{ d.}$, the *Answer* to the First Part of the *Question*.

Then the next Work will be, to find what *Principal* or *Sum*, being put out Ten Years, at 6 per Cent. &c. will amount to $418\text{ l. } 14\text{ s. } 6\frac{1}{2}\text{ d.}$ Here is given $A = 418,6783$, $R = 1,06$, $t = 10$; to find P , per *Theorem 2*, Page 610.

Thus, $R^{10} = 1,790847$) $418,6783 = A$ ($233,7884 = 233\text{ l. } 15\text{ s. } 9\text{ d.}$, the present Worth of 75 l. per Annum in Reversion, &c. as was required.

Example 2. What Annuity or Yearly Rent, to be entered upon Ten Years hence, and then to continue Seven Years, may be purchased for $233\text{ l. } 15\text{ s. } 9\text{ d.}$ Ready Money, at 6 per Cent. &c. Compound Interest?

In the 1st Work of this *Question* there is given, $P = 233,7884$, $R = 1,06$, and $t = 10$ (the Time which the Annuity is not to be entered upon); to find A , per *Theorem 1*, Page 610.

Thus, $PR^t = 233,7884 \times 1,790847 = 418,6783 = A$, the Amount of $233\text{ l. } 15\text{ s. } 9\text{ d.}$ put to Interest Ten Years, at 6 per Cent. &c. Then, for the Second Work of the *Question*, there is given $P = 418,6783$, $R = 1,06$, and $t = 7$ (the Time that the Annuity is to be enjoyed); to find u , per *Theorem 2*. of this *Section*.

$$\begin{aligned}\text{Thus, } PR^t \times R &= 418,6783 \times 1,50363 \times 1,06 = 667,3095 \\ &\quad - PR^t = 418,6783 \times 1,50363 = 629,5372\end{aligned}$$

$$R^t - 1 = 0,50363 \quad 37,7723 \quad (75 = u).$$

That is, $u = 75\text{ l.}$ the Yearly Rent required by the *Question*.

These Two Examples of finding P and u do fully show the Method that must be used in resolving the two General, and, indeed, the most useful *Questions* about Annuities or Leases in Reversion. And if there be Occasion, either the Rate, or the Time, viz. R or t , may be found by a due Application of their respective *Theorems*.

Note, That which hath been done in the two last Sections about Annuities or Yearly Rents, &c. at 6 per Cent. may also be done for any Rate of Interest, by applying the Difference of the Rate (viz. x), as directed in the First Section of this Chapter.

Now, because that Rents and Annuities, &c. are usually paid either by Quarterly or Half-Yearly Payments, and the Method of computing them by the Pen may be thought a little troublesome; I have inserted the following Tables of the Amounts of 1 l. for each, at 6 per Cent.

Half Years = t .	Annuities of 1 <i>l.</i> at 6 per Cent. Compound Interest.	Half Years = t .	Annuities of 1 <i>l.</i> at 6 per Cent. Compound Interest.	Half Years = t .	Annuities of 1 <i>l.</i> at 6 per Cent. Compound Interest.
1	1.029,563,014,1	11	1.377,787,559,2	21	1.843,790,552,3
2	1.06	12	1.418,59,112,2	22	1.898,298,558,3
3	1.061,336,794,9	13	1.460,454,812,7	23	1.954,417,985,3
4	1.1236	14	1.503,630,259,0	24	2.012,196,471,8
5	1.156,817,002,6	15	1.548,082,101,7	25	2.071,683,064,4
6	1.191,016	16	1.593,848,074,5	26	2.132,928,260,1
7	1.226,226,022,8	17	1.640,967,027,6	27	2.195,984,048,3
8	1.262,476,96	18	1.689,478,958,9	28	2.260,903,955,7
9	1.299,799,584,2	19	1.739,425,049,3	29	2.327,743,091,2
10	1.338,225,577,6	20	1.790,847,696,5	30	2.396,558,100,1

Quarterly Amounts.

Quarters of a Year = t .	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.	Quarters of a Year = t .	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.	Quarters of a Year = t .	Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.
1	1.014,673,846,1	21	1.357,802,493,8	41	1.817,126,319,9
2	1.029,563,014,1	22	1.377,787,559,2	42	1.843,790,552,3
3	1.044,670,663,4	23	1.398,005,001,9	43	1.870,846,050,9
4	1.06	24	1.418,519,112,2	44	1.898,298,558,3
5	1.075,554,276,9	25	1.439,334,243,5	45	1.926,153,898,9
6	1.091,336,794,9	26	1.460,454,812,7	46	1.954,417,985,3
7	1.107,350,903,2	27	1.481,885,302,0	47	1.983,096,814,0
8	1.1236	28	1.503,630,259,0	48	2.012,196,471,8
9	1.140,087,533,5	29	1.525,694,297,8	49	2.041,723,133,0
10	1.156,817,002,6	30	1.548,082,101,7	50	2.071,683,064,4
11	1.173,791,957,4	31	1.570,798,420,3	51	2.102,082,622,8
12	1.191,016	32	1.593,848,074,5	52	2.132,928,260,1
13	1.208,492,785,6	33	1.617,235,955,7	53	2.164,226,521,1
14	1.226,226,022,8	34	1.640,967,027,6	54	2.195,984,048,3
15	1.244,219,474,8	35	1.665,046,325,3	55	2.228,207,580,1
16	1.262,476,96	36	1.689,478,958,9	56	2.260,903,955,7
17	1.281,002,352,7	37	1.714,270,113,3	57	2.294,080,112,3
18	1.299,799,584,2	38	1.739,425,049,3	58	2.327,743,091,2
19	1.318,872,643,3	39	1.764,949,104,8	59	2.361,900,034,9
20	1.338,225,577,6	40	1.790,847,696,5	60	2.396,558,193,1

Either of these *Tables* may also be made useful for any proposed *Rate of Interest*; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* = x , &c.

As, for Instance, Suppose any of the aforesaid *Questions* about *Annuities* or *Rents*, &c. were to be computed at 8 *per Cent. per Annum*.

Then $1,08 - 1,06 = 0,02 = x$, for *Yearly Payments*, as before.
 Consequently 2) $0,02$ ($0,01 = x$, for *Half Year's Payments*.
 Or 4) $0,02$ ($0,005 = x$, for *Quarterly Payments*.

Now these *Values* of x , although they are not really true, yet they may serve indifferently well for small *Rents*; as I have already said, *Page* 621. But if you would work exactly :

Then $\sqrt{1,08} = 1,0392304845$ &c.
 — $\sqrt{1,06} = 1,0295680141$ *Vide Table, Page* 628.

Difference = $0,0096624704 = x$, for $\frac{1}{2}$ *Yearly Payments*.

And $\sqrt{} : \sqrt{1,08} = 1,0194263092$ &c.
 — $\sqrt{} : \sqrt{1,06} = 1,0146738461$ See the *Last Table*.

Their Difference $0,0047524631 = x$, for *Quarterly Payments*.

These are the true *Values* of x , which, being *involved* with their respective *Amounts* (as before for *Years*, &c.) according as the *Question* requires, the *Result* will be the *Answer* at 8 *per Cent.* &c. The like may be done for any other *Rate*, either *Greater* or *Less* than 6.

Now, although the *Method* used here (and in *Page* 612 and 613, &c.) be really true (by which the *Tables* calculated only for 6 *per Cent.* are made effectual for all *Rates* of *Compound Interest*), yet it was rather proposed to show what may possibly be performed by the *Pen*, without a great many *Tables* of several *Rates*, than intended for common *Practice*.

For it must needs be confessed, that *Tables*, calculated on Purpose for any designed *Rate of Interest*, are much more ready and useful in common *Practice*. And therefore, since the *Legislative Power* have thought fit to reduce the *Rate of Interest*, and hath settled it by an *Act of Parliament*, at 5 *per Cent.*, I have therefore been at the *Trouble* (*which was not a little*) to calculate the following *Tables* for that *Rate*; but do not think it convenient to take the *Tables* at 6 *per Cent.* out of the *Book*, because the *Examples* are all suited to them; and not only so, but they may be found useful in the taking of *Leases* for *Houses*, &c. For, in those *Cases*, the *Purchaser* is allowed more *Interest* for his *Purchase Money*, than the common *Rate* paid upon the *Loan of Money*.

Here

Here follow New *Tables* of the *Amounts* of One Pound at the *Rate* of 5 per Cent. per Annum Compound Interest, for *Years*, *Half Years*, *Quarters*, *Months*, and *Days*.

I. The Table of the Yearly Amounts of 1l. &c.					
Years = t.	The Amounts of 1l. &c.	Years = t.	The Amounts of 1l. &c.	Years = t.	The Amounts of 1l. &c.
1	1.05 = R	14	1.979,931,60	27	3.732,456,32
2	1.1025 = RR	15	2.078,928,18	28	3.920,129,14
3	1.157,625 = R ³			29	4.116,135,99
4	1.215,506,25	16	2.182,874,50	30	4.321,942,39
5	1.276,281,56	17	2.297,018,42		
		18	2.406,619,23	31	4.538,039,49
6	1.340,095,64	19	2.526,950,19	32	4.764,741,47
7	1.407,100,42	20	2.653,297,70	33	5.003,188,54
8	1.477,455,44			34	5.253,347,97
9	1.551,328,22	21	2.785,962,59	35	5.516,015,36
10	1.628,894,63	22	2.925,260,72		
		23	3.071,523,75	36	5.791,816,13
11	1.710,339,36	24	3.225,099,94	37	6.081,406,94
12	1.795,856,33	25	3.386,354,94	38	6.385,477,29
13	1.885,649,14	26	3.555,672,69	39	6.704,751,15

II. The Table of the Half-Yearly Amounts of 1l. &c.					
Half Years = t.	The Amounts of 1l. &c.	Half Years = t.	The Amounts of 1l. &c.	Half Years = t.	The Amounts of 1l. &c.
1	1.024,695,07	11	1.307,799,13	21	1.669,120,30
2	1.05	12	1.340,95,64	22	1.710,339,36
3	1.075,929,83	13	1.373,189,40	23	1.752,576,32
4	1.1025	14	1.407,100,42	24	1.795,856,33
5	1.129,726,32	15	1.441,848,87	25	1.840,205,13
6	1.157,625	16	1.477,455,44	26	1.885,649,14
7	1.186,212,64	17	1.513,941,32	27	1.932,215,39
8	1.215,506,25	18	1.551,328,22	28	1.970,931,60
9	1.245,523,27	19	1.580,538,38	29	2.028,826,16
10	1.276,281,56	20	1.628,894,63	30	2.078,928,18

III. The

III. The Table of the Quarterly Amounts of 1 <i>l.</i> &c.					
Quarters = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.	Quarters = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.	Quarters = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.
1	1.012,272,23	21	1.291,944,39	41	1.648,884,80
2	1.024,695,07	22	1.307,799,43	42	1.669,120,31
3	1.037,270,37	23	1.323,849,05	43	1.689,604,14
4	1.05	24	1.340,095,64	44	1.710,339,36
5	1.062,885,85	25	1.356,541,61	45	1.731,329,04
6	1.075,929,83	26	1.373,189,40	46	1.752,576,32
7	1.089,133,89	27	1.390,041,51	47	1.774,084,35
8	1.1025	28	1.407,100,42	48	1.795,856,33
9	1.116,030,14	29	1.424,368,69	49	1.817,895,19
10	1.129,726,32	30	1.441,848,87	50	1.840,205,13
11	1.143,590,59	31	1.459,543,58	51	1.862,788,56
12	1.157,625	32	1.477,455,14	52	1.885,649,14
13	1.171,831,64	33	1.495,587,12	53	1.908,790,27
14	1.186,212,64	34	1.513,941,32	54	1.932,215,39
15	1.200,770,12	35	1.532,520,76	55	1.955,927,99
16	1.215,506,25	36	1.551,328,22	56	1.979,931,60
17	1.230,423,23	37	1.570,366,48	57	2.004,229,78
18	1.245,523,27	38	1.589,638,38	58	2.028,826,16
19	1.260,808,62	39	1.609,146,80	59	2.053,724,39
20	1.276,281,56	40	1.628,894,63	60	2.078,928,18

IV. The Table of the Monthly Amounts of 1 <i>l.</i> &c.					
Months = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.	Months = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.	Months = <i>t.</i>	The Amounts of 1 <i>l.</i> &c.
1	1.004,074,12	5	1.020,537,28	9	1.037,270,37
2	1.008,164,85	6	1.024,695,07	10	1.041,496,34
3	1.012,272,23	7	1.028,869,81	11	1.045,739,53
4	1.016,396,96	8	1.033,061,55	12	1.05

NOTE: The Amount of one Pound, for one Day, is 1,0001336807225. &c. (found as that in Page 616); but, in the following Table, I take only Nine of those Figures, as being sufficient in Practice, for computing the Interest of any Sum not exceeding One Hundred Millions of Pounds.

V. The

V. The Table of the Daily Amounts of *l. &c.*

Days = <i>t.</i>	The Amounts of <i>l. &c.</i>	Days = <i>t.</i>	The Amounts of <i>l. &c.</i>	Days = <i>t.</i>	The Amounts of <i>l. &c.</i>
1	1.000,133,68	41	1.005,495,58	81	1.010,886,23
2	1.000,267,38	42	1.005,630,00	82	1.011,021,37
3	1.000,401,09	43	1.005,764,43	83	1.011,156,52
4	1.000,534,83	44	1.005,898,88	84	1.011,291,69
5	1.000,668,58	45	1.006,033,35	85	1.011,426,88
6	1.000,802,35	46	1.006,167,84	86	1.011,562,09
7	1.000,936,14	47	1.006,302,34	87	1.011,697,32
8	1.001,069,94	48	1.006,436,87	88	1.011,832,56
9	1.001,203,77	49	1.006,571,41	89	1.011,967,83
10	1.001,337,61	50	1.006,705,97	90	1.012,103,11
11	1.001,471,47	51	1.006,840,55	91	1.012,238,41
12	1.001,605,35	52	1.006,975,14	92	1.012,373,72
13	1.001,739,24	53	1.007,109,75	93	1.012,509,06
14	1.001,873,15	54	1.007,244,38	94	1.012,644,41
15	1.002,007,08	55	1.007,379,03	95	1.012,779,78
16	1.002,141,03	56	1.007,513,70	96	1.012,915,17
17	1.002,275,00	57	1.007,648,39	97	1.013,050,58
18	1.002,408,99	58	1.007,783,09	98	1.013,186,00
19	1.002,542,99	59	1.007,917,81	99	1.013,321,45
20	1.002,677,01	60	1.008,052,55	100	1.013,456,91
21	1.002,811,05	61	1.008,187,31	101	1.013,592,39
22	1.002,945,10	62	1.008,322,08	102	1.013,727,88
23	1.003,079,18	63	1.008,456,87	103	1.013,863,40
24	1.003,213,27	64	1.008,591,68	104	1.013,998,93
25	1.003,347,38	65	1.008,726,51	105	1.014,134,48
26	1.003,481,51	66	1.008,861,36	106	1.014,270,05
27	1.003,615,65	67	1.008,996,23	107	1.014,405,64
28	1.003,749,82	68	1.009,131,11	108	1.014,541,25
29	1.003,884,00	69	1.009,266,01	109	1.014,676,87
30	1.004,018,20	70	1.009,400,93	110	1.014,812,52
31	1.004,152,42	71	1.009,535,87	111	1.014,948,18
32	1.004,286,65	72	1.009,670,82	112	1.015,083,86
33	1.004,420,91	73	1.009,805,79	113	1.015,219,55
34	1.004,555,18	74	1.009,940,79	114	1.015,355,27
35	1.004,689,47	75	1.010,075,79	115	1.015,491,00
36	1.004,823,76	76	1.010,210,83	116	1.015,626,75
37	1.004,958,10	77	1.010,345,87	117	1.015,762,52
38	1.005,092,45	78	1.010,480,93	118	1.015,898,31
39	1.005,226,81	79	1.010,616,02	119	1.016,034,12
40	1.005,361,19	80	1.010,751,12	120	1.016,169,94

Days

Days = t.	The Amounts of l. &c.	Days = t.	The Amounts of l. &c.	Days = t.	The Amounts of l. &c.
121	1.016,305,78	161	1.021,754,39	201	1.027,232,21
122	1.016,441,74	162	1.021,890,98	202	1.027,360,53
123	1.016,577,52	163	1.022,027,58	203	1.027,506,86
124	1.016,713,49	164	1.022,164,21	204	1.027,644,22
125	1.016,849,33	165	1.022,300,85	205	1.027,781,60
126	1.016,985,27	166	1.022,437,51	206	1.027,918,99
127	1.017,121,22	167	1.022,574,19	207	1.028,056,40
128	1.017,257,19	168	1.022,710,89	208	1.028,193,84
129	1.017,393,17	169	1.022,847,61	209	1.028,331,29
130	1.017,529,18	170	1.022,984,34	210	1.028,468,75
131	1.017,665,21	171	1.023,121,09	211	1.028,606,24
132	1.017,801,25	172	1.023,257,87	212	1.028,743,75
133	1.017,937,31	173	1.023,394,66	213	1.028,881,27
134	1.018,073,38	174	1.023,531,47	214	1.029,018,81
135	1.018,209,48	175	1.023,668,29	215	1.029,156,37
136	1.018,345,59	176	1.023,805,14	216	1.029,293,95
137	1.018,481,73	177	1.023,942,00	217	1.029,431,54
138	1.018,617,88	178	1.024,078,88	218	1.029,569,16
139	1.018,754,05	179	1.024,215,78	219	1.029,706,79
140	1.018,890,24	180	1.024,352,70	220	1.029,844,45
141	1.019,026,44	181	1.024,489,64	221	1.029,982,12
142	1.019,162,67	182	1.024,626,59	222	1.030,119,80
143	1.019,298,91	183	1.024,763,56	223	1.030,257,51
144	1.019,435,17	184	1.024,900,55	224	1.030,395,24
145	1.019,571,45	185	1.025,037,56	225	1.030,532,98
146	1.019,707,75	186	1.025,174,59	226	1.030,670,74
147	1.019,844,06	187	1.025,311,64	227	1.030,808,52
148	1.019,980,39	188	1.025,448,70	228	1.030,946,32
149	1.020,116,75	189	1.025,585,78	229	1.031,084,14
150	1.020,253,12	190	1.025,722,88	230	1.031,221,97
151	1.020,389,50	191	1.025,860,00	231	1.031,359,83
152	1.020,525,91	192	1.025,997,14	232	1.031,497,70
153	1.020,662,34	193	1.026,134,30	233	1.031,635,59
154	1.020,798,78	194	1.026,271,47	234	1.031,773,50
155	1.020,935,24	195	1.026,408,66	235	1.031,911,43
156	1.021,071,72	196	1.026,545,88	236	1.032,049,38
157	1.021,208,22	197	1.026,683,10	237	1.032,187,34
158	1.021,344,73	198	1.026,820,15	238	1.032,325,33
159	1.021,481,27	199	1.026,957,62	239	1.032,463,33
160	1.021,617,82	200	1.027,094,90	240	1.032,601,35

Days = t.	The Amounts of l. &c.	Days = t.	The Amounts of l. &c.	Days = t.	The Amounts of l. &c.
241	1.032,739,39	281	1.038,276,09	321	1.043,842,49
242	1.032,877,44	282	1.038,414,89	322	1.043,982,03
243	1.033,015,52	283	1.038,553,71	323	1.044,121,59
244	1.033,153,61	284	1.038,692,54	324	1.044,261,17
245	1.033,291,73	285	1.038,831,39	325	1.044,400,77
246	1.033,429,86	286	1.038,970,27	326	1.044,540,38
247	1.033,568,01	287	1.039,109,16	327	1.044,680,02
248	1.033,706,17	288	1.039,248,17	328	1.044,819,67
249	1.033,844,36	289	1.039,386,99	329	1.044,959,34
250	1.033,981,57	290	1.039,525,94	330	1.045,099,03
251	1.034,120,79	291	1.039,664,91	331	1.045,238,74
252	1.034,259,03	292	1.039,803,89	332	1.045,378,47
253	1.034,397,29	293	1.039,942,89	333	1.045,518,22
254	1.034,535,57	294	1.040,081,91	334	1.045,657,98
255	1.034,673,87	295	1.040,220,95	335	1.045,797,77
256	1.034,812,18	296	1.040,360,01	336	1.045,937,57
257	1.034,950,52	297	1.040,499,08	337	1.046,077,39
258	1.035,088,87	298	1.040,638,18	338	1.046,217,23
259	1.035,227,24	299	1.040,777,29	339	1.046,357,09
260	1.035,365,63	300	1.040,916,42	340	1.046,496,97
261	1.035,504,04	301	1.041,055,57	341	1.046,636,86
262	1.035,642,47	302	1.041,194,74	342	1.046,776,78
263	1.035,780,91	303	1.041,333,93	343	1.046,916,71
264	1.035,919,38	304	1.041,473,14	344	1.047,056,67
265	1.036,057,86	305	1.041,612,36	345	1.047,196,64
266	1.036,196,36	306	1.041,751,60	346	1.047,336,63
267	1.036,334,88	307	1.041,890,86	347	1.047,476,64
268	1.036,473,42	308	1.042,030,15	348	1.047,616,66
269	1.036,611,97	309	1.042,169,44	349	1.047,756,71
270	1.036,750,55	310	1.042,308,76	350	1.047,896,77
271	1.036,889,14	311	1.042,448,10	351	1.048,036,86
272	1.037,027,75	312	1.042,587,45	352	1.048,176,96
273	1.037,166,38	313	1.042,726,83	353	1.048,317,08
274	1.037,305,03	314	1.042,866,22	354	1.048,457,22
275	1.037,443,70	315	1.042,005,63	355	1.048,597,38
276	1.037,582,39	316	1.043,145,06	356	1.048,737,56
277	1.037,721,09	317	1.043,284,51	357	1.048,877,75
278	1.037,859,82	318	1.043,423,97	358	1.049,017,97
279	1.037,998,56	319	1.043,563,46	359	1.049,158,20
280	1.038,137,32	320	1.043,702,97	360	1.049,298,45
				361	1.049,438,72
				362	1.049,579,01
				363	1.049,719,32
				364	1.049,859,65
				365	1.049,999,99
				366	1.05

I think

I think it needless to say any Thing of the Use of these *Tables*, because I take it for granted, that whoever understands the Work of the foregoing *Examples*, at 6 per Cent. cannot but know how to make use of these *Tables* at 5 per Cent. as Occasion requires.

Thus far concerning *Annuities*, or *Leases*, &c. that are *limited* by any assigned *Time*; and it is only such that can be computed by *Theorems* or certain *Rules*. However, it may not perhaps be *unacceptable*, to insert a brief Account of some *Estimates* that have been reasonably made, by *two* very ingenious *Persons*, about the Proportion or Difference of *Men's Lives*, according to their several *Ages*; which may be of good Use in computing the *Values* of *Annuities*, or taking of *Leases* for *Lives*, &c.

Of the Values of Annuities for Lives.

Sir William Petty, in his Discourse made before the Royal Society (*Anno* 1674), concerning the Use of *Duplicate Proportion*, in the Life of Man and it's *Duration*; saith, that it is found by *Experience* that there are more Persons living of between 16 and 26 Years Old, than of any other *Age* or *Decade* of Years in the whole Life of Man (*viz.* 70 or 80 Years). His Reason for that Assertion I shall omit; but, supposing it true, he thence infers, that the *Square-Roots* of every Number of Men's Ages under 16 (whose *Square-Root* is 4), compared with the said Number 4, doth shew the *Proportion* of the Likelihood of such Men's reaching the *Age* of 70 Years.

As, for Example, it is 4 Times more likely, that One of 16 Years Old should live to 70, than a New-Born Babe: It is 3 Times more likely, that One of 9 Years Old should attain the *Age* of 70, than the said Infant, &c.

On the other Hand, it is 5 to 4, that One of 25 Years Old will die before One of 16; and 6 to 5, that One of 36 will die before One of 25. And so on according to the *Square-Roots* of any other declining *Age*, compared with (4,6) the *Square-Root* of 21, which is the *Year* of *Perfection* according to the Sense of our Law, and the *Age* for whose Life a *Lease* is most valuable.

2. The ingenious and great Mathematician, Doctor Edmund Halley, (*in Philosoph. Transact. Numb.* 196.) doth, with great Industry and Skill, draw an Estimate of the Proportion of Men's Lives, from the Monthly Tables of the Births and Funerals in Breslaw, the Capital City of the Province of Silesia; or, as the Germans call it, Schlesia. Whence he proves, that it is 80 to 1 a Person of 25 Years Old will not die in a Year; that it is $5\frac{1}{2}$ to 1, that a Man of 40 will live 7 Years; that a Man of 50 Years Old may reasonably expect to live 27 or 28 Years, &c.

Now from these and the like *Proportions* he justly infers that the *Price* of *Insurance* upon *Lives* ought to be regulated, there being a great Difference between the *Life* of a Man of 20, and one of 50. For *Example*: It is 100 to 1, that a Man of 20 dies not in a *Year*, and but 38 to 1, for a Man of 50 *Years* of *Age*. And upon these also depends the *Valuation* of *Annuities* for *Lives*; for it is plain, that the *Purchaser* ought to pay only such a *Part* of the *Value* of any *Annuity*, as he hath *Chances* that he is living.

And for that Purpose he hath taken the Pains (*which was not a little*) to compute the following *Table* (that shews the *Value* of *Annuities*) for every Fifth *Year* of *Age* to the 70th.

Age.	Year's Purchase.	Age.	Year's Purchase.	Age.	Year's Purchase.
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same ingenious Gentleman proceeds on, and shews how to estimate or find the *Value* of *Two Lives*, and then of *Three Lives*, which, being too long a Discourse to be recited here, I have, for Brevity's Sake, omitted it; and shall only add this serious Observation,

Viz. How unjustly we repine at the Shortness of our *Lives*, and think ourselves wronged if we attain not to *Old Age*; whereas it appears, that the *One Half* of those, that are BORN, die in Seventeen Years' Time. For, by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616. So that, instead of murmuring at what we call a *Short Life*, we ought to account it as a great Blessing that we have survived, perhaps by many Years, that *Period* of *Life* whereat the *One Half* of the whole Race of Mankind does not arrive.

SECT. IV. Of Purchasing Freehold, or Real Estates, at Compound Interest.

ALL *Freehold* or *Real Estates*, are supposed to be purchased or bought to continue for ever (*viz. without any limited Time*); therefore the Business of computing the true *Value* of such *Estates* is grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually decreasing, *ad Infinitum*.

Thus, let *P*, *u*, *R*, denote the same *Data* as in the last Section. Then the *Series* will be, $\frac{u}{R}$, $\frac{u}{RR}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, and so on in $\div$ until the last Term = 0.

$= 0$. Then will $P - 0$ (viz. P) be the *Sum* of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Consequents*; therefore it will be $u : \frac{u}{R} :: P : P - \frac{u}{R}$, which produces $PR - u = P$. *

This *Equation* affords the following *Theorems*.

Theorem 1. $PR - P = u$. *Theorem 2.* $\left\{ \frac{u}{R-1} = P \right.$

Theorem 3. $\left\{ \frac{P+u}{P} = R \right.$

Example. Suppose a *Freehold Estate* of 75*l.* *Yearly Rent* were to be sold; what is it worth, allowing the *Buyer* 6 per Cent. *&c.* *Compound Interest* for his *Money*?

In this *Question* there is given $u = 75$, $R = 1,06$; to find P , per *Theorem 2*. Thus, $R - 1 = 0,06$ $75 = u$ (1250*l.* = P , the *Answer* required. And so on for any of the rest, as *Occasion* requires. But if the *Rent* is to be paid, either by *Quarterly*, or *Half-Yearly Payments*;

Then $R = \sqrt{1,06}$ for *Half-Yearly* } *Payments at 6 per Cent.*
And $R = \sqrt[4]{1,06}$ for *Quarterly* }

Or $\left\{ \begin{array}{l} R = 1,08 \text{ for Yearly} \\ R = \sqrt{1,08} \text{ for Half-Yearly} \\ R = \sqrt[4]{1,08} \text{ for Quarterly} \end{array} \right\}$ *Payments at 8 per Cent.*

The like is to be understood for any other proposed *Rate of Interest*, either greater or less than 6 per Cent.

The *Application* of these *Theorems* to *Practice* is so very easy, that it is needless to insert more *Examples*.

* For $u \times P - \frac{u}{R}$ will be $= \frac{u}{R} \times P$, that is, $uP - \frac{uu}{R}$ will be $= \frac{uP}{R}$; and consequently (multiplying all the terms into R), $uPR - uu$ will be $= uP$, and (dividing all the terms by u), $PR - u$ will be $= P$. Q. E. D.

And we will find that the sum of the squares of the sides of a triangle is equal to the square of the hypotenuse.

The following theorem is also true:

$$a^2 + b^2 = c^2$$

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Let us now consider the case of a right-angled triangle. We will find that the sum of the squares of the two sides is equal to the square of the hypotenuse.

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The above theorem is also true for any right-angled triangle.

The Application of these theorems to the study of the properties of the triangle is of great importance.

Let us now consider the case of a right-angled triangle. We will find that the sum of the squares of the two sides is equal to the square of the hypotenuse.

CLAVIS USURÆ:

OR,

A KEY TO INTEREST,

BOTH

Simple and Compound:

CONTAINING

Practical *Rules*, plainly expressed in *Words at length*; whereby all the various *Cases* of *Interest*, and *Annuities*, or *Leases*, either in Possession, or Reversion, and Purchasing *Freehold Estates*, &c. may very easily be Resolved, both by the *Pen* and a small *Table of Logarithms*, hereunto annexed, for all *Rates* of *Interest*, and *Times* of *Payment* whatsoever; illustrated by Variety of *Examples*.

To which is Added,

Rules to be Observed in Estimating the *Value* of *Annuities*, or *Leases*, and *Insurances* for *Lives*, &c.

ALSO,

The Business of *Rebate* or *Discompt*, and the *Equation* of *Payments* (very useful for Merchants and other Dealers) is here *Rectified* and truly *Determined*.

By JOHN WARD,

AUTHOR OF THE YOUNG MATHEMATICIAN'S GUIDE, ETC.

CLASSIC ESSAYS
OF
A KEY TO INTEREST

By John Ward

The author of this work, who has spent many years in the study of the human mind, and who has observed the various phases of human nature, has endeavored to present a clear and concise view of the principles of interest, and to show how they may be applied to the various branches of human life.

It is the object of this work to show the value of interest, and to show how it may be applied to the various branches of human life.

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JOHN WARD
AUTHOR OF THE "KEY TO INTEREST"

To the Honourable
Sir *JOHN WENTWORTH*,
OF
NORTH-ELMS-HALL,
IN THE
West-Riding of Yorkshire,
BARONET;

This TRACT, as an Acknowledgment of Great Favours and
Obligations received,

Is most humbly Dedicated,
and Presented,
By

JOHN WARD.

JOHN W. BARNETT

JOHN W. BARNETT

JOHN W. BARNETT

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JOHN W. BARNETT

THE
P R E F A C E.

IT may be here expected that I should (according to the Custom of Prefaces) give the Reader some Account of the ensuing Pages, and of the Motives which induced me to Publish them; more especially, because there are already so many Books which Treat upon the same Subject.

It is true indeed, the Business of Interest and Annuities, &c. has been handled by several Authors; and I, amongst the rest, did adventure to Cast in my Mite upon that Subject, in a Compendium of Algebra, Published Anno 1695, which appears to be so well Approved of, that Doctor John Harris hath Transcribed and Inserted that whole Discourse into his Lexicon Technicum; or, The Universal English Dictionary. Since then I wrote somewhat fuller upon the same Subject, in the Algebräick Part of the Young Mathematician's Guide, Published Anno 1707.

But even that, and all that I have hitherto seen upon this Subject, seems to fall short of what might be desired: For most Authors, which treat of Interest and Annuities, &c. do perform their Computations by the Help of particular Tables, Calculated to several Rates of Interest and Discompt.

And those Authors which perform their Computations in Compound Interest, &c. by Logarithms, seem (to me) to be too general and short in their Theorems, by which those Calculations are performed, especially in respect of the different Times of Payment, &c.

And what I have heretofore done has been by Algebräick Theorems, to be performed either by the Pen, or with the help of a small Table, Calculated on purpose; which was indeed rather to shew what may possibly be performed by the Pen (without a great many Tables of several Rates) than for common Practice: Nor can it be supposed, that every Reader, that may have Occasion to peruse the Business of Interest, is fitly qualified to understand those Theorems so well, as to apply them to Practice.

And for that Reason I was requested by an Ingenious Gentleman (to whom I stand highly Obligated) to Explain those Theorems, and Reduce them into Practical Rules, expressed in Words at length, that so they might become of general Use; the which I

have accordingly done along with the Theorems, so plainly, that (I presume) any One, who is but a little versed in the first Four Rules of Arithmetick, may be able to Resolve all the various Cases relating to Interest, both Simple and Compound, with those of Annuities, or Leases, &c. for any proposed Time, and at any given Rate of Interest, both by the Pen, and a small Table of Logarithms hereunto annexed. Also, I have added a New Method of finding the true Equated Time, for the Discharging of several Sums at one entire Payment, without Loss either to the Debtor, or Creditor; a thing which hath hitherto been imperfect.

And in order to render the whole Work as plain and easy to be understood as possibly I can, I have given Two or Three Examples in every Case, for different Rates of Interest, and Times of Payment, with their Operations at large: For, although Rules are never so well expressed in Words (according to the Author's Sense), yet there may (and perhaps will) arise some seeming Difficulties in them, which Examples will help to Explain and render Easy.

To be brief; When this Treatise was perused in Manuscript by a Person that has very good Skill in the Business of Interest, &c. it gave such ample Satisfaction, that I was prevailed with to Publish it; and accordingly I have adventured to send it Abroad into the World, in hopes it will be found useful, which is the chief thing desired,

LONDON,
November 21st, 1709.

By

J. WARD.

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A
KEY TO INTEREST,

BOTH
Simple and Compound, &c.

CHAP. I.
Of Decimal Arithmetick.

I DO here take it for granted, that the Reader is well acquainted with *Vulgar Arithmetick*, especially the common Rules; viz. *Addition, Subtraction, Multiplication, and Division of whole Numbers*: But, because the Computations of *Interest, and Values of Annuities, &c.* cannot well be performed by *Logarithms*, or indeed by any other Method whatsoever, without some Knowledge of those Rules in *Decimal Fractions or Parts*, which perhaps he is not so well acquainted with; I therefore thought it convenient to give a brief Account thereof.

SECT. I. *Notation of Decimals.*

ANY thing which is called One, (whether it be *Coin, Weight, Measure, or Time, &c.*) as one *Foot, one Yard, one Pound, one Shilling, one Year, or one Day, &c.* we conceive it to be *Divided* into Ten equal Parts, and every one of those Parts are supposed to be sub-divided into other Ten equal Parts; and so on by a *Decimal Division* (viz. by Tens) *ad infinitum*.

The *Unit* or 1, being thus divided by Imagination into 10, 100, 1000, or 10000, &c. equal Parts at pleasure, any *Number* of those Parts may be as easily expressed and set down as whole *Numbers*, they being only distinguished and known from whole Numbers, by a *Comma, or Point*, as in the following Table.

Whole Numbers.						Decimal Parts.					
5	4	3	2	1	0	1	2	3	4	5	6
Hundreds of Thousands &c.	Tens of Thousands	Thousands	Hundreds	Tens	Unit's Place	Parts of Ten, or $\frac{1}{10}$	Parts of a Hundred	Parts of 1000	Parts of 10000	Parts of 100000	Parts of 1000000 &c.

From this Table it does plainly appear,

1. That, as whole Numbers do increase or become greater from the *Unit's* Place towards the Left Hand, by a ten-fold Proportion: so *Decimal* Parts do decrease or become less from the *Unit's* Place towards the Right Hand, in the same Proportion, viz. by *Tens*.

2. That the *Decimal* Parts are only separated and known from whole Numbers by a *Point*, or *Comma*; and take their Denomination from their Distance below the *Unit's* Place towards the Right Hand.

Thus, $\begin{cases} 0,5 & \text{is } 5 \text{ parts of } \textit{Ten}. \\ 0,25 & \text{is } 25 \text{ parts of a } \textit{Hundred}. \\ 0,657 & \text{is } 657 \text{ parts of a } \textit{Thousand}, \text{ \&c.} \end{cases}$

3. *Cyphers* being annexed to *Decimal* Parts, alter not their *Values*: That is, those *Cyphers* do neither increase nor decrease the Value of the Parts they are annexed to.

Thus, 0,5, 0,50, 0,500, 0,5000, &c. are each of them but five Tenth parts of a Unit, or 1.

4. But *Cyphers* prefixed before *Decimal* Parts do decrease their *Value*, by removing them further from the *Comma*, or *Unit's* Place.

Thus, $\begin{cases} 0,5 & \text{is } 5 \text{ parts of } \textit{Ten}. \\ 0,05 & \text{is } 5 \text{ parts of a } \textit{Hundred}. \\ 0,005 & \text{is } 5 \text{ parts of a } \textit{Thousand}. \end{cases}$

Consequently the true *Value* of all *Decimal* Parts is easily known by their Distance from the *Unit's* Place (as above): Which being well understood, the following *Rules* will be found very easy.

SECT. III. *Addition and Subtraction of Decimals.*

HAVING set down all the proposed Numbers in their respective Places; viz. every *Figure*, as well of the *Decimal* Parts, as of the whole *Numbers*, directly underneath those of the same *Value* (or *Name*); which may be very easily done, if the *Commas*, or separating *Points*, are carefully placed directly one under another;

Then { Add, or Subtract them as if they were all whole Numbers; and from their Sum, or Difference, Cut off, by the separating Comma, so many Places of Decimal Parts, as there are in any of the given Numbers.

Examples in Addition.

Let it be required to find the Sum of 43,25 and 2,45 and 29,9 and 7,054. These Numbers, being rightly set down, will stand

Thus, 43,25	Again, 378,0009	} Add
2,45	59,025	
29,9	470,9	
7,054	92,0741	
Sum 82,654	Sum 1000,0000	

Examples in Subtraction.

From 753,25	From 75,094	From 543,000
Take 469,25	Take 68,596	Take 267,546
Rem. 284,00	6,498	275,454

Note, Addition and Subtraction do mutually prove each other, as in whole Numbers; of which I suppose it needless to insert Examples.

SECT. III. *Multiplication of Decimals.*

WHETHER the *Factors* (viz. the Numbers) proposed to be *Multiplied* together, are either all *Decimal* Parts, or *Decimals* joined to whole *Numbers*, *Multiply* them together as if they were all whole Numbers, and for the true *Value* of their *Product*, observe this *Rule*.

Rule. { Cut off (viz. separate) with a Comma, so many Places of Decimal Parts in the Product as there are in both the Factors counted together.

Examples.

Suppose it were required to *Multiply* 21,3 with 4,56.

The given Factors $\left\{ \begin{array}{r} 21,3 \\ 4,56 \end{array} \right.$ 1278 1065 852 Product 97,128	Or $\left\{ \begin{array}{r} 4,56 \\ 21,3 \end{array} \right.$ 1368 456 912 Product 97,128
--	--

Again, Let it be required to *Multiply* 4573 into 7,546, and 36,4078 into 549,3. The Work may stand

Thus, $\left\{ \begin{array}{r} 4573 \\ 7,546 \end{array} \right.$ 27438 18292 22865 32011 34507,858	Multiply $\left\{ \begin{array}{r} 36,4078 \\ 549,3 \end{array} \right.$ 1092234 3276702 1456312 1820390 Products 19998,80454
---	--

Note, It sometimes happens, that in *Multiplying* Parts with Parts only, there will not be so many *Figures* in their Product as there ought to be Places of Parts by the *Rule*: In that Case you must supply the Defect by *prefixing Cyphers* to their Product; as in these following

Examples.

$\left\{ \begin{array}{r} 0,2365 \\ 0,2435 \end{array} \right.$ 11825 7095 9460 4730 0,05758775	Multiply $\left\{ \begin{array}{r} 0,03472 \\ 0,02364 \end{array} \right.$ 13888 20832 10416 6944 Products 0,0008207808
--	--

SECT. IV. *Division of Decimals.*

DIVISION of *Decimals* is performed in all respects like that of whole *Numbers*; and for discovering the true *Value* of the *Quotient*, observe this General *Rule*.

Rule. { *The Places of Decimal Parts in the Divisor and Quotient being counted together, must always be equal in Number with those in the Dividend.*

This General Rule admits of Four Cases.

Case 1. When the Places of *Decimal Parts* in the *Divisor* and *Dividend* are equal in Number, then the *Quotient* will be a whole Number; as in the Two following Examples.

Dividend.
Divisor 7,54) 7140,38 (947 the *Quotient*.
 6786

3543
 3016

5278

5278

(0)

Again, If it were required to Divide 2,6925 by 0,0075.

Dividend.
Divisor 0,0075) 2,6925 (359 *Quotient*.

Case 2. When the Number of Places of *Decimal Parts* in the *Dividend*, exceeds the number of those in the *Divisor*, *Cut off* the Excess for *Decimal Parts* in the *Quotient*.

Examples.

7,54) 71,4038 (9,47. And 0,75) 2,6925 (3,59.

Case 3. When there are not so many Places of *Decimal Parts* in the *Dividend*, as there are in the *Divisor*, then annex *Cyphers* to the *Dividend*, to make them equal, and the *Quotient* will be a whole Number; as in *Case 1*.

Examples.

Let us suppose it were required to Divide 1565,7 by 3,684, and 3615 by 5,784, the Work must stand thus:

3,684) 1565,700 (425. And 5,784) 3615,000 (625.

Case 4. If, after the Division is finished, there are not so many Figures in the *Quotient*, as there ought to be Places of *Decimal Parts*, by the *General Rule*, you must then supply their Defect, by *prefixing* *Cyphers* before the *Quotient* Figures.

Examples.

Examples.

Let it be required to Divide 3,5532 by 756.

$$\begin{array}{r}
 756 \overline{) 3,5532} \quad (47 \\
 \underline{3024} \\
 5292 \\
 \underline{5292} \\
 (0)
 \end{array}$$

Here, when the Division is finished, there are but *Two* Figures in the Quotient, whereas there ought to be *Four* Places of Decimal Parts, by the General *Rule*; therefore two Cyphers must be prefixed before the Quotient Figures, thus, 0,0047.

That is, 756) 3,5532 (0,0047 is the true Quotient required.

Suppose it were required to Divide 0,0007475 by 0,575; viz. all Decimal Parts; then it will be 0,575) 0,0007475 (0,0013, the Quotient, &c.

Note, Multiplication and Division do mutually prove the Truth of each other: That is, If in *Multiplication* you Divide the Product by either of the Factors, the Quotient will be the other Factor, if the Work be true.

So in *Division*, If you Multiply the Quotient and the Divisor together, their Product (*being Added to what remained after the Division*) will be the Dividend, if that Work be true.

SECT. V. To Reduce (or rather Change) *Vulgar Fractions* into *Decimal Parts*, and the Contrary.

Rule. { Annex Cyphers to the Numerator of the given Fraction; then Divide it by the Denominator, and the Quotient will be the Decimal Parts sought.

Examples.

Suppose it were required to Reduce $\frac{3}{4}$ into Decimal Parts: Then it will be, 4) 3,00 (0,75, the Decimal Parts required. And $\frac{1}{2}$ is 0,5; thus, 2) 1,0 (0,5, the Decimal. Or $\frac{1}{4}$ is 0,25; thus, 4) 1,00 (0,25, the Decimal.

And thus may the Decimal Parts *equivalent* to any known Part, or Parts of *Coin, Weight, Measures, or Time, &c.* be easily found, if you first reduce the given Parts of the *Coin, or Time, &c.* into a *Vulgar Fraction*, whose Denominator is the Number of those known Parts contained in the Integer, and the given Parts it's Numerator.

Examples in Coin.

Suppose it were required to find the Decimal Parts of a Pound Sterling, equal to 17s. 6d.

First, because 20s. make one Pound, therefore 17s. is $\frac{17}{20}$ of a Pound; and because there are 240 Pence in a Pound, therefore 6d. is $\frac{6}{240}$ of a Pound.

Then

Then 20) 17,00 (0,85 is the Decimal of 17s.
And 240) 6,000 (0,025 is the Decimal of 6d.

Consequently their Sum 0,875 will be the Decimal Parts of a Pound equivalent to 17s. 6d. as was required, &c.

Example in Time.

Let it be required to find the Decimal Parts of a Year equal to 126 Days.

First, because 365 Days do make a common Year, therefore 126 Days is $\frac{126}{365}$ Parts of a Year.

Then 365) 126,0000 (0,3452 will be the Decimal Parts of a Year equivalent to 126 Days, &c.

The like is to be understood in *Reducing*, or *Changing* the known Parts of any proposed Integer into Decimal Parts, equal to those known Parts; or at least so near the Truth, as it may be thought necessary to approach.

Now the contrary Operations to these, *viz.* to find the *Value* of any given Decimal Parts of *Coin*, or *Time*, &c. is only the *Converse* of the Work above, and may be performed by this following Rule.

Rule. { *Multiply the given Decimal Parts with such a Number of Units as are contained in the Integer to which the given Decimal does belong, and the Product will be the Value of the said Decimal.*

Examples.

1. Let 25,7875 signify 25*l.* and the Decimal Parts of 1*l.* How many Shillings and Pence do these Decimal Parts denote? That is, How many Shillings and Pence are equal to 0,7875 Decimal Parts of a Pound *Sterling*?

First, $0,7875$
Multiplied with 20 the Shillings in 1*l*.

Gives 15,7500, { or 15 Shillings, and 0,7500
Decimals of 1s.

Multiply the Decimals 0,7500 with 12 the Pence in 15.

150
75

Gives Pence 9,0000 Hence the Answer is,

That 25*l.* 1*5s.* 9*d.* is the same with 25,7875 Decimals of a Pound.

2. Suppose the same 25,7875 to signify 25 Years, and the Decimal Parts of one Year. How many *Days, Hours, &c.* would those Decimal Parts denote?

Answer, 25 Years, 287 Days, 10 Hours, and 30 Minutes.

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Thus,

Thus, $25,7875$
Multiplied with 365 the Days in one Year,

39375
 47250
 23625

Gives Days $287,4375$ and Decimal Parts of 1 Day,
Multiply with 24 the Hours in 1 Day,

17500
 8750

Gives Hours $10,5000$ and 0,5 parts of 1 Hour,
Multiply with 60 the Minutes in 1 Hour,

Minutes $30,0000$

And thus may any proposed Number of Decimal Parts be *Reduced* or *Changed* into the known Parts of what they represent, *viz.* whether they be Parts of *Coin, Weight, Measures, or Time, &c.* a due Regard being had to the Number of Units which are contained in the Denomination of the Thing to which the Decimal Parts belong.

CHAP. II.

The *Definition* and *Use* of *Logarithms* in *General*.

SECT. I. *Definitions*.

LOGARITHMS are a Sort of Artificial Numbers, so adapted to correspond with Natural Numbers, that the *Addition* and *Subtraction* of them do exactly answer to the *Multiplication* and *Division* of those Natural Numbers they are adapted to.

That is, If any two given Numbers are either to be *Multiplied*, or *Divided*, the Logarithms of those Numbers being accordingly *Added*, or *Subtracted*, their *Sum*, or *Difference*, will be the Logarithm of that Natural Number, which is the Product, or Quotient, of such *Multiplication*, or *Division*.

And that the Value of any Product, or Quotient, so found by the Sum, or Difference, of two Logarithms may be truly known, every Logarithm of the Prime Numbers, *viz.* of 1, 10, 100, 1000, &c. in Whole Numbers; and of

0,1,

0,1, 0,01, 0,001, 0,0001, &c. in Decimal Parts, is the *Index* or *Characteristick* to all the intermediate Logarithms which are between them: As in this

T A B L E.

Whole Numbers.	Logarithms.	Decimal Parts.	Logarithms.
1	0.000000	1	0.000000
10	1.000000	0,1	9.000000
100	2.000000	0,01	8.000000
1000	3.000000	0,001	7.000000
10000	4.000000	0,0001	6.000000
&c.	&c.	&c.	&c.

This Table shows some of the principal *Logarithms*, or *Characteristicks*, both of Whole Numbers, and of Decimal Parts, with so many Cyphers annexed to each of them (*like Decimal Parts*) as the Number of Places are in the following *Table of Logarithms*: From whence it will be easy to perceive, That,

If 0 be made the Logarithm of 1. And 1 be made the Logarithm of 10. Then all the intermediate Logarithms, which belong to the Natural Numbers, between 1 and 10, (*viz. the Logarithms of 2, 3, 4, 5, 6, &c.*) must needs be Decimals less than 1. And, for the same Reason, That,

If 1 be the Logarithm of 10. And 2 be made the Logarithm of 100. Then all the intermediate Logarithms, which belong to the Natural Numbers, between 10 and 100, will be a 1, with Decimal Parts annexed to it: And consequently all the Logarithms of the Natural Numbers, between 100 and 1000, will be a 2, with Decimal Parts annexed to it; and so on for higher Numbers, as in the Table.

Now all these principal Logarithms, *viz. 0, 1, 2, 3, 4, &c.* of Whole Numbers, are called *Affirmative Indices* or *Characteristicks*.

And those which belong to Decimal Parts or *Fractions*, *viz. 9, 8, 7, 6, &c.* are called *Negative Indices* or *Characteristicks*; being distinguished from the Affirmative Indices by the Negative Sign — set over their Heads.

For, If 0 be the Logarithm of 1 (*as above*). Then it follows, that all the Logarithms of Fractions (*viz. of Numbers less than 1,*) must needs be less than 0. That is, they must be *Negative Numbers*.

From what hath been here said, it will be easy to deduce the General Rule, by which the Distance of the Natural Numbers from the Unit's Place is always known.

Viz. { That every Index or Characteristick is less by an Unit, or 1, than the Number of Places of Figures in the Natural Numbers to which it belongs.

As, for instance, in the following *Numbers*, wherein I suppose 5381 to be a Whole Number; whose Logarithm is 3.730863. Now, if this Natural Number 5381 be otherwise taken or varied in it's Places, then the *Logarithms* will stand thus :

<i>Natural Numbers.</i>	<i>Logarithms.</i>
5381 —————	3.730863
538,1 —————	2.730863
53.81 —————	1.730863
5.381 —————	0.730863
0,5381 —————	9.730863
0,05381 —————	8.730863
0,005381 —————	7.730863 &c.

In these Examples, you see that the Logarithms are all the same, save only the *Indices* or *Characteristicks* are altered according to the Distance of the first Figure of the Natural Number from the Unit's Place; which being once well understood, it will be easy to find the Logarithm of any given Number in the *Table of Logarithms*, and to prefix it's proper *Index* to it.

Or, if any Logarithm, with it's Index, be given; to find it's correspondent Number, so far as the *Table of Logarithms* extends; and, upon occasion, to one or two Places further, with a very small trouble.

SECT. II. To find the *Logarithm* of any given *Number*.

THE first Page of the annexed *Tables of Logarithms* contains all the Natural Numbers, in their proper Order, from 1 to 100. And against every one of those Numbers is placed it's *Logarithm*, with it's *Index* or *Characteristick* before it.

Thus, against the Number 28 is it's *Log.* 1.447158. And against the Number 89 is it's *Log.* 1.949390. And so on for the rest.

In the first Column of all the following Pages (*under Num.*) the Natural Numbers proceed, in their due Order, from 100 to 1000. And, in the next Column (*under 0*), against every one of those Numbers, is the Decimal Part of it's Logarithm, without any Index; to which you must prefix it's proper Index, according as the Natural Number you make use of requires.

As, for instance; against the Number 856 (*under 0*) is 932474. To which, if 2, the Index of 856, be prefixed, it will become 2.932474, the completed Logarithm of 856.

The other nine Columns of each Page, after the first Page, contain the Logarithms of all the Numbers from 1000 to 10000. These Columns are distinguished

distinguished on their Tops with the Figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. So that, to find the Logarithm of any Number between 1000 and 10000; as, suppose of 5468, you must look for the Three first Figures, viz. 546, in the first Column under *Num.*, and for the fourth, or last, Figure, viz. 8, you must look at the Top of the Page: Then, in the Column under the last Figure 8, and right over-against the Three first Figures 546, there is 737829. To which, if 3, the Index of 5468, be prefixed, you will then have 3.737829, the completed Logarithm of 5468, as was required. And so for any other Logarithm of any proposed Number not exceeding 10000.

But if the proposed Number be above 10000 (*which is the Limit of the annexed Table.*) Then the Logarithm of that Number must be found by help of the Common Difference of the Logarithms, which is in the last Column of every Page under *Diff.*

Thus, $\left\{ \begin{array}{l} \text{Find the Logarithm of the first Four Figures of the given Number,} \\ \text{without it's Index (as above), and Multiply the Common Difference} \\ \text{which stands against that Logarithm (under Diff.) with the other Figures} \\ \text{of the given Number, Casting off so many Figures of that Product as} \\ \text{there are in the Multiplier.} \\ \text{Then Add the Remaining Figures of that Product, to the Logarithm of} \\ \text{the first Four Figures, and to their Sum prefix the proper Index, and} \\ \text{you will have the completed Logarithm required.} \end{array} \right.$

Example.

Suppose it were required to find the Logarithm of 698476. First, the Logarithm of 6984 is found in the Table (*as above*) to be 844104, and against it, under *Diff.* is 62. This 62, being multiplied with 76 (*the other two Figures of the given Number*), produces 4712. Cut off the 12 (*viz. the Two last Figures*), and then Add the 47 to the Logarithm last found, and the Sum will be 844151; to which prefixing 5, the proper Index of the given Number 698476, it will be 5.844151, the Logarithm of 698476, as was required.

SECT. III. To find the Number to any given Logarithm.

OMIT the Index or Characteristick of the given Logarithm, and then seek it in the Table of Logarithms; and if it can be exactly found there, then the Number in the first Column (*under Num.*), with that on the Top over the Logarithm, will be the Number required. But if the given Logarithm (*without it's Index*) cannot be exactly found in the Table, then the proper Number agreeing to that Logarithm, may be found by the help of the Common Difference of the Logarithms.

Thus,

Thus, { *From the given Logarithm Subtract the next Less, and to the Remainder Annex Cyphers; and then Divide it by the Common Difference found against the next less Logarithm, under Diff., and the Quotient will be a Number that must be Annexed to the Number already found against the next less Logarithm, according as the Index of the given Logarithm denotes.*

Example.

Suppose 5.660279 were a given Logarithm, and it were required to find the natural Number answering to it.

Here the Number sought must consist of Six Places of Figures in Whole Numbers, as appears by it's Index 5. Which being omitted, I seek in the Table of Logarithms for 660279; but not finding it exactly there, I take the next Less to it, *viz.* 660201, which stands under 3, and against 457. Therefore I conclude, that the first Four Figures of the Number sought must be 4573; and the Common Difference found against 660201, under *Diff.*, is 95.

Then, from the given Logarithm 660279
I Subtract the next Less, *viz.* 660201

And there remains 78

To which Annexing Two Cyphers, (*because there is yet wanting two Places of Figures,*) it will become 7800. The which being Divided by the Common Difference 95, the Quotient will be 82. Thus, 95) 7800 (82, which must be annexed to 4573 (*found before*), and the Sum will be 457382, the Number that answers to the given Logarithm 5.660279, as was required.

Thus one may, without much Trouble, find the Logarithm of any given Number (*very near*), although it exceed the Limits of the Table by 1, 2, or 3 Places of Figures; and also the Number agreeing to any given Logarithm, without the help of such a Table of *Proportional Parts* as is usually inserted along with the Table of Logarithms for that Purpose.

SECT. IV. To perform *Multiplication by Logarithms.*

THE Multiplication of any Two given Numbers together, may be performed by Logarithms.

Thus, { *Add the respective Logarithms of the given Numbers together, and their Sum will be the Logarithm of the Product required; due regard being had to the true ordering of their Indices; which admits of Three Cases.*

Case 1.

Case 1. If the Indices are both Affirmative, their Sum, with what arises from the Addition of their Logarithms, will be an Affirmative Index.

Examples.

	Numbers.	Logarithms.	
Multiplicand	7564	3.878752	} Add
Multiplier	75	1.875061	
Product	567300	5.753813	

Again,

Multiplicand	75,64	1.878752	} Add
Multiplier	7,5	0.875061	
Product	567,300	2.753813	

Case 2. If one of the Indices be Affirmative, and the other be Negative; and if their Sum be above 10. Then Cast off 10, and the Remainder will be an Affirmative Index: But if their Sum be Less than 10, it will be a Negative Index.

Examples.

	Numbers.	Logarithms.	
Multiplicand	75,64	1.878752	} Add
Multiplier	0,75	9.875061	
Product	56,7300	1.753813	

Again,

Multiplicand	75,64	1.878752	} Add
Multiplier	0,0075	7.875061	
Product	0,5673	9.753813	

Case 3. If both the Indices are Negative, and their Sum be above 10. Then Cast off 10, and the Remainder will be a Negative Index. If their Sum be just 10, Add 1 to it. If it be under 10, Add 10 to it, and that Sum will be a Negative Index.

The Two Last Parts of this *Case* seldom come into Practice. I shall therefore only give an Example of the First.

Example.

Example.

	<i>Numbers.</i>	<i>Logarithms.</i>	
<i>Multiplicand</i>	0,0347	8.540329	} Add
<i>Multiplier</i>	0,0236	8.372912	
<i>Product</i>	0,00081892	6.913241	

SECT. V. *Divison performed by Logarithms.*

To divide one Number by another, is only the Converse of the Last Work, and is performed by Logarithms.

Thus, { *Subtract the Logarithm of the Divisor from the Logarithm of the Dividend, and the Remainder will be the Logarithm of the Quotient; with this Consideration, That*

When a Lesser Number Divides a Greater, the Index of the Quotient-Logarithm will be Affirmative: But if a Greater Number divides a Lesser, then the Index of the Quotient-Logarithm will be Negative, and there must be 10 Added to the Index of the Logarithm of the Dividend, when Subtraction cannot be made without it.

Examples.

	<i>Numbers.</i>	<i>Logarithms.</i>	
<i>Dividend</i>	567300	5.753813	} Subtract
<i>Divisor</i>	7564	3.878752	
<i>Quotient</i>	75	1.875061	

Again,

<i>Dividend</i>	68,6	1.836324	} Subtract
<i>Divisor</i>	78,4	2.894316	
<i>Quotient</i>	0,0872	8.942008	

Again,

	<i>Numbers.</i>	<i>Logarithms.</i>	
<i>Dividend</i>	56,73	1.753813	} Subtract
<i>Divisor</i>	0,75	9.875061	
<i>Quotient</i>	79,64	1.878752	

SECT.

SECT. VI. To *Extract* the *Square* and *Cube-Roots*, &c. by *Logarithms*.

If the Logarithm of any given Number be Divided by 2, the Quotient will be the Logarithm of the *Square-Root* of that Number. And if the Logarithm of any Number be Divided by 3, the Quotient will be the Logarithm of the *Cube-Root* of that Number. And so on for higher Powers.

Examples.

Suppose it were required to Extract the *Square-Root* of 6968, whose Logarithm is 3.843108.

Then 2) 3.843108 (1.921554. This Quotient is the Logarithm of 83.4745, &c., which is the *Square-Root* of 6968, as was required.

Again; Let it be required to Extract the *Cube-Root* of 6968, whose Logarithm is 3.843108, as before.

Then 3) 3.843108 (1.281036. This Quotient is the Logarithm of 19.1, &c. which is the *Cube-Root* of 6968, as was required.

And, in the same Manner, the *Biquadrate Root*, or that of the Fourth Power, may be found, if the Logarithm of the given Number be Divided by 4. And if the Logarithm of any given Number be Divided by 5, the Quotient will be the Logarithm of the *Surfsolid Root*, or that of the 5th Power, and so on for the 6th, 7th, 8th, or any proposed Root of a single Power, how high soever it be, provided the Index of the Logarithm of the given Number be Affirmative. But,

The Dividing of Logarithms, whose Indices are Negative, (by 2, 3, 4, 5, &c.) and to determine the true Quotient-Index hath been thought so difficult a Work, that the *Learned* and *Ingenious* Mr. *Oughtred* contrived a Table on Purpose to perform it. *Vide Key of the Mathematicks*, Page 168.

Now, according to the Negative Indices I have here made Use of, that Work may be very easily performed.

Thus, $\left\{ \begin{array}{l} \text{If you are to Divide a Logarithm that hath a Negative Index by 2,} \\ \text{Add } \overline{10} \text{ to the given Index: If by 3, then Add } \overline{20} \text{ to the given Index:} \\ \text{If by 4, then Add } \overline{30} \text{ to the given Index, \&c.} \end{array} \right.$

That is, Still increasing the Index of the given Logarithm by $\overline{10}$, as the Divisor doth increase by an Unit or 1.

Examples.

Suppose it were required to Extract the *Square-Root* of 0.054756, whose Logarithm is $\overline{8}.738432$.

Then 2) $\overline{18.738432}$ ($\overline{9.369216}$). This Quotient is the Logarithm of 0,234, which is the *Square-Root* of 0,054756, as was required.

Again, Let it be required to Extract the *Cube-Root* of 0,054756, whose Logarithm is $\overline{8.738432}$, as before.

Then 3) $\overline{28.738432}$ ($\overline{9.579477}$). This is the Logarithm of 0,3797, &c. which is the *Cube-Root* of 0,054756, as was required.

Once more; Suppose it were required to Extract the Second *Sur-solid Root* or that of the Seventh Power, out of 0,054756, whose Logarithm is $\overline{8.738432}$, as above.

Then 7) $\overline{68.738432}$ ($\overline{9.819776}$). This is the Logarithm of 0,66035, &c. which is the *Root* required, &c. *

Thus far may suffice concerning the Nature and Use of the Table of Logarithms in general; which being a little considered of, it will be easy to Apply them to the following *Calculations*.

CHAP.

* The Author ought to have given a demonstration of this rule for dividing Logarithms of which the indexes, or characteristicks are negative, and the decimal parts are affirmative, in order to find the square-root, or the cube-root, or the seventh root, or any other root, of a fraction. But these roots may be found without supposing any Logarithms to have negative numbers for their indexes, or characteristicks, by proceeding in the manner following.

If it be required to find the square-root of the decimal fraction 0,054,756, let this fraction be multiplied by 100, or the square of 10. And the product will be the number 5,4756; which is greater than 1, and therefore has a logarithm that is altogether affirmative, and is not composed of a negative whole number in it's index, and an affirmative decimal fraction. This logarithm will be 0,738,432. Therefore the logarithm of the square-root of 5,4756 will be $= \frac{0,738,432}{2} = 0,369,216$, which is the logarithm of the number 2,34. Therefore the square-root of 5,4756 is 2,34. But the square-root of 5,4756, or of $100 \times 0,054,756$, is equal to 10 times the square-root of the fraction 0,054,756. Therefore 2,34 will be equal to 10 times the said square-root; and consequently the said square-root will be $= \frac{2,34}{10} = 0,234$. Q. E. I.

Secondly, if it be required to extract the cube-root of the same fraction 0,054,756, let the said fraction be multiplied by 1000, or the cube of 10. And the product will be the number 54,756, which is greater than 1, and therefore has an affirmative logarithm. This logarithm will be 1,738,432. Therefore the logarithm of the cube-root of 54,756 will be $= \frac{1,738,432}{3} = 0,579,477$, which is the logarithm of the number 3,797. Therefore the cube-root of 54,756 is 3,797. But, since 54,756 is $= 1000 \times 0,054,756$, it's cube-root will be $= 10$ times the cube-root of 0,054,756. Therefore 3,797 is $= 10$ times $\sqrt[3]{0,054,756}$; and consequently $\sqrt[3]{0,054,756}$ is $= \frac{3,797}{10} = 0,3797$. Q. E. I.

Thirdly, if it be required to extract the 7th root of the same fraction 0,054,756, let the said fraction be multiplied by 10,000,000, or the 7th power of 10. And the product will be 547,560, which is a whole number, and therefore has an affirmative logarithm. This logarithm will be 5,738,432.

C H A P. III.

The Calculation of *Questions* in *Simple Interest* and *Annuities*; performed both by the *Pen*, and by *Logarithms*.

BEFORE we proceed to the following Computations, it may be convenient to premise a few useful Things that will help to Explain and Shorten the Work. And, first, Of a few common *Characters*; *Viz.*

- + Signifies *More*, or the Affirmative Sign of *Addition*.
- Signifies *Less*, or the Negative Sign of *Subtraction*.
- × Signifies *Into* or *With*, and is the Sign of *Multiplication*.
- = Signifies *Equal* to, or the Sign of *Equality*.

Log. Denotes the Completed *Logarithm* of any Number.

And when the *Ratio* of the *Rate* of *Interest* is mentioned, it signifies only the Simple Interest of 1*l.* for one Year, at any proposed Rate of Interest *per Cent.* which may be thus found by the *Rule of Three*.

As 100 : Is to 6 :: So is 1 : To 0,06, the *Ratio* of the Rate of 6 *per Cent. per Annum*.

Or, As 100 : Is to 7 :: So is 1 : To 0,07, the *Ratio* of the Rate of 7 *per Cent. per Annum, &c.*

The which may also be found by *Logarithms*.

5.738,432. Therefore the logarithm of the 7th root of the number 547,560 will be = $\frac{5.738,432}{7}$ = 0.819,776, which is the logarithm of the number 6.6035 &c. Therefore the 7th root of 547,560 is = 6.6035 &c. But, since 547,560 is = 10,000,000 × 0.054,756 = $10^7 \times 0.054,756$, it's seventh root will be = 10 times the 7th root of the fraction 0.054,756. Therefore 6.6035 &c will be = 10 × $\sqrt[7]{0.054,756}$, and consequently $\sqrt[7]{0.054,756}$ will be = $\frac{6.6035 \text{ \&c}}{10}$ = 0.66035 &c. Q. E. I.

And, in like manner, if it be required to find the *m*th root of the same fraction 0.054,756, (*m* being any whole number whatsoever,) let the said fraction be multiplied by 10^m , and let the logarithm of the product $10^m \times 0.054,756$ be found. Then divide this logarithm by *m*: and the quotient will be the logarithm of the *m*th root of $10^m \times 0.054,756$; and the number corresponding to this logarithm will be the said *m*th root of $10^m \times 0.054,756$. This number will be = 10 × $\sqrt[m]{0.054,756}$; and consequently the 10th part of it will be = $\sqrt[m]{0.054,756}$. Q. E. I.

And thus we may avoid all the mysteries of negative logarithms and arithmetical complements.

F. M.

Thus, $\left\{ \begin{array}{l} \text{From the Logarithm of the given Rate of Interest, Subtract the} \\ \text{Logarithm of 100 (viz. 2.000000), and the Remainder will be the} \\ \text{Logarithm of the Ratio of that Rate.} \end{array} \right.$

Example.

Suppose the given Rate of Interest to be that of 6 per Cent. per Annum :
Then

The Logarithm of 6 is 0.778151 } Subtract
The Logarithm of 100 is 2.000000

The Remainder is the Log. $\overline{8.778151}$ of 0.06.

That is, 0.06 is the Ratio of 6 per Cent. &c.

And thus may the Ratio of any other proposed Rate of Interest per Cent. be easily obtained.

But, because it is the Logarithms of those Ratios, that are of Use in the following Calculations relating to Simple Interest, I have here annexed a small Table of several Rates of Interest, with their Ratios, and the Logarithms of those Ratios.

Rates of Interest per Cent.	Ratios of those Rates.	Logarithms of those Ratios.	Rates of Interest per Cent.	Ratios of those Rates.	Logarithms of those Ratios.
3	0,03	$\overline{8.477121}$	7	0,07	$\overline{8.845098}$
4	0,04	$\overline{8.602060}$	$7\frac{1}{2}$	0,075	$\overline{8.875061}$
$4\frac{1}{2}$	0,045	$\overline{8.653212}$	8	0,08	$\overline{8.903090}$
5	0,05	$\overline{8.698970}$	$8\frac{1}{2}$	0,085	$\overline{8.929419}$
$5\frac{1}{2}$	0,055	$\overline{8.740363}$	9	0,09	$\overline{8.954242}$
6	0,06	$\overline{8.778151}$	$9\frac{1}{2}$	0,095	$\overline{8.977724}$
$6\frac{1}{2}$	0,065	$\overline{8.812913}$	10	0,1	$\overline{9.000000}$

These Things being premised, we may proceed to the following Work ; and, first, of Money forborn at any Rate of Simple Interest, &c.

SECT. I. Of Simple Interest.

IN the *Young Mathematician's Guide*, Page 245, (from whence the following Rules are deduced,) I have made Use of Letters to denote the several Parts of the Question,

Viz.

Viz. $\left\{ \begin{array}{l} P \text{ Signifies any Principal or Sum put to Interest.} \\ T \text{ The Time of it's continuing at Interest.} \\ R \text{ The Ratio of the Rate of Interest per Cent.} \\ A \text{ The Amount of the Principal and it's Interest.} \end{array} \right.$

Any three of these Parts being given, the other may be found by Help of this General

$$\text{Theorem, } TRP + P = A.$$

This Theorem admits of four Cases, or Variety of Questions.

Case 1. If P , T , and R , are given, thence to find A . That is, If any Principal, with the Time of it's being at Interest, and the Rate of Interest per Cent. per Annum are given, To find the Interest, and the Amount.

This Question I take to be of the most general Use of any that Occurs in the whole Business of Simple Interest; and may be performed thus:

First, by Common *Arithmetick*.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Principal, the Time, and the Ratio of the Rate, all} \\ \text{three together; and their Product will be the Interest: To which Add} \\ \text{the Principal, and the Sum will be the Amount required.} \end{array} \right.$

Example.

What Sum will 567*l.* 10*s.* Amount to in Nine Years, at the Rate of 6 per Cent. per Annum?

Here is given $P = 567,5$, $T = 9$, and $R = 0,06$; to find A ; which, by the Rule, is done thus:

$$\begin{array}{rcl} \text{First, } P = 567,5 & \} & \text{Multiply} \\ \text{And } T = 9 & \} & \\ \hline \text{Product } 5107,5 & & \end{array} \quad \begin{array}{rcl} \text{Again, } 5107,5 & \} & \text{Multiply} \\ R = 0,06 & \} & \\ \hline 306,45 & = & 306*l.* 9*s.* \end{array}$$

That is, 306*l.* 9*s.* is the Interest of 567*l.* 10*s.* for Nine Years: Then 306*l.* 9*s.* + 567*l.* 10*s.* = 873*l.* 19*s.* = A , the Amount required.

The same performed by *Logarithms*.

Thus, $\left\{ \begin{array}{l} \text{To the Log. of the Principal, Add the Log. of the Time, and the} \\ \text{Log. of the Ratio; their Sum will be the Log. of the Interest; to which} \\ \text{Add the Principal, \&c. as above.} \end{array} \right.$

That is, in the same *Example*.

$$\begin{array}{rcl} \text{Thus, } P = 567,5 & \text{it's Log. is } 2,753966 & \\ \text{And } T = 9 & \text{it's Log. is } 0,954242 & \\ R = 0,06 & \text{it's Log. is } 8,778151 & \\ \hline & & \text{Add} \end{array}$$

The Sum is the Logarithm 2.486359 of 306,45.

That:

That is, 306*l.* 9*s.* is the Interest, as before, to which the Princip. 567*l.* 10*s.* being Added,

the Sum is 873*l.* 19*s.* = *A*, the Amount required, as above.

Case 2. When *A*, *T*, and *R*, are given; to find *P*. That is, To find what Principal or Sum, being put to Interest any assigned Time, will amount to a proposed Sum in that Time, at any given Rate of Interest *per Cent. per Annum, &c.*

First, by the *Pen* only.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Time with the Ratio of the Rate, and to their Product} \\ \text{Add 1. Then Divide the proposed Amount by that Sum, and the Quotient} \\ \text{will shew the Principal required.} \end{array} \right.$

Example.

*What Principal or Sum of Money, being put to Interest for Nine Years, will Amount to (or Raise a Stock of) 873*l.* 19*s.* at 6 per Cent, &c.?*

Or thus; Suppose a Debt of 873*l.* 19*s.* were not to be paid until Nine Years hence; What would it be worth in Ready Money; the Creditor allowing the Rate of 6 per Cent. Discount to the Debtor?

In this Question there is given, $A = 873,95$, $T = 9$, and $R = 0,06$; to find *P*.

Thus, $\left. \begin{array}{l} T = 9 \\ R = 0,06 \end{array} \right\} \text{Multiply} \quad \begin{array}{l} \text{Then } 0,54 + 1 = 1,54 \\ \text{And } 1,54) 873,95 \text{ (} 567,5 = P. \end{array}$

Product 0,54

That is, 567,5 = 567*l.* 10*s.* is the Principal (or Ready Money) required.

The same may be performed by *Logarithms*.

Thus, $\left\{ \begin{array}{l} \text{Add the Log of the Time to the Log. of the Ratio of the Rate, and} \\ \text{to the Number found by their Sum Add 1. Then if the Log. of that} \\ \text{Sum be Subtracted from the Log. of the proposed Amount, there will} \\ \text{Remain the Log. of the Principal required.} \end{array} \right.$

That is, in the same Example,

Thus, $\left. \begin{array}{l} T = 9 \text{ it's Log. is } 0,954242 \\ \text{And } R = 0,06 \text{ it's Log. is } 8,778151 \end{array} \right\} \text{Add}$

The Sum is this Logarithm, $\overline{9,732393}$ of 0,45. To which Adding 1, it will become 1,45.

Then

Then $A = 873,95$; it's Log. is 2.941487 } Subtract
 And $1,54$; it's Log. is 0.187521 }

Remains the Logarithm 2.753966 of $567,5 = P$.

That is, $567\frac{1}{2}$ l. 10s. is the Principal required by the Question, as above.

Case 3. Suppose P , T , and A , were given; to find R , viz. Having any Principal, with the Time of it's being at Interest, and the Sum it is proposed to Raise, or Amount to, in that Time, given; thence to find the Rate of Interest *per Cent. per Annum*.

First, by the *Pen* only.

Rule. { If the Difference between the proposed Amount and the Principal, be
 Divided by the Product of the Principal Multiplied into the Time, the
 Quotient will shew the Ratio of the Rate of Interest required.

Example.

At what Rate of Interest per Cent. will $567\frac{1}{2}$ l. 10s. Raise a Stock, or Amount to $873\frac{1}{2}$ l. 19s. in Nine Years' Time?

Here is given $P = 567,5$, $T = 9$, and $A = 873,95$; to find R ; which, by the Rule, is

Thus, $873,95 - 567,5 = 306,45$, the Dividend.

And $567,5 \times 9 = 5107,5$, the Divisor.

Then $5107,5 \div 306,45 = 0,06 = R$, the Ratio of the Rate of Interest.

And As $1 : 100 :: 0,06 : 6$, So is $100 : 6$, the Rate of Interest *per Cent.* &c. as was required.

Or, by Logarithms,

Thus, { From the proposed Amount Subtract the Principal (as above); then
 from the Log. of the Remainder Subtract the Sum of the Logarithms of
 the Principal and the Time: and there will Remain the Log. of the Ratio
 of the Rate, &c.

That is, in the same Example,

$873,95 - 567,5 = 306,45$; it's Log. 2.486359

And $P = 567,5$; it's Logarithm is 2.753966 } Add
 $T = 9$; it's Logarithm is 0.954242 }

From the first Logarithm Subtract this Log. 3.708208

And there remains the Logarithm $\overline{8}.778151$ of $0,06$.

Which shews the Rate of Interest to be 6 *per Cent. per Annum*, as before by the *Pen*.

Case 4. Having P , R , and A , given; to find T . That is, To find the Time in which any given Principal will amount to, or Raise, any proposed Sum, the Rate of Interest *per Cent.* being also given.

First, by the *Pen* only.

Rule. $\left\{ \begin{array}{l} \text{If the Difference between the proposed Amount and the Principal, be} \\ \text{Divided by the Product of the Principal Multiplied into the Ratio of the} \\ \text{Rate; the Quotient will shew the Time required.} \end{array} \right.$

Example.

In what Time will 567l. 10s. Amount to (or raise a Stock of) 873l. 19s. at 6 per Cent. per Annum?

In this Question, there is given $P = 567,5$, $R = 0,06$, and $A = 873,95$, to find T ; which is done thus:

First, $873,95 - 567,5 = 306,45$, the Dividend.

And $567,5 \times 0,06 = 34,05$, for the Divisor.

Then $34,05 \overline{) 306,45}$ ($9 = T$, the Time; *Viz.* Nine Years will be the Time required.

The same performed by *Logarithms*.

Thus, $\left\{ \begin{array}{l} \text{From the proposed Amount Subtract the Principal (as before); Then} \\ \text{from the Logarithm of the Remainder Subtract the Sum of the Loga-} \\ \text{ritbms of the Principal and Ratio of the Rate; and there will remain} \\ \text{the Logarithm of the Time.} \end{array} \right.$

That is, in the Last Example;

Thus, $873,95 - 567,5 = 306,45$; it's Log. $\underline{2.486359}$

$P = 567,5$; it's Logarithm is $\underline{2.753966}$
 $R = 0,06$; it's Logarithm is $\underline{8.778151}$ } Add

From the first Logarithm Subtract this Log. $\underline{1.532117}$

There remains the Logarithm of $9 = T$, 0.954242

Which shews, that the Time sought is just Nine Years.

If the Work of these Four Examples, and the Rules by which they are performed, be well understood, they will be sufficient (*notwithstanding there is really but One Question, only it is varied according to the several Cases,*) to shew how any Question of the like Kind may be truly Resolved, at any proposed Rate of Simple Interest, and for any assigned Time; especially if the Time given (or sought) does consist of Compleat or Whole Years.

But if the Time given (or sought) does not consist of Whole Years, (as most generally it does not, it being either Less than a Year, or Years, and some Parts of a Year, as *Weeks*, *Months*, or *Quarters*, &c.) Then the odd Time, less than a Compleat Year, must be Reduced (or Converted) into Decimal Parts of a Year (*per Sect. V. p. 652.*) And, unless such Parts of a Year chance to be just $\frac{1}{4}$, $\frac{1}{2}$, or $\frac{3}{4}$ of a Year, then the best Way will be to reduce the Odd Time into Days, and then work with the Decimal Parts of a Year that are equivalent to that Number of Days.

And for the easy and ready finding out of the true Number of Days that are contained between any Two assigned Times less than a Year, and the Decimal Parts of a Year that are equal to those Days, I have here inserted Two small Tables.

The Use of the following Table of Months and Days, is no more but thus: Find the first proposed Month at the Top, or Head, of it's respective Part of the Table; and in the same Column under it, look for the other Month; and by it stands the Number of Days required, according to the Title of the Table.

As, for Example; From the 1st, 5th, 10th, 17th, or any other Day in *April*, To the 1st, 5th, 10th, 17th (*viz. to the same*) Day in *December*, is just 244 Days.

Or, from the 4th, 7th, or 12th, &c. of *October*, To the 4th, 7th, or 12th, &c. of *August*, is just 304 Days. And so the true Number of Days that are between any Two of the same Days in the proposed Months, may be found by Inspection only.

But if the Two given Days of the Month are different, then their Difference must be Added to the Number found in the Table: As, suppose, between the 4th of *October*, and the 25th of *August*. Here, because the 25th of the one Month, is 21 Days more than the 4th Day of the other Month; therefore the Number of Days required will be $304 + 21$; *viz.* 325 Days, and so for any other Two Months and Number of Days in any proposed Time throughout the Year; as in this Table.

A Table that shews, by Inspection only, the true Number of Days, from every Day in any Month, to the same Day in any other Month, throughout the whole Year.

January	February	March	April	May	June
Feb. 31	Mar. 28	Apr. 31	May 30	June 31	July 30
Mar. 59	Apr. 59	May 61	June 61	July 61	Aug. 61
Apr. 90	May 89	June 92	July 91	Aug. 92	Sep. 92
May 120	June 120	July 122	Aug. 122	Sep. 123	Oct. 122
June 151	July 150	Aug. 153	Sep. 153	Oct. 153	Nov. 153
July 181	Aug. 181	Sep. 184	Oct. 183	Nov. 184	Dec. 183
Aug. 212	Sep. 212	Oct. 214	Nov. 214	Dec. 214	Jan. 214
Sep. 243	Oct. 242	Nov. 245	Dec. 244	Jan. 245	Feb. 245
Oct. 273	Nov. 273	Dec. 275	Jan. 275	Feb. 276	Mar. 273
Nov. 304	Dec. 303	Jan. 306	Feb. 306	Mar. 304	Apr. 304
Dec. 334	Jan. 334	Feb. 337	Mar. 334	Apr. 335	May 334
Jan. 365	Feb. 365	Mar. 365	Apr. 365	May 365	June 365

July	August	September	October	November	December
Aug. 31	Sep. 31	Oct. 30	Nov. 31	Dec. 30	Jan. 31
Sep. 62	Oct. 61	Nov. 61	Dec. 61	Jan. 61	Feb. 62
Oct. 92	Nov. 92	Dec. 91	Jan. 92	Feb. 92	Mar. 90
Nov. 123	Dec. 122	Jan. 122	Feb. 123	Mar. 120	Apr. 121
Dec. 153	Jan. 153	Feb. 153	Mar. 151	Apr. 151	May 151
Jan. 184	Feb. 184	Mar. 181	Apr. 182	May 181	June 182
Feb. 215	Mar. 212	Apr. 212	May 212	June 212	July 212
Mar. 243	Apr. 243	May 242	June 243	July 242	Aug. 243
Apr. 274	May 273	June 273	July 273	Aug. 273	Sep. 274
May 304	June 304	July 303	Aug. 304	Sep. 304	Oct. 304
June 335	July 334	Aug. 334	Sep. 335	Oct. 334	Nov. 335
July 365	Aug. 365	Sep. 365	Oct. 365	Nov. 365	Dec. 365

A Table for the ready finding the Decimal Parts of a Year, Equal to any Number of Days, &c.

Days.	Dec. Parts.	Days.	Dec. Parts.	Days.	Dec. Parts.
1	= 0,002740	10	= 0,027397	100	= 0,273973
2	= 0,005479	20	= 0,054794	200	= 0,547945
3	= 0,008219	30	= 0,082192	300	= 0,821918
4	= 0,010959	40	= 0,109589	365	= 1,000000
5	= 0,013699	50	= 0,136986		
6	= 0,016438	60	= 0,164383	$\frac{1}{4}$ of a Year	= 0,25
7	= 0,019178	70	= 0,191781	$\frac{1}{2}$ a Year	= 0,5
8	= 0,021918	80	= 0,219178	$\frac{3}{4}$ of a Year	= 0,75
9	= 0,024657	90	= 0,246575		

The Use of this Table is thus :

If the proposed Number of Days can be exactly found in the Table (under *Days*), their Decimal Parts are also found against them by Inspection only.

But if the true Number of Days cannot be exactly found there, then both they, and their Decimal Parts, must be collected out of the Table at twice, or thrice, according as their Number requires.

As, for Example : Suppose it were required to find the Decimal Parts of a Year equal to 135 Days ?

	<i>Days.</i>	<i>Dec. Parts.</i>	
Then	100	= 0,273973	} Add all these together.
	30	= 0,082192	
	5	= 0,013699	

Hence 135 = 0,369864 the Decimal Parts required.

And thus may the Decimal Parts of a Year, equivalent to any given Number of Days, be very easily found to Six Places of Figures; but, for Common Business, it may suffice to work with only Four of those Places; and in small Sums, 2, or 3 Places (according to Discretion) may be near enough to the Truth.

Or the Decimal Parts, equivalent to any given Number of Days, may be found by the Logarithms.

Thus, $\left\{ \begin{array}{l} \text{From the Log. of the given Number of Days Subtract the Log. of} \\ 365 \text{ Days (viz. 2,562293), and there will remain the Log. of the} \\ \text{Decimal Parts equal to that Number of Days.} \end{array} \right.$

Example.

Let it be required to find the Decimal Parts of a Year equal to 135 Days; as before.

Here is given 135; it's Logarithm is 2.130334 } Subtract
Days in a Year 365; it's Logarithm is 2.562293 }

And there remains the Logarithm $\overline{9.568041}$ of 0,369863, the Decimal Parts equal to 135 Days, &c. as above.

These Things being understood, it will be as easy to Calculate any Question relating to any of the foregoing Cases, when the Time given, or sought, is either Less than a Year, or Years and Parts of a Year; as it is for those in Whole Years only.

As, for Instance, in *Case 1*. Suppose it were required to find *What Sum 780l. 15s. would Amount to in 3 Years and 179 Days, at Five and a Half per Cent.?*

In this Question there is given $P = 780,75$, $R = 0,055$, and $T = 3,4904$, found by the Table.

$$\begin{array}{l} \text{Thus, Three Years} = 3,000000 \\ \text{Days } \left\{ \begin{array}{l} 100 = 0,273973 \\ 70 = 0,191781 \\ 9 = 0,024657 \end{array} \right\} \text{Add} \end{array}$$

Hence 3 Years and 179 Days is $= 3,490411 = T$.

$$\begin{array}{l} \text{Then } P = 780,75 ; \text{ it's Log. } 2,892512 \\ T = 3,4904 ; \text{ it's Log. } 0,542875 \\ R = 0,055 ; \text{ it's Log. } 8,740363 \end{array} \left. \vphantom{\begin{array}{l} P \\ T \\ R \end{array}} \right\} \text{Add}$$

The Sum is the Logarithm $2,175750$ of 149,88.

That is, 149,88 = 149l. 17s. 7d. is the Interest.
To which Add the Principal 780l. 15s. 0d.

And the Sum is 930l. 12s. 7d. the *Amount* required.

Again, That I may make all as plain as I can, take an *Example* in *Case 4*; viz. To find the Time.

Let it be required to find, *In what Time 780l. 15s. would Amount to the Sum of 930l. 12s. 7d. at the Rate of 5½ per Cent. per Annum?*

In this Example there is given $A = 930,63$, $R = 0,055$, and $P = 780,75$; to find T : Which is done thus:

$$930,63 - 780,75 = 149,88 ; \text{ it's Log. } 2,175750$$

$$\begin{array}{l} \text{And } P = 780,75 ; \text{ it's Log. } 2,892512 \\ R = 0,055 ; \text{ it's Log. } 8,740363 \end{array} \left. \vphantom{\begin{array}{l} P \\ R \end{array}} \right\} \text{Add}$$

From the first Logarithm Subtract this Log. $1,632875$

And there remains the Log. $0,542875$ of 3,4904.

That is, 3 Years, and 0,4904 Decimal Parts of a Year. But 0,4904 are = 179 Days, per Rule in Page 653. Consequently 3 Years and 179 Days is the Time required by the Question.

Or the Decimal Parts of a Year may be otherwise Reduced into Days, by the Help of the last Table;

As

As in this Example ; From 0,4904 the given Parts of a Year,
Take the next less Tabular Number 0,2740 = 100 Days.

There remains 0,2164
Again, From the Remainder take 0,1918 = 70 Days.

And there remains this 0,0246 = 9 Days.

So that 0,4904 Parts of a Year are = 179 Days, by the Table, &c. as before.

Or the Decimal Parts of a Year may be Reduced into Days by the Logarithms ;

Thus, $\left\{ \begin{array}{l} \text{To the Logarithm of the proposed Parts Add the Logarithm of 365,} \\ \text{the Days in a Common Year, and the Sum will be the Logarithm of} \\ \text{the Days required.} \end{array} \right.$

For Instance, In the Last Example, wherein

The given Decimal Parts are 0,4904 ; it's Log. $\overline{9.690550}$ } Add
The Days in a Year are 365 ; it's Log. 2.562293 }

The Sum is this Logarithm 2.252843 of 179, &c.

SECT. II. Of Annuities, or Pensions, &c. in Arrears, Computed at Simple Interest.

I SHALL here make Use of the same Letters to denote the several Parts of the Question, as in the afore said *Young Mathematician's Guide*, Page 248.

Viz.

U Denotes the Annuity or Pension, &c. *viz.* either Yearly, Half-Yearly, or Quarterly Rents.

T The Time of it's being unpaid ; *viz.* the Number of all the Payments that are in Arrear.

A The Amount of the Annuity and it's Interest, *viz.* the Sum of all the Arrears due.

R The Ratio of the Rate, *viz.* the Interest of 1*l.* as before.

$$\text{Then will } \left\{ \frac{TTR - TR}{2} = \frac{A - TU}{U} \right.$$

From this Equation are deduced the following Rules, which admit of Four Cases.

Note.

Note. *I do here take it for granted, that the Reader doth by this Time very well know how to perform Multiplication and Division of any given Numbers by Logarithms (as directed in Chap. II. and practised in all the Examples of the Last Section). And therefore I shall, for Brevity's Sake, omit setting-down in Words at Length (as in the Last Section) how the following Cases are to be Resolved by Logarithms; that Work being so easily understood, by having a due Regard to their respective Rules, that it seems to me to be wholly needless to repeat in Words how it is to be done; but it will be sufficient only to set down the Work, at large, of all the Examples, according to the Import of their respective Rules, by which they are Computed with the Pen only.*

Case 1. Having U , T , and R , given; to find A . That is, If any Annuity, or Rent, with the Time of it's being Unpaid, and the Rate of Interest per Cent. be given; thence to find what Sum all those Arrears will Amount to in that Time; allowing any assigned Rate of Simple Interest, for every particular Payment as it becomes due.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Time, the Time less 1, and Half the Ratio of the Rate, all} \\ \text{three together, and to their Product Add the Time; Then Multiply that} \\ \text{Sum with the Annuity: and the Product will shew the Amount required.} \end{array} \right.$

Example.

Suppose 356*l.* Annuity, or Yearly Rent, be forborn, or unpaid, for Nine Years; What Sum will all those Arrears amount to in that Time, allowing 6 per Cent. per Annum Interest for each Payment, as it becomes due?

In this Example there is given $U = 356$, $T = 9$, and $R = 0,06$; to find A .

First, by the Pen only.

Let the Work be prepared thus, $9 - 1 = 8$, and $\frac{1}{2}R = 0,03$.

Then $9 \times 8 = 72$; and $72 \times 0,03 = 2,16$; to which Adding the Time, it will be $2,16 + 9 = 11,16$. And 11,16, Multiplied with 356, is 3972,96 = A .

That is, 3972*l.* 19*s.* 2½*d.* will be the Amount, as was required.

The same performed by Logarithms.

First, $T = 9$; it's Logarithm is 0.954242
And $T - 1 = 8$; it's Logarithm is 0.903090 } Add
Again, $\frac{1}{2}R = 0,03$; it's Logar. is 8.477121

Their Sum is the Logarithm 0.334453 of 2.16.

Then $2,16 + 9 = 11,16$; it's Logarithm is 1.047664 } Add
And $U = 356$; it's Logarithm is 2.551450

The Sum is the Logarithm 3.599114 of 3972,96.

That

That is, 3972*l.* 19*s.* 2½*d.* is = A , the Amount, or Sum, of all the Arrears, as before by the Pen.

Case 2. When A , T , and R , are given; to find U . That is, To find what Annuity, or Yearly Rent, being forborn, or unpaid, for any assigned Time, will Amount to a proposed Sum, allowing any given Rate of Interest *per Cent.* for every Payment as it becomes due.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Time, the Time less 1, and Half the Ratio of the Rate,} \\ \text{all three together; and to their Product Add the Time (as before). Then} \\ \text{Divide the proposed Amount by that Sum: and the Quotient will shew the} \\ \text{Annuity, or Yearly Rent, required.} \end{array} \right.$

Example.

Suppose it were required to find, *What Annuity, or Yearly Rent, will Amount to (or raise a Stock of) 3972*l.* 19*s.* 2½*d.* allowing 6 per Cent. for every Payment, as it becomes due?*

In this Question there is given $A = 3972,96$, $T = 9$, and $R = 0,06$; to find U ; which may be thus done.

First, by the Pen only; Thus,

The Work prepared is $9 - 1 = 8$, and $\frac{1}{2}R = 0,03$. Then $9 \times 8 = 72$, and $72 \times 0,03 = 2,16$; which, being added to the Time, is $9 + 2,16 = 11,16$ for a *Divisor*.

Then 11,16) 3972,96 (356 = U .

Viz. 356*l.* will be the Yearly Rent required.

The same performed by *Logarithms*.

Thus, $T = 9$; its Logarithm is 0.954242
 $T - 1 = 8$; its Logarithm is 0.903090
 And $\frac{1}{2}R = 0,03$; its Logarithm is 8.477121 } Add

The Sum is the Logarithm 0.334453 of 2,16.
 Again, $A = 3972,96$; its Logarithm is 3.599113
 And $9 + 2,16 = 11,16$; its Logarithm is 1.047664 } Subtract.

There remains the Logarithm 2.551449 of 356 = U .

That is, 356*l.* is the Annuity, or Yearly Rent, as before.

Case 3. When U , A , and R , are given; to find T . That is, To find the Time in which any Annuity, or Yearly Rent, being forborn or unpaid, will Amount to any proposed Sum; allowing any given Rate of Interest *per Cent.*, &c. for each Payment, as it becomes due.

Subtract the Ratio of the Rate from 2; then Divide the Remainder by twice the Ratio, and call the Quotient x.
Next, Square that Quotient (viz. Multiply it with itself), and call that Square xx.
Rule. *Then Divide Twice the proposed Amount by the Product of the Annuity Multiplied with the Ratio, and to the Quotient Add the Square Number called xx: Then Extract the Square Root of that Sum, and from that Root Subtract the Number called x, and the Remainder will shew the Time sought.*

Example.

What Time will 356l. Yearly Rent require to raise the Sum of 3972l. 19s. 2^d. at 6 per Cent. &c. for each Payment as it becomes due?

In this Question there is given $U = 356$, $R = 0,06$, and $A = 3972,96$; to find T .

And, first, by the Pen only.

1st, $2 - 0,06 = 1,94$, and $2R = 0,12$ $1,94$ ($16,166 = x$).

Next, $16,166 \times 16,166 = 261,361$, the Number called xx .

Then $2A = 7945,92$, and $356 \times 0,06 = 21,36$.

And $21,36$ $7945,92$ ($372,003$).

Then $372,003 + 261,361 = 633,364$, whose Square Root is $25,166$.
 Lastly, $25,166 - 16,166$ leaves $9 = T$; *Viz.* 9 Years, is the Time required by the Question.

The same found by *Logarithms*.

First, $2 - 0,06 = 1,94$; it's Log. $0,287802$ }
 And $2R = 0,12$; it's Logarithm is $9,079181$ } Subtract

There Remains the Logarithm $1,208621$ of $16,166$, called x .
 Multiplied with 2

The Product is the Logarithm $2,417242$ of $261,361$, called xx .

Again; $2A = 7945,92$; it's Logarithm $3,900145$

And $U = 356$; it's Logarithm is $2,551449$ }
 $R = 0,06$; it's Logarithm is $8,778151$ } Add

From the 1st Logarithm Subtract this Log. $1,329600$

And there Remains the Logarithm $2,570545$ of $372,003$.

Then $372,003 + 261,361 = 633,364$; it's Log. $2,801585$
 The Half of that Logarithm is $1,400792$

Lastly,

Lastly, the Number which belongs to the Half Logarithm is

$$\begin{array}{r} 25,166 \\ \times = 16,166 \\ \hline \end{array}$$

Remains $9,000 = T$; the Number of Years fought, &c.

Case 4. If U , T , and A , are given; To find R . That is, Having any Annuity, or Yearly Rent, with the Time of it's being Unpaid, and the Sum it is proposed to Amount to in that Time given; Thence to find, what Rate of Interest *per Cent.* must be allowed for every Payment, as it becomes due.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Annuity with the Time, and Multiply that Product with} \\ \text{the Time again; And make Half the Difference of those Two Products a} \\ \text{Divisor.} \end{array} \right.$
 $\left\{ \begin{array}{l} \text{Next, Subtract the first Product from the proposed Amount; then} \\ \text{Divide the Remainder by the aforesaid Divisor. And the Quotient will be} \\ \text{the Ratio of the Rate of Interest required.} \end{array} \right.$

Example.

Suppose 356l. Yearly Rent, being forborn, or unpaid, Nine Years, be proposed to Raise the Sum of 3972l. 19s. 2½d. What Rate of Interest *per Cent.* must be allowed for every Payment, as it becomes due?

In this Question there is given $U = 356$, $T = 9$, and $A = 3972,96$; to find R . Which may be thus found.

First, by the *Pen* only.

Thus, $356 \times 9 = 3204$; and $3204 \times 9 = 28836$.

Next, $28836 - 3204 = 25632$; it's Half is 12816 for the Divisor.

And $3972,96 - 3204 = 768,96$, for the Dividend.

Then $12816 \overline{) 768,96}$ ($0,06 = R$, the Ratio.

And it will be, As 1 : is to $0,06$:: So is 100 : To 6 , the Rate of Interest *per Cent.*, &c. as was required.

The same may be done by *Logarithms*.

Thus, $U = 356$; it's Log. 2.551450
 And $T = 9$; it's Log. 0.954242 } Add these two Logarithms together.

Their Sum will be 3.505692; it's Number is 3204
 To this Sum Add the Logar. 0.954242

Their Sum is 4.459934; it's Number is 28836

The Difference of these Two Numbers is 25632

And the $\frac{1}{2}$ of 25632 is 12816 for the Divisor.

From $A = 3972,96$ take the 1st Number 3204,
 and there remains 768,96; it's Logarithm 2.885904 } Subtract
 and 12816, the Divisor; it's Logarithm 4.107752 }

There Remains the Logarithm $\bar{8}.778152$ of $0,06 = R$.

Then $1 : 0,06 :: 100 : 6$, the Rate of Interest *per Cent*, &c.

Thus you have all the Four Cases relating to *Annuities*, or *Rents*, &c. in Arrears, with their Examples in Yearly Payments. But if the *Annuities*, or *Rents*, are to be paid by Half-Yearly, or Quarterly Payments, as most generally they are,

Then,

1. Instead of the Ratio of the given Rate of Interest, you must take the $\frac{1}{2}$ of that Ratio for Half-Yearly Payments, and the $\frac{1}{4}$ of it for Quarterly Payments, &c.

2. And you must take the $\frac{1}{2}$ of the Yearly Rent for Half-Yearly Payments, and the $\frac{1}{4}$ of it for Quarterly Payments, &c.

3. But, instead of the proposed Number of Years, you must take Twice that Number for Half-Yearly Payments, and Four Times that Number for Quarterly Payments, &c. As in the following Examples.

Examples in Half-Yearly Payments.

Suppose 356l. per Annum Annuity, payable every Half-Year, were forborn, or unpaid, Nine Years; What wou'd all those Arrears Amount to, at the Rate of 6 per Cent. per Annum, &c.?

In this Question there is given $U = 178$, viz. the $\frac{1}{2}$ of 356l.; $R = 0,03$, the $\frac{1}{2}$ of the Ratio of 6 per Cent.; and $T = 18$, viz. 9×2 , the Number of Half-Years in Nine Years: Thence to find A (per Rule at Case 1.)

First, to Prepare the Work, $18 - 1 = 17$; and $\frac{1}{2}R = 0,015$.

Then $18 \times 17 = 306$; and $306 \times 0,015 = 4,59$. To which Add the Time, viz. $4,59 + 18 = 22,59$. And then $22,59 \times 178$ gives $4021,02 = A$, the Amount or Sum of all the Arrears, as was required.

The same Example in Quarterly Payments.

That is, 356l. a Year, to be paid every Quarter, being forborn Nine Years, What Sum will it Amount to, allowing 6 per Cent, &c. for every Payment, as it becomes due?

Now here it will be $U = 89$, viz. the $\frac{1}{4}$ of 356l.; $R = 0,015$, viz. the $\frac{1}{4}$ of 0,06, the Ratio of the given Rate *per Cent*, &c.; and $T = 36$, viz. 9×4 , the Number of Quarters in Nine Years: Thence to find A , as before.

To

To Prepare the Work, $36 - 1 = 35$; and $\frac{1}{4}R = 0,0075$.

Then $36 \times 35 = 1260$; and $1260 \times 0,0075 = 9,45$; to which Add the Time, viz. $9,45 + 36 = 45,45$, and then $45,45 \times 89$ gives $4045,05 = A$, the Amount or Sum of all the Arrears at Quarterly Payments.

By comparing the Work of these Two Examples with that in Page 674, it may be observed, That Half-Yearly Payments are more advantageous than Yearly Payments; and Quarterly are yet more advantageous than Half-Yearly:

For Yearly Payments Amount but to 3972*l.* 19*s.* 2½*d.*

But Half-Yearly Payments Amount to 4021*l.* 00*s.* 3½*d.*

And Quarterly Payments Amount to 4045*l.* 01*s.* 00*d.*

If the Two last Examples be well considered and understood, it will be easy to conceive how to Compute any Question in the other Three Cases, when the Payments are either Half-Yearly, or Quarterly; and therefore I shall omit inserting Examples, and proceed to the next Section.

SECT. III. The Present Worth of Annuities and Pensions, &c. Computed at Simple Interest.

IN this Section I shall make Use of these Letters to denote the several Parts of the Question:

Viz. $\left\{ \begin{array}{l} U \text{ Denotes the Annuity or Rent,} \\ T \text{ The Time of it's Continuance,} \\ R \text{ The Ratio of the Rate of Interest,} \end{array} \right\} \text{ As before.}$

And P Denotes the Present Worth of the Annuity.

$$\text{Then } TTRU - TRU + 2TU = 2P + 2TRP.$$

(Vide the aforelaid Young Mathematician's Guide, Page 251.)

And from this Equation are deduced the following Rules, which also admits of Four Cases, as in the last Section.

Case 1. Having U , T , and R , given; To find P .

That is, Having any Annuity, or Yearly Rent, and the Time of it's Continuance, given; To find the Present Worth of that Annuity, or Lease, &c. allowing any given Rate of Simple Interest *per Cent.* to the Purchaser for his Ready Money.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Time, the Time less 1, and Half the Ratio of the Rate, all} \\ \text{three together, and to their Product Add the Time; Then Multiply that} \\ \text{Sum with the Annuity, (So far the Work is just the same as in the last} \\ \text{Section.) Then Divide that Product by the Product of the Ratio into the} \\ \text{Time, Added to 1, and the Quotient will shew the Present Worth required.} \end{array} \right.$

4 R 2

Example.

Example.

Suppose a Lease of 250l. per Annum, the Rents to be paid Half-Yearly, were to be Lett for 21 Years; What may the Present Worth of that Lease be, at the Rate of 5 per Cent, &c.

In this Question there is given $U = 125$, viz. the $\frac{1}{2}$ of 250; $R = 0,025$, the $\frac{1}{2}$ of 0,05, the Ratio of 5 per Cent.; and $T = 42$ (viz. 21×2), the Number of Half-Years in 21 Years; To find P .

First, the Work Prepared stands thus, $42 - 1 = 41$; and $\frac{1}{2}R = 0,0125$.

Then $42 \times 41 = 1722$; and $1722 \times 0,0125 = 21,525$. Next, $21,525 + 42 = 63,525$; which, Multiplied with 125, the Half-Yearly Payment, is 7940,625 for the Dividend.

Again, $42 \times 0,025 = 1,05$, and $1,05 + 1 = 2,05$, for the Divisor.

Then $2,05) 7940,625$ ($3873,4756 = P$).

Viz. 3873l. 9s. 6d. will be the Present Worth of such a Lease, as was required.

The same performed by *Logarithms*.

First, $T = 42$; it's Logarithm is 1.623249	} Add
And $T - 1 = 41$; it's Logarithm is 1.612784	
$\frac{1}{2}R = 0,0125$; it's Logarithm is <u>8.096910</u>	

The Sum is the Logarithm 1.332943 of 21,525.

And $21,525 + 42 = 63,525$; it's Logarithm is 1.802945	} Add
$U = 125$; it's Logarithm is <u>2.096910</u>	

The Sum is 3.899855, the Logarithm of the Dividend.

Again, $T = 42$; it's Logarithm is 1.623249	} Add
And $R = 0,025$; it's Logarithm is <u>8.397940</u>	

The Sum is the Logarithm 0.021189 of 1,05.

And $1,05 + 1 = 2,05$, for the Divisor.

Then the Logarithm of the Dividend is 3.899855	} Subtract
And the Logarithm of 2,05, the Divisor, is <u>0.311754</u>	

There Remains the Logarithm 3.588101 of 3873,47.

That is, 3873l. 9s. 4 $\frac{1}{2}$ d. is here found to be the present Worth of that Annuity; as before, *Ferè*.

Case 2. When P , T , and R , are given; to find U . Viz. To find what Annuity, or Lease, &c. may be Purchased for any proposed Sum, to continue any assigned Time, and at any given Rate of Interest.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Time, the Ratio of the Rate, and the Present Worth,} \\ \text{(viz. the Purchase Money,) all three together; and to their Product} \\ \text{Add the Present Worth; make that Sum a Dividend.} \\ \text{Then Multiply the Time, the Time less 1, and Half the Ratio} \\ \text{together; and to their Product Add the Time; make that Sum a Di-} \\ \text{visor; the Quotient arising from thence will shew the Annuity, or Rent,} \\ \text{required.} \end{array} \right.$

Example.

What Annuity, or Rent, to be paid Half-Yearly, and to continue 21 Years, may be purchased for the Sum of 3873l. 9s. 6d. at the Rate of 5 per Cent.?

In this Example there is given $P = 3873,475$; $T = 42$, viz. 21×2 , the Half-Years in 21 Years; and $R = 0,025$, the $\frac{1}{2}$ of the Ratio 0,05 of 5 per Cent.; To find U , the Half-Yearly Rent.

The Work prepared is $42 - 1 = 41$; and $\frac{1}{2}R = 0,0125$.

Then $3873,475 \times 42 = 162685,95$, and $162685,95 \times 0,025$ gives 4067,14875; and $4067,14875 + 3873,475 = 7940,62375$, for the Dividend.

Again, $42 \times 41 = 1722$, and $1722 \times 0,0125 = 21,525$, and $21,525 + 42 = 63,525$, for the Divisor.

Then $63,525 \mid 7940,62375$ ($125 = U$, the Half-Yearly Rent; consequently $125 \times 2 = 350$ l. will be the Annuity or Rent per Annum, as was required.

The same done by Logarithms.

First, $P = 3873,475$; it's Logarithm is 3.588101
And $T = 42$; it's Logarithm is 1.623249
 $R = 0,0125$; it's Logarithm is 8.397940 } Add

The Sum is the Logarithm 3.609290 of 4067,15.

And $4067,15 + 3873,475 = 7940,625$, the Dividend.

Again, $T = 42$; it's Logarithm is 1.623249
 $T - 1 = 41$; it's Logarithm is 1.612784
And $\frac{1}{2}R = 0,0125$; it's Logarithm is 8.096910 } Add

The Sum is the Logarithm 1.332943 of 21,525.

And $21,525 + 42 = 63,525$ for the Divisor.

Then the Dividend 7940,625; it's Logarithm is 3.899854
The Divisor 63,525; it's Logarithm is 1.802945 } Subtract

There remains the Logarithm 2.096909 of 125.

Viz. 125l. is the Half-Yearly Rent, &c. as before.

Case 3.

Case 3. When U , P , and R , are given; To find T . That is, when any Annuity, with it's proposed Value or Purchase, and the Rate of Interest *per Cent*, are given; Thence to find how Long the Purchaser ought to enjoy it.

To the Present Worth Add Half the Annuity; then Multiply that Sum with the Ratio, and Subtract the Product from the Annuity; Then Divide the Remainder by the Product of the Annuity Multiplied with the Ratio, and call the Quotient x .
Next, Square that Quotient (viz. Multiply it with itself), and call that Square xx .
Then Divide the Present Worth by the same Divisor, viz. by the Product of the Annuity into the Ratio; and to Twice that Quotient Add the Square Number called xx ; Then Extract the Square Root of their Sum, and from that Root Subtract the Number called x ; The Remainder will shew the Time sought.

Rule.

Example.

Suppose one would Lay out 3873l. 9s. 6d. Ready Money for an Annuity of 250l. per Annum, to be paid by Half-Yearly Payments; How long may he enjoy that Annuity, to be allowed but 5 per Cent. per Annum, Simple Interest, for his Money?

Or thus, which is the same Thing :

In what Time will 250l. per Annum Pay off a Debt of 3873l. 9s. 6d. by Half-Yearly Payments, allowing the Creditor 5 per Cent. Interest for his Money, until the Debt be Discharged?

In this Question there is given $P = 3873,475$; $U = 125$, viz. the Half-Yearly Payments; and $R = 0,025$, viz. the $\frac{1}{40}$ of the Ratio of 5 per Cent. Thence to find T , the Number of Half-Years.

First, $\frac{1}{2}U = 62,5$, and $62,5 + 3873,475 = 3935,975$.

Then $3935,975 \times 0,025 = 98,4$ *ferè*, and $125 - 98,4$, Leaves 26,6 for the first Dividend.

And $125 \times 0,025 = 3,125$, for the Common Divisor.

Then 3,125) 26,600 (8,512, the Number called x .

And $8,512 \times 8,512 = 72,454$, the Number called xx .

Again, the Common Divisor is 3,125.

And 3,125) 3873,475 (1239,512.

And twice 1239,512 is 2479,024
 The Number called xx is 72,454 } Add

And the Square Root of 2551,478 is 50,512.

Lastly,

Lastly, $50,512 - 8,512$ (*the Number called x*) leaves $42 = T$, the Number of Half-Years; consequently the Half of 42, *viz.* 21, will be the Time, or Number of Years, required by the Question.

This Case may also be Resolved by Logarithms; but it is attended with some seeming Difficulties, which requires a due Consideration of the Rule.

As, for Instance, in the Last Example, wherein $P = 3873,475$; $U = 125$; and $R = 0,025$; To find T .

$$\begin{array}{r} 62,5 + 3873,475 = 3935,975; \text{ it's Logarithm } 3.595053 \\ R = 0,025; \text{ it's Logarithm is } 8.397940 \end{array} \left. \vphantom{\begin{array}{r} 62,5 + 3873,475 \\ R = 0,025 \end{array}} \right\} \text{Add}$$

The Sum is 1.992993, which is the Logarithm of 98.4.

Further, $125 - 98.4$ is $= 26,6$, the Dividend. And the Logarithm of this Dividend 26,6 is 1.424,882.

$$\begin{array}{r} \text{Again, } U \text{ is } = 125; \text{ it's Logarithm is } 2.096,910 \\ \text{And } R \text{ is } = 0,025; \text{ it's Logarithm is } 8.397,940 \end{array} \left. \vphantom{\begin{array}{r} U \text{ is } = 125 \\ R \text{ is } = 0,025 \end{array}} \right\} \text{to be Added together.}$$

Their Sum is 0.494,850, which is the Logarithm of the Number 3,125, the Divisor.

This Logarithm 0.494,850 is now to be Subtracted from the Logarithm of the Dividend, to wit, 1.424,882

And the Remainder will be 0.930,032, which is the Logarithm of 8,512.

Therefore 8,512 will be the Quotient of the said Division, or will be the Number called x .

And the Double of the last Logarithm is 1.860064, which is the Logarithm of 72,454, the Number called xx .

$$\begin{array}{r} \text{Again, } P = 3873,475; \text{ it's Logarithm is } 3.588100 \\ \text{And the Logarithm of the Divisor is } 0.494850 \end{array} \left. \vphantom{\begin{array}{r} P = 3873,475 \\ \text{Logarithm of Divisor} \end{array}} \right\} \text{Subtract}$$

There remains the Logarithm 3.093250 of 1239,512.

$$\begin{array}{r} \text{Twice } 1239,512 \text{ is } 2479,024 \\ \text{The Number called } x \text{ } 72,454 \end{array} \left. \vphantom{\begin{array}{r} 1239,512 \\ 72,454 \end{array}} \right\} \text{Add}$$

The Sum is 2551,478; it's Logarithm is 3.406792.

The Half of the last Logarithm is 1.703396;

It's Number is 50,512; from which Subtract the Number called $x = 8,512$

Remains $42,000 = T$, the Number of Half-Years, &c. as before.

Case 4. When U , P , and T , are given; To find R .

That is, when any Annuity, with the Time of it's Continuance, and it's proposed Present Worth for that Time, are given; Thence to find what Rate of Interest *per Cent.* is allowed the Purchaser.

Rule.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Annuity with the Time, and from the Product Subtract} \\ \text{the Present Worth; then make Twice that Remainder a Dividend.} \\ \text{Next, Add the Annuity to Twice the Present Worth, and from their} \\ \text{Sum Subtract the first Product; then Multiply the Remainder with the} \\ \text{Time, and make the Product a Divisor; by which Divide the aforesaid} \\ \text{Dividend; and the Quotient will shew the Ratio of the Rate of Interest} \\ \text{required.} \end{array} \right.$

Example.

Suppose 3873l. 9s. 6d. were given for an Annuity of 250l. per Annum; to be paid Half-Yearly, and to continue 21 Years; What Rate of Interest per Cent, &c. is allowed the Purchaser?

Here is given $P = 3873,475$; $U = 125$, the Half-Yearly Payments; and $T = 42$, the Number of Half-Years in 21 Years.

Then $125 \times 42 = 5250$, the first Product;

And $5250 - 3873,475 = 1376,525$; it's Double is 2753,05, for the Dividend.

Next, $2P = 7746,95$, and $7746,95 + 125 = 7871,95$, and $7871,95 - 5250$ (the first Product) leaves 2621,95.

Again, $2621,95 \times 42 = 110121,9$ for a Divisor.

Then $110121,9 \div 2753,05$ ($0,025 = R$, the Ratio of the Rate per Cent. for Half-Yearly Payments; consequently $0,025 \times 2 = 0,05$ is the Ratio of the Rate per Cent. per Annum. And then it will be,

As 1 : To 0,05 :: So is 100 : To 5, the Rate of Interest, as was required.

The same may be done by *Logarithms*.

Thus, $U = 125$; it's Logarithm is 2.096910 } Add
And $T = 42$; it's Logarithm is 1.623249 }

The Sum is the Logarithm 3.720159 of 5250.

And $5250 - 3873,475 = 1376,525$, the Double whereof is 2753,05; it's Logarithm is 3.439814, to be Reserved.

Next, $2P = 7746,95$, and $7746,95 + 125 = 7871,95$, and $7871,95 - 5250$, the first Product or Number above,

There remains 2621,95; it's Logarithm is 3.418624 } Add
The Time $42 = T$; it's Logarithm is 1.623249 }

Subtract this Logarithm 5.041873 Sum
From the Reserved Logarithm, viz. 3.439814

And there remains the Logarithm 8.397941 of 0,025.

Viz. $0,025 = R$, the Ratio of the Rate per Cent. for Half-Yearly Payments, &c. as above.

These

These Four Examples may be sufficient to shew how any other of the like Kind may be Resolved, either for whole Years, or those of Quarterly Rents, &c.; provided the Purchased Rent, Lease, or Annuity, is to commence immediately.

Of the Present Values of Annuities in Reversion.

But if it be required to find the present Value, or Worth of any Rent, or Annuity, &c. in Reversion; That is, when it is not to be Entered upon until after some Time, or Number of Years, are past.

Then $\left\{ \begin{array}{l} \text{You must first find what the Proposed Annuity, or Rent, \&c. would} \\ \text{be worth for the given Time of it's Continuance, as if it were to be} \\ \text{immediately Entered upon:} \\ \text{And then you must find what Principal, or Sum, being forborn at} \\ \text{Interest during the Time of that Reversion, would Amount to, or Raise,} \\ \text{the aforesaid Value; That Principal will be the Sum which should be} \\ \text{paid for the proposed Annuity in Reversion.} \end{array} \right.$

Example.

There is the Reversion of a Lease of 175l. per Annum, to be Lett for Eleven Years, which are to commence after Nine Years are expired; It is required to find the present Worth of that Lease, allowing the Purchaser 6 per Cent. for his Ready Money?

The first Work in this Question must be to find what 175l. per Annum, to continue 11 Years, is worth in Ready Money, supposing it were to be immediately Entered upon: As in Case 1, Page 680.

That is, here is given $U = 175$; $T = 11$; and $R = 0,06$; To find P , the Present Worth.

The Work Prepared will stand thus, $11 - 1 = 10$, and $\frac{1}{2}R = 0,03$.

Then $11 \times 10 \times 0,03 = 3,3$; and $3,3 + 11 = 14,3$.

Next, $14,3 \times 175 = 2502,5$, for a Dividend.

Again, $11 \times 0,06 = 0,66$, and $0,66 + 1 = 1,66$, the Divisor.

Then $1,66 \overline{) 2502,5}$ ($1507,53 = P$, the Present Worth, if the Lease were to commence immediately; but, because it is not so, Therefore

The next Work must be to find what Principal or Sum, being put out to Interest for Nine Years, at 6 per Cent, &c. will Amount to 1507,53l.; the which will be found (*per Case 2, Page 666.*) to be 978,9162 = 978l. 18s. 3 $\frac{1}{4}$ d. which is the true present Worth of that Lease in Reversion, as was required.

2. Or, if it be required to find what Annuity, &c. in Reversion, may be Purchased for any proposed Sum, and at any given Rate of Interest per Cent, when the Time during which the Annuity is not to be Entered upon, and the Time of it's Continuance, are both given.

Then *The first Work must be to find what the Sum, proposed to be laid out in the Reversion, would Amount to in that Time wherein the Annuity is not to be in Possession, as if it were forborn at Interest during that Time, at any given Rate per Cent. (by Rule 1, Page 665.)*

And the second Work will be to find what Annuity, &c. that Amount will purchase (as in Case 2, Page 680.); and that will be the Answer to the Question.

These Two are the most general and useful Questions that relate to Purchasing Annuities, or Leases in Reversion: Not but that, if there be Occasion, either the Time, or the Rate of Interest, may be found, by a due Application of their respective Rules.

I know that Sir Samuel Moreland, and several other Authors, that Treat upon the Subject of Interest and Annuities, do assert, That neither the Arrears, nor the present Worths of Annuities, can be truly Computed at Simple Interest, but only at Compound Interest; and therefore, say they, all Computations of Annuities, or Leases, &c. at Simple Interest are to be laid aside as useless.

It is true, indeed, that, in Purchasing of Annuities, or Taking of Leases, either to be in present Possession, or in Reversion; it is usual to allow the Purchaser Compound Interest for his Ready Money, notwithstanding it be not lawful to Lett out Money at Interest upon Interest, viz. at Compound Interest.

However, that we may the better judge whether such Computations as are made of Annuities in Arrears at Simple Interest, be used or no: Let us here see and compare how near they will agree with the Ninety-Nine Years' Annuities, settled by the late Acts of Parliament.

A Remark on the late Annuities for 99 Years granted by Act of Parliament.

Those Annuities were Sold at Sixteen Years' Value, or Purchase; That is, 1600*l.* Ready Money Bought 100*l.* per Annum, to continue 99 Years, and so in Proportion for a Greater, or Lesser, Sum.

Now it may be reasonable to suppose, that the Parliament designed, at the Settling of those Annuities at that Value, to allow the Purchaser after the Rate of Six per Cent. per Annum, Simple Interest, for his Ready Money, according to the former Laws relating to Interest: If so, the Interest of 1600*l.* would be 96*l.* per Annum for ever (as by Case 1, Page 665.); consequently there will remain but 4*l.* per Annum for 99 Years in Lieu of the Principal or Purchase-Money: That is, there will be but $99 \times 4*l.* = 396*l.*$ repaid instead of the 1600*l.*, if it be only received without any further consideration.

But, if we Compute what that 4*l.* per Annum would Amount to, if it were forborn in Arrears for 99 Years, allowing 6 per Cent. for every Payment, as it becomes due, &c. (per Case 1, Page 674.) Then will that Amount undoubtedly shew what Sum will be truly repaid for the 1600*l.* first paid for that Annuity.

And,

And, in order to that, there is given $U = 4$, $T = 99$, and $R = 0,06$; the Ratio of the Rate of 6 *per Cent.*; to find A , the Amount.

Which, Prepared for the Work, is $99 - 1 = 98$; and $\frac{1}{2}R = 0,03$.

Then $99 \times 98 = 9702$, and $9702 \times 0,03 = 291,06$; to find A , the Amount.

Next, $291,06 + 99 = 390,06$, and $390,06 \times 4 = 1560,24$, equal A , viz. 1560*l.* 4*s.* 9½*d.* is the Sum that will in effect be received for the 1600*l.* over and above the 96*l.* *per Annum*, Interest.

Now, in this Calculation, I take for granted what I have elsewhere proved; (*Vide Young Mathematician's Guide, Page 248.*) viz. That if any Yearly Rent be forborn, or Let run in Arrears, and any Rate of Simple Interest be allowed for every Payment, as it becomes due; the Amount, or Sum, of all those Payments will be the very same, as if every Year's Rent were actually received, and immediately put out to Interest at the same Rate.

Whence it follows, That although 396*l.* be all the Money that is actually received in the Space of 99 Years, over, or above, the 96*l.* *per Annum*, Interest; yet, according to the Nature of Interest, there is effectually 1560*l.* 4*s.* 9½*d.* received; supposing the Annuity to be paid but Yearly, which is indeed paid Quarterly, and therefore the Sum will be still nearer to 1600*l.*; which plainly shews that the Computations of Annuities, or Rents run in Arrears at Simple Interest, are not to be wholly rejected as useless.

But I cannot pretend to say much in the Behalf of such Computations as are made about the present Worth of Annuities, or Leases, &c. at Simple Interest, nor are they to be relied on in Practice: However, because this Tract should not be wanting in that Part, I thought it convenient to shew (*although but briefly*) how those Computations may be performed.

And I shall, in the next Chapter, handle the Business of Purchasing Annuities, &c. more fully; especially that Part of it which relates to Annuities, or Leases in Reversion.

C H A P. IV.

The Calculation of Questions in Compound Interest, and Annuities; performed by the Help of Logarithms.

COMPOUND INTEREST is that which is produced from any Principal and it's Interest put together, as the Interest of that Principal becomes due:

That is, at every Payment, Or rather at the Times when the Payments become due, there is still created a New Principal, by the Increase of the growing Interest; and therefore it is called *Interest upon Interest*, or *Compound Interest*; which is grounded upon a Series of Geometrical Proportionals continued, as I have plainly demonstrated in the *Young Mathematician's Guide*, Page 253, &c.

In all Computations of this Kind, whether they are about Money forborn at Interest, or those relating to Annuities, &c. we generally make Use of the Amount, or Produce, of One Pound and it's Interest for One Year, instead of the Ratio of the given Rate of Interest; as before in Simple Interest.

And that Amount of 1*l.* is no more but the Ratio of the given Rate Added to a Unit, or 1.

For the Ratio is the Interest of 1*l.* (*Vide Page 663.*) Consequently, if 1*l.* be Added to it, the Sum will be the Amount of 1*l.* (*per Rule at Case 1, Page 665.*) Or,

The Amount of 1*l.* for One Year, at any given Rate of Interest *per Cent*, may be found by this Proportion;

Viz.

As 100 : 106 :: 1 : 1,06, the Amount of 1*l.* at 6 *per Cent*.

Or 100 : 107 :: 1 : 1,07, the Amount of 1*l.* at 7 *per Cent*.

And so on for any other given Rate of Interest.

Or the aforesaid Amounts of 1*l.* may be otherwise found by *Logarithms*;

Thus, $\left\{ \begin{array}{l} \text{Add the proposed Rate of Interest to 100, and from the Logarithm} \\ \text{of that Sum Subtract the Logarithm of 100 (viz. 2.000000), the} \\ \text{Remainder will be the Logarithm of the Amount of 1*l.* at that Rate of} \\ \text{Interest per Cent.} \end{array} \right.$

Example.

Suppose the given Rate of Interest to be that of 6 *per Cent*. per Annum.

Then $100 + 6 = 106$; it's Logarithm is 2.025306 } Subtract
And the Logarithm of 100 is 2.000000

There Remains the Logarithm 0.025306 of 1,06.

Viz. 1,06 is the Amount of 1*l.* at 6 *per Cent*, &c. And so for any other Rate of Interest.

And, because the Logarithms of those Amounts of 1*l.* are of the greatest Use in the following Calculations, I have here annexed a small Table of several Rates of Interest, with the Amounts of 1*l.* *per Annum* at those Rates, and the Logarithms of those Amounts.

Rates

Rates of Interest per Cent. l.	The Amounts of 1l. Viz. R.	Logarithms of those Amounts, Viz. of R.	Rates of Interest per Cent. l.	The Amounts of 1l. Viz. R.	Logarithms of those Amounts, Viz. of R.
3	1,03	0.012,837	7	1,07	0.029,384
4	1,04	0.017,033	7½	1,075	0.031,408
4½	1,045	0.019,116	8	1,08	0.033,424
5	1,05	0.021,189	8½	1,085	0.035,430
5½	1,055	0.023,252	9	1,09	0.037,426
6	1,06	0.025,306	9½	1,095	0.039,414
6½	1,065	0.027,350	10	1,1	0.041,393

These Things being understood, you may proceed to the Resolving of Questions about Money forborn at Compound Interest.

SECT. I. Of Compound Interest, Or Money Lett out at Interest upon Interest.

I SHALL here make Use of the same Letters to denote the several Parts of the Question, as before in Page 665, except *R* only, and that I shall use the small Letter *t*, instead of the great Letter *T*, to denote the Time.

P Denotes any Principal put to Interest.
 Viz. $\begin{cases} t & \text{The Time of it's Continuance at Interest.} \\ A & \text{The Amount, or the Principal and it's Interest.} \end{cases}$

And *R* Denotes the Amount of 1l. for One Year.

That is, the Ratio of the given Rate more 1.

Then it will be, $PR^t = A$.

This Equation admits of the same Four Cases, or Variety of Questions, as that of Simple Interest in Page 665. And here it may not be amiss to make Use of the same Examples, that we may the more readily see the Difference that is between Simple Interest and Compound Interest.

Case 1. If *P*, *t*, and *R*, are given; To find *A*, the Amount.

Rule. $\begin{cases} \text{First Multiply the Logarithm of } R, \text{ the Amount of 1l. at the given} \\ \text{Rate per Cent, with the Time; Then to that Product Add the Loga-} \\ \text{rithm of the proposed Principal: and their Sum will be the Logarithm of} \\ \text{the Amount required.} \end{cases}$

Example.

Example.

What Sum will 567l. 10s. Amount to in Nine Years, at the Rate of 6 per Cent. per Annum?

Hence $R = 1,06$; it's Logarithm is $0.025,306$
 And $t = 9$ } Multiply

Their Product is $0.227,754$
 Also, $P = 567,5$; it's Logarithm is $2.753,966$ } Add

The Sum is the Logarithm $2.981,720$ of $958,78 = A$.

That is, $958l. 15s. 7d.$ will be the Amount required. Consequently $958l. 15s. 7d. - 567l. 10s. = 391l. 5s. 7d.$ will be the Interest only, which is more than $306l. 9s.$, or the Simple Interest of that Principal for the same Time, by $84l. 16s. 7d.$ See Page 666.

Case 2. When A , t , and R , are given; To find P .

Rule. { Multiply the Logarithm of R , the Amount of 1l. at the given Rate per Cent, with the Time (as before); Then Subtract that Product from the Logarithm of the proposed Amount: and there will Remain the Logarithm of the Principal required.

Example.

What Principal, or Sum of Money, being put out to Interest for Nine Years, will Amount to (or raise the Sum of) 958l. 15s. 7d. at the Rate of 6 per Cent. per Annum?

Or thus; There is a Debt of $958l. 15s. 7d.$ which is not due until Nine Years hence; but it is agreed to be paid in present Money. What Sum must the Creditor Receive, allowing the Rebate, or Discount, of 6 per Cent. per Annum to the Debtor for his Ready Money.

Here $A = 958,78$; it's Logarithm $2.981,720$

And $R = 1,06$; it's Logarithm is $0.025,306$
 Also, $t = 9$ } Multiply.

Subtract this Logarithm $0.227,754$ from the first.

There Remains the Logarithm $2.753,966$ of $567,5 = P$.

That is, $567l. 10s.$ is the Principal (or Ready Money), which was required.

Case 3. Suppose P , t , and A , are given; To find R .

Rule. $\left\{ \begin{array}{l} \text{From the Logarithm of the proposed Amount Subtract the Logarithm} \\ \text{of the proposed Principal; Then Divide the Remainder by the Time, and} \\ \text{the Quotient will be the Logarithm of } R, \text{ the Amount of 1l. \&c. as in} \\ \text{this Example.} \end{array} \right.$

Example.

At what Rate of Interest per Cent, &c. will 567l. 10s. Raise a Stock, or Amount to, 958l. 15s. 7d. in Nine Years' Time?

Here is given $A = 958,78$; it's Logarithm is 2.981,720 } Subtract
And $P = 567,5$; it's Logarithm is 2.753,966 }

Also, $t = 9$. Then 9) 0.227,754 (0,025,306, the Logarithm of 1,06 = R ; then 1,06 — 1 = 0,06, the Ratio. And it will be, As 1 : 0,06 :: So is 100 : To 6, the Rate of Interest per Cent, &c. (See Page 667.)

Case 4. Having P , R , and A , given; To find t .

Rule. $\left\{ \begin{array}{l} \text{From the Logarithm of the proposed Amount Subtract the Logarithm} \\ \text{of the proposed Principal (as before), and Divide their Difference by the} \\ \text{Logarithm of } R, \text{ the Amount of 1l. at the given Rate: and the Quotient} \\ \text{will shew the Time required.} \end{array} \right.$

Example.

In what Time will 567l. 10s. Amount to (or Raise a Stock of) 958l. 15s. 7d. at the Rate of 6 per Cent. per Annum?

Here is given $A = 958,78$; it's Logarithm is 2.981,720 } Subtract
And $P = 567,5$; it's Logarithm is 2.753,966 }

Also, $R = 1,06$; it's Logarithm is 0,025,306) 0.227,754 (9 = t .

Viz. Nine Years will be the Time required by the Question.

These Four Examples are sufficient to shew how all Questions of the like Kind may be truly Resolved; if the Time given (*or sought*) be whole Years; but if it be not, then the Odd Time, whether it be under, or above, a Compleat Year, must be Reduced into Decimal Parts of a Year, (See Page 669.) by Help of the Two Tables in Page 670; and then it will be as easy to Resolve any of the foregoing Cases, when the Time given (*or sought*) is in Parts of a Year, as it is for those in whole Years.

Example 1. *When the Time is Less than a Year. Let it be required to find the Amount of 375l. for 210 Days, at 6 per Cent, &c.*

In

In this Question, there is given $P = 375$, $R = 1,06$, and $t = 210$ Days $= 0,5753$ Decimal Parts of a Year, *per Table, Page 670*; Thence to find A ; as in *Case 1.* of this *Section*.

First, $R = 1,06$; it's Logarithm is $0,025306$
And $t = 0,5753$ } Multiply

Their Product is $0,014558$
Again, $P = 375$; it's Logarithm is $2,574031$ } Add

The Sum is the Logarithm $2,588589$ of $387,78 = A$.

That is, $387\text{ l. } 15\text{ s. } 7\text{ d.}$ will be the Amount required.

Example 2. Suppose it were required to find what 265 l. would Amount to in Five Years and 135 Days; at the Rate of 6 per Cent, &c.

In this Question is given $P = 265$, $R = 1,06$, and $t = 5,3698$; To find A , the Amount; as before.

Thus, $R = 1,06$; it's Logarithm is $0,025306$
And $t = 5,3698$ } Multiply

Their Product is $0,135888$
Again, $P = 265$; it's Logarithm is $2,423246$ } Add

The Sum is the Logarithm $2,559134$ of $362,355 = A$.

Viz. $362\text{ l. } 7\text{ s. } 1\text{ d.}$ will be the Amount, or Sum, required.

Example 3. Let it be required to find what Principal, or Sum, put to Interest at 6 per Cent, will Amount to (or Raise a Stock of) 362 l. 7 s. 1 d. in Five Years and 135 Days.

Here is given $A = 362,355$, $R = 1,06$, and $t = 5,3698$; To find P , as in *Case 2.*

Thus, $A = 362,355$; it's Logarithm $2,559,134$ of the Dividend.

And $R = 1,06$; it's Logarithm is $0,025,306$
 $t = 5,3698$ } Multiply

Then Subtract this Logarithm $0,135,888$ from the 1st Logarithm.

And there Remains the Logarithm $2,423,246$ of $265 = P$.

Viz. 265 l. will be the Principal, or Sum, required.

If

If the Work of these Three Examples be well understood, it must needs be easy to find proper Answers to all Questions that can be proposed in the other Cases, viz. for any proposed Time, and Rate of Interest, according as the Date requires.

SECT. II. Of *Annuities*, or *Rents*, &c. that are in Arrear, Computed at *Compound Interest*.

I SHALL here make Use of the same Letters to represent the same Things as in Page 673. Save only that R doth here denote the Amount of $1l.$, and that t , instead of T , denotes the Time, as in the last Section.

Viz. $\begin{cases} U = \text{the Annuity, or Rent.} \\ t = \text{the Time of it's being Unpaid.} \\ A = \text{the Amount, or Sum of all the Arrears.} \end{cases}$

And $R =$ the Amount of $1l.$ &c. as in Page 689.

Then will $A - U = AR - UR^t$.

From this Equation are deduced the Four following Cases, and the Rules by which they are Resolved.

N. B. The following Cases admit of some Variety, in Respect to the true Intervals of the Times, which are betwixt the several Payments; viz. it must be well considered whether the Payments are to be made Yearly, or at the End of some Part of a Year; as the $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{12}$, $\frac{1}{365}$, That is, Half-Yearly, Quarterly, Monthly, Weekly, or Daily: the which must be very carefully minded when the Question is proposed, in any of the following Cases, or the Answers will not be true.

Case 1. Having U , t , and R , given; To find A .

That is, If any Annuity, or Rent, &c. with the Time of it's being unpaid, and the Rate of Interest per Cent, are given; Thence to find what Sum all those Arrears will Amount to in that Time, allowing Compound Interest for every particular Payment, as it becomes due.

Rule. $\begin{cases} \text{Multiply the Logarithm of } R \text{ (the Amount of } 1l.) \text{ with the Time;} \\ \text{and to that Product Add the Logarithm of the Annuity; Then find the} \\ \text{Number which belongs to that Sum, and Subtract the Annuity from that} \\ \text{Number; (So far the Work is general to all Times of Payment.)} \\ \text{Then, if the Remainder (*) be Divided by } R - 1 \text{ (viz. by the Ratio of} \\ \text{the Rate of Interest per Cent.), the Quotient will shew the Sum, or} \\ \text{Amount, of all the Arrears for Yearly Payments.} \end{cases}$

Example I.

Suppose 356l. Annuity, or Yearly Rent, were forborn, or unpaid, Nine Years; What would the Sum of all those Arrears Amount to at 6 per Cent, &c.?

Here is given $U = 356$, $t = 9$, and $R = 1.06$; To find A .

Thus, $R = 1.06$; it's Logarithm is $0.025,306$ } Multiply
 $t = 9$

The Product is $0.227,754$ } Add
 Again, $U = 356$; it's Logarithm is $2.551,450$

The Sum is the Logarithm $2.779,204$ of $601,4545$.

And $601,4545 - 356 = 245,4545$; $R - 1 = 0.06$.

Then 0.06 $245,4545$ ($4090,91 = A = 4090l. 18s. 2\frac{1}{2}d.$ being the Sum of all the Arrears for Yearly Payments.

Note. Here we may observe the Difference that is betwixt the Arrears Computed at Simple Interest, and at Compound Interest. See the same Example in Page 674.

Or, the Amount may be also found by Logarithms without the last Division; for, having once formed the Dividend and Divisor (as before), then the Difference of their Logarithms will shew the Amount.

Thus, $245,4545$; it's Logarithm is $2.389,972$ } Subtract
 And $R - 1 = 0.06$; it's Logarithm is $8.778,151$

There Remains the Logarithm $3.611,821$ of $4090,91 = A$. And so on, as before.

But if the Payments are to be made at any Time Less than a Compleat Year; as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{12}$, &c.

Then { You must take the like Part of the aforesaid Remainder (*), and make it a new Dividend: And you must also take the same Part of the Logarithm of R ; and the Number answering to that Part, being made less by an Unit, or 1, will be a new Divisor (instead of $R - 1$), by which if the new Dividend be Divided, the Quotient will shew the Sum of all the Arrears required.

Example II. In Half-Yearly Payments.

Suppose the aforesaid 356l. per Annum were to be paid by Half-Yearly Payments, viz. 178l. every Half-Year, and it were unpaid Nine Years (as before); What Sum would all those Arrears Amount to then, at 6 per Cent, &c.?

Here

Also, $U = 356$; Their Product is 0.227,754 } Add
 it's Logarithm is 2.551,450 }

But here the $\frac{1}{2}$ of $245,4545 = 122,72725$ is the New Dividend.

Then $0,029563) 122,72725 (4151,38 = A.$

Here, also, the Amount may be obtained by Logarithms, without the last Division; having once formed the New Dividend, and it's proper Divisor, as before.

There Remains the Logarithm $\overline{3.618,190}$ of $4151,38 = A$.

Viz. 415*l.* 7*s.* 7*d.* = *A*, the Amount, £*c.* as before.

Example III. In Quarterly Payments.

Let it be required to find the Amount, or Sum, of all the Arrears that would be due from 245l. per Annum, to be paid by Quarterly Payments (viz. 61l. 5s. per Quarter), supposing it to be forborn, or unpaid, Ten Years and a Half, at $5\frac{1}{2}$ per Cent, &c.

Again, $U = 245$; The Product is 0.244,146 } Add
 it's Logarithm is 2.389,166 }

Their Sum is the Logarithm $2.633,312$ of $429,846$.

And $429,846 - 245 = 184,846$. So far the Work is still the same as in Whole Years.

But now it must be the $\frac{1}{4}$ of $184,846 = 46,211\frac{1}{2}$ for the New Dividend.

4 T 2

Next,

Next, $R = 1,055$; the $\frac{1}{4}$ of it's Logarithm is $0.005,813$, whose Number is $1,01348$; and $1,01348 - 1 = 0,01348$ for the New Divisor.

Then $0,01348) 46,2115 (3428,15 = A.$

That is, $3428\text{ l. } 3\text{ s.}$ will be the Amount, or Sum, of all those Arrears, as was required.

Or the last Division may be avoided, and the Amount may be found by the Logarithms; Thus,

<i>Dividend</i> $46,2115$; it's Logarithm is $1.664,750$	} Subtract
<i>Divisor</i> is $0,01348$; it's Logarithm is $8.129,690$	

There Remains the Logarithm $3.535,060$ of $3428,15 = A.$

Viz. $3428\text{ l. } 3\text{ s.}$ is the Amount required, as above.

The same may be done for *Monthly*, *Weekly*, or *Daily* Payments, Care being taken in forming the New Dividend and Divisor, according to their respective Times.

Case 2. When A , t , and R , are given; To find U .

That is, to find what Annuity, or Yearly Rent, being forborn, or unpaid, any assigned Time, will Amount to a proposed Sum, allowing any given Rate of Interest *per Cent.* for every Payment, as it becomes due.

Note. *This Case, as well as the Last, admits of some Variety in Respect to the Times of Payment; viz. Whether they are Yearly, Half-Yearly, or Quarterly, &c.; the which must be carefully observed, as the following Rule directs.*

Rule. $\left\{ \begin{array}{l} \text{Multiply the Logarithm of } R \text{ (the Amount of 1 l.) with the given} \\ \text{Time; and the Number answering to their Product, being made less by 1,} \\ \text{will be the Divisor (for all Times of Payment at that Rate of Interest).} \\ \text{Next, Multiply the proposed Amount, or Sum, with } R - 1, \text{ viz. with} \\ \text{the Ratio of the given Rate; Then Divide their Product by the aforesaid} \\ \text{Divisor, and the Quotient will shew the Annuity, or Rent, for Yearly} \\ \text{Payments.} \end{array} \right.$

Example I.

What Annuity, or Yearly Rent, being forborn, or unpaid, Nine Years, will Amount to (or Raise a Stock of) 4090 l. 18 s. 2½ d. at the Rate of 6 per Cent. for every Payment, as it becomes due?

Here is given $A = 4090,91$, $t = 9$, and $R = 1,06$; To find U .

First, $R = 1,06$; it's Logarithm is $0.025,306$	} Multiply
And $t = 9$	

The Product is the Logarithm $0.227,754$ of $1,68948$.

And $1,68948 - 1 = 0,68948$ for the Divisor.

Again,

$$\begin{array}{r} \text{Again, } A = 4090,91 \\ \text{And } R - 1 = \quad 0,06 \end{array} \left. \vphantom{\begin{array}{r} A \\ R-1 \end{array}} \right\} \text{Multiply}$$

The Product is 245,4546 for the Dividend.

Then 0,68948) 245,4546 ($356 = U$.

That is, 356 $\%$ will be the Annuity, or Yearly Rent, as was required to be paid by Yearly Payments.

Or, when the Divisor has been found (*as above*), the Rest of the Work may be performed by Logarithms only;

$$\begin{array}{r} \text{Thus, } A = 4090,91; \text{ it's Logarithm is } 3,611,821 \\ \text{And } R - 1 = 0,06; \text{ it's Logarithm is } 8,778,151 \end{array} \left. \vphantom{\begin{array}{r} A \\ R-1 \end{array}} \right\} \text{Add}$$

$$\begin{array}{r} \text{Sum is } 2,389,972 \\ \text{The Divisor is } 0,68948; \text{ it's Logarithm is } 9,838,522 \end{array} \left. \vphantom{\begin{array}{r} \text{Sum} \\ \text{Divisor} \end{array}} \right\} \text{Subtract}$$

There Remains the Logarithm 2,551,450 of $356 = U$.

Viz. 356 $\%$ will be the Yearly Annuity, or Rent, &c.

The Rule, when the Payments are Half-Yearly, or Quarterly, &c.

But, if the Payments are required to be paid either Half-Yearly, or Quarterly, &c. as in the last Case;

Then, $\left\{ \begin{array}{l} \text{Instead of } R - 1, \text{ you must take the respective Part of the Logarithm} \\ \text{of } R \text{ (viz. of the Amount of 1l.), and with the Number answering to} \\ \text{that Part, being made less by 1, Multiply the proposed Amount; Then} \\ \text{Divide that Product by the aforesaid Divisor: and the Quotient will shew} \\ \text{what Annuity, or Rent, must be paid at the proposed Time of each Payment.} \end{array} \right.$

Example II. In Half-Yearly Payments.

What Annuity, or Rent, to be paid by Half-Yearly Payments, being forborn, or unpaid, during Nine Years, will Amount to the Sum of 4151l. 7s. 7d. allowing 6 per Cent. for every Payment, as it becomes due?

Here is given $A = 4151,38$, $t = 9$, and $R = 1,06$; To find U .

$$\begin{array}{r} \text{First, } R = 1,06; \text{ it's Logarithm is } 0,025,306 \\ \text{And } t = \quad 9 \end{array} \left. \vphantom{\begin{array}{r} R \\ t \end{array}} \right\} \text{Multiply}$$

The Product is the Logarithm 0 227,754 of 1,68948.

And 1,68948 $- 1 = 0,68948$ for the Divisor, as before, in the last Example.

But, instead of the Multiplier $R - 1$, we must find a different Multiplier, as follows. R is $= 1,06$; the $\frac{1}{2}$ of it's Logarithm is 0,012653; of which the Number is 1,029563; and $1,029563 - 1 = 0,029563$, is the Multiplier. And

And it must be $A = 4151,38$
 Multiplied with $0,029563$

The Product is $122,72724$, the New Dividend.

Then $0,68948$) $122,72724$ ($178 = U$, viz. 178% will be one of the Half-Yearly Payments of the Annuity, or the Rent required.

Or, when we have found the Divisor, and the New Multiplier that is to be used instead of $R - 1$; Then the Work may be done by the Logarithms,

Thus, $A = 4151,38$; it's Logarithm is $3.618,190$ } Add
 Multiplier is $0,029563$; it's Logarithm is $8.470,749$

Sum $2.088,939$ } Subtract
 The Divisor is $0,68948$; it's Logarithm is $9.838,520$

There Remains the Logarithm $2.250,419$ of $178 = U$.

Viz. 178% is one of the Half-Yearly Payments, &c. as before.

Example III. In Quarterly Payments.

Let it be required to find what Annuity, or Rent, to be paid by Quarterly Payments, being forborn, or unpaid, Ten Years and a Half, would Amount to the Sum of 3428l. 3s. at Five and a Half per Cent. per Annum, for every Payment, as it becomes due?

Here is given $A = 3428,15$, $t = 10,5$, and $R = 1,055$; To find U , the Quarterly Payment.

First, $R = 1,055$; it's Logarithm is $0.023,252$ } Multiply
 And $t = 10,5$

The Product is the Logarithm $0.244,146$ of $1,75447$.

Then $1,75447 - 1 = 0,75447$, the Divisor.

Again, $R = 1,055$; the $\frac{1}{4}$ of it's Logarithm is $0.005,813$; it's Number is $1,01348$, and $1,01348 - 1 = 0,01348$, the Multiplier.

Next, $A = 3428,15$
 Multiplied with $0,01348$

Produces $46,21146$, the Dividend.

Then $0,75447$) $46,21146$ ($61,25 = U$.

That is, 61% 5s. will be one of the Quarterly Rents, or Payments; consequently $61,25 \times 4 = 245\%$ per Annum will be the Annuity required.

Or,

Or, by *Logarithms*, Thus;

$A = 3428,15$; it's Logarithm is $3.535,060$
And $0,01348$; it's Logarithm is $8.129,690$ } Add

Sum is $1.664,750$
Again, $0,75447$; it's Logarithm is $9.877,642$ } Subtract

There Remains the Logarithm $1.787,108$ of $61,25 = U$.

Viz. 61*l.* 5*s.* will be the Rent, to be paid every Quarter, &c.

The like may be done in finding the Monthly, Weekly, or Daily Annuities, Care being taken in Forming the New Multipliers and Divisors proper for the Times given.

Case 3. Having U , A , and R , given; To find t .

That is, To find the Time in which any proposed Annuity, or Rent, will Amount to any assigned Sum, at any given Rate of Interest *per Cent*, &c.

Rule. $\left\{ \begin{array}{l} \text{Multiply the proposed Amount with } R - 1, \text{ viz. with the Ratio of} \\ \text{the given Rate; and Divide that Product by the Annuity, and to the} \\ \text{Quotient Add 1. Then, if the Logarithm of that Sum be Divided by} \\ \text{the Logarithm of } R \text{ (viz. the Amount of 1*l.*), the Quotient will shew} \\ \text{the Time for Yearly Payments.} \end{array} \right.$

Example I.

*In what Time will 356*l.* Yearly Rent, or Annuity, Amount to the Sum of 4090*l.* 18*s.* 2½*d.* at the Rate of 6 per Cent, &c.*

Here is given $U = 356$, $A = 4090,91$, and $R = 1,06$; To find t .

Thus, $A = 4090,91$
And $R - 1 = 0,06$ } Multiply

$U = 356$) $245,4546$ ($0,68948$.

And $0,68948 + 1 = 1,68948$; it's Logarithm is $0,227,754$.

Then $R = 1,06$; it's Logarithm is $0,025,306$) $0,227,754$ ($9 = t$.

Viz. 9 Years will be the Time required in the Question.

Or

Or otherwise, by *Logarithms*.

Thus, $A = 4090,91$; it's Logarithm is $3,611,821$ } Add
 And $R - 1 = 0,06$; it's Logarithm is $8,778,151$ }

Again, $U = 356$; it's Logarithm is $2,551,450$ } Subtract
 Sum $2,389,972$ }

There Remains the Logarithm $9,838,522$ of $0,68948$.

And $0,68948 + 1 = 1,68948$; it's Logarithm is $0,227754$.

Then (*as before*) $1,06$; it's Logarithm is $0,025306$ $0,227754$ ($9 = t$).

Viz. 9 Years will be the Time required, if the Payments are to be paid but once a Year.

But if they are to be paid either *Half-Yearly*, or *Quarterly*, &c. as before in the two last Cases;

Then, $\left\{ \begin{array}{l} \text{Instead of making } R - 1, \text{ the Multiplier as above, you must take} \\ \text{such a Part of the Logarithm of } R, \text{ (the Amount of } 1\text{), as the given} \\ \text{Time requires; and with it's Number, made Less by } 1, \text{ Multiply the} \\ \text{proposed Amount; Then Divide that Product by the Sum which ought to} \\ \text{paid at it's respective Time; and to that Quotient Add } 1; \text{ and so proceed} \\ \text{as the Rule directs.} \end{array} \right.$

Example II. In Half-Yearly Payments.

In what Time will 356l. per Annum Amount to the Sum of 4151l. 7s. 7d. If it is to be paid by Half-Yearly Payments, viz. 178l. every Half-Year; and to be allowed 6 per Cent, &c. for every Payment, as it becomes due?

Here is given $U = 178$, *viz.* the $\frac{1}{2}$ of 356, $A = 4151,38$, and $R = 1,06$; To find t , the Number of Years.

First, $R = 1,06$; the $\frac{1}{2}$ of it's Logarithm is $0,012653$, and it's Number is $1,029563$; then $1,029563 - 1 = 0,029563$, the Multiplier, instead of $R - 1$.

And it will be $A = 4151,38$
 Multiplied with $0,029563$

The Product is $122,72724$ for the Dividend.

Then $U = 178$ $122,72724$ $0,68948$; to which add 1 , and it will be $1,68948$; it's Logarithm is $0,227754$.

And $R = 1,06$; it's Logarithm is $0,025306$ $0,227754$ ($9 = t$).

Viz. 9 Years will be the Time sought.

Or,

Or, having once formed the Multiplier, then the Work may be done by Logarithms,

$$\begin{array}{rcl} \text{Thus, } A = 4151,38; \text{ it's Logarithm is } 3.618,190 & \} & \text{Add} \\ \text{Multiplier is } 0,029563; \text{ it's Logarithm is } 8.470,749 & \} & \\ \hline & \text{Sum} & 2.088,939 \\ \text{Again, } U = 178; \text{ it's Logarithm is } 2.250,418 & \} & \text{Subtract} \\ \hline & & 9.838,521 \end{array}$$

There Remains the Logarithm 9.838,521 of 0,68948.

And $0,68948 + 1 = 1,68948$; it's Logarithm is 0.227,754; $R = 1,06$; it's Logarithm is 0.025,306) 0.227,754 ($9 = t$, the Number of Years fought. As before.

Example III. In Quarterly Payments.

In what Time will 245l. per Annum, to be paid Quarterly (viz. 61l. 5s. per Quarter), Amount to the Sum of 3428l. 3s., allowing $5\frac{1}{2}$ per Cent. for the Forbearance of the Payments, as they become due?

Here is given $U = 61,25$ (viz. the $\frac{1}{4}$ of 245l.); $R = 1,055$, and $A = 3428,15$; to find t , the Time.

First, $R = 1,055$; the $\frac{1}{4}$ of it's Logarithm is 0.005,813; it's Number is 1,01348, and $1,01348 - 1 = 0,01348$, the Multiplier;

And $A = 3428,15$ the Multiplicand,
Multiplied with 0.01348

Their Product is 46,21146 the Dividend.

Then $U = 178$) 46,21146 (0,75447; to which Add 1, and it is 1,75447; it's Logarithm is 0.244,146.

Then $R = 1,055$; it's Logarithm is 0.023,252) 0.244,146 ($10,5 = t$.

Viz. Ten Years and a Half will be the Time required by the Question.

The same may be found by Logarithms, as in the last Example.

Case 4. Having U , t , and A , given; To find R .

That is, If any Annuity, with the Time of it's being Unpaid, and the Sum it is proposed to Amount to in that Time, are given; Thence to find the Rate of Interest *per Cent.* that must be allowed for every Payment, as it becomes Due.

The Solution of this Question is more difficult than any hitherto has been ; and the Equation by which the Value of R must be found is this,

$\frac{A}{U} - 1 = \frac{A}{U} R - R'$, which is called an Affected Equation*, and requires an Analytical Solution to find the Value of R , (as in the *Young Mathematician's Guide*, Page 268,) which cannot otherwise be found by any General Rule, that can be prescribed here, as in the precedent Cases, but only by Approaching to it with Trials. And, in order to perform that Approximation with the least Trouble, I shall here propose the following Method, as the easiest that I can at present think of.

Which is thus :

First, Divide the proposed Amount by the Annuity, and Reserve the Quotient. Next, Subtract 1 from that Quotient, and call the Remainder x .

Then you must make choice (by guess) of such a Number for the Value of R , as you think will be the Amount of 1l. at the Rate of the Interest sought ; and
Multiply

* By *affected*, or (as the word is now more frequently written,) *affected*, equations in Algebra we are to understand all such equations as contain more than one power of the unknown quantity x , in contra-distinction to such equations as involve only one power of the said quantity, which are called *pure* equations. The reason and origin of these expressions may be explained as follows. They were introduced into Algebra by the celebrated French Algebräist Monsieur Viète, or (as he is more frequently called,) *Vieta*, who flourished about the year 1580, and was the great improver of modern Algebra. This great writer has many expressions peculiar to himself, and which have not been adopted by subsequent Algebräists, or, at least, by such of them as have written since the publication of Harriot's Algebra in the year 1631, and of Des Cartes's Geometry in the year 1637. Amongst these are the following expressions.—He calls a set of quantities that are in continued geometrical proportion, (such as the quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c,) a set of *scalar* quantities, or *magnitudines scalares*; and, when there are several of these *scalar* quantities connected with each other by the signs + and −, or by Addition and Subtraction, (as in the compound quantity $x^5 + ax^4 - b^2x^3$;) he calls the highest quantity, or that which is farthest in the scale of quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c, (to wit, the quantity x^5 in the said compound quantity $x^5 + ax^4 - b^2x^3$;) the *power* of the fundamental quantity x , or of the second term in the said scale; and he calls the lower scalar quantities, which are involved in the second and third terms of the said compound quantity $x^5 + ax^4 - b^2x^3$, to wit, the quantities x^4 and x^3 , (or, in our present language, the inferior powers of x ;) scalar quantities of a *parodic* degree to x^5 , or the *power* of the fundamental quantity x . This word *parodic* I take to be derived (though Vieta does not tell us so,) from the Greek words *παρά* and *ὁδός*, which signify *near* and *a way*, or *road*, because these inferior scalar quantities, x^3 and x^4 , lie in the way as you pass along in the scale of the aforesaid quantities 1, x , x^2 , x^3 , x^4 , x^5 , x^6 , x^7 , &c, from 1 to x^5 , which he calls the *power* of x in the said compound quantity $x^5 + ax^4 - b^2x^3$. These inferior scalar quantities x^3 and x^4 are therefore *parodic*, or *situated in the way to*, or are *leading to*, the said *power*, or higher scalar quantity, x^5 . He then proceeds to define a *pure power* and an *affected power*, and tells us, that a *pure power* is a scalar quantity that is not affected with, or mixed with, any *parodic*, or *inferiour* scalar quantity, and that an *affected power* is a scalar quantity that is mixed, or connected by Addition, or Subtraction, with one, or more, *inferiour*, or *parodic*, scalar quantities, combined with co-efficients that raise them to the same dimension as the *power* itself, or make them *homogeneous* to it, and consequently capable of being added to it, or subtracted from it. Thus x^5 alone is a *pure power* of x , namely, it's fifth power; and $x^5 + ax^4 - b^2x^3$ is an *affected power* of x , namely, it's fifth power *affected by*, or *connected with*, the two *parodic*, or *inferiour*, scalar quantities, x^3 and x^4 , which are multiplied into bb and a , in order to make them *homogeneous* to, or of the same dimension with, x^5 itself, and consequently capable of being added to it, or subtracted from it. And he seems to have used the word

Multiply that Number into the reserved Quotient. Also, you must Multiply the Logarithm of that supposed Number with the Time; and find the Number which belongs to their Product: Then, if the Difference between that Number and the aforesaid Product of R into the Reserved Quotient, be Equal to (viz. be the same with) the Number called x , you have hit upon the true Value of R . But if they are not equal, then another Trial must be made, &c. As in the following Examples.

Example I. In Yearly Payments.

There is an Annuity, or Rent, of 356l. per Annum, and it has been forborn, or unpaid, Nine Years; And there is 4090l. 18s. 2½d. given for the Amount, or Sum, of all the Arrears; it is required to find what Rate of Interest per Cent. is allowed for each Payment, as it became Due?

Here is given $U = 356$, $t = 9$, and $A = 4090.91$; To find R , the Amount of 1l.

Thus, 356×4090.91 (11,49132, the Quotient to be Reserved; from which Subtract 1, and it will be 10,49132, which is the Number called x : And so far the Work will always be certain, it being only that of the first Side of the Equation.

affected in speaking of such a compound quantity as $x^5 + ax^4 - b^2x^3$, because the magnitude of the highest power of x , to wit, x^5 , was changed, or altered, or made greater, or less, than it would otherwise be, by the addition of the parodic quantity ax^4 , and the subtraction of the parodic quantity b^2x^3 ; which increase, or diminution, or change, in the principal quantity x^5 he seemed to think might be well expressed by saying it was *affected* by it's connection with the said parodic quantities. There are some expressions in his book that satisfy me that this was the idea he annexed to the word *affected*. See Schooten's edition of Vieta's Works, published at Leyden in Holland, in the year 1646, pages 3 and 4.

This, then, being the meaning of the expressions a *pure power* and an *affected power*, the meaning of the corresponding expressions of a *pure equation* and an *affected equation* follows from it of course: a *pure equation* signifying an equation in which a pure power of an unknown quantity is declared to be equal to some known quantity; such as the equation $x^5 = 79$; and an *affected equation* signifying an equation in which a power of an unknown quantity affected by, or connected, either by Addition or Subtraction, with, some inferior powers of the same unknown quantity, (multiplied into proper co-efficients in order to make them *homogeneous* to the said highest power of the said unknown quantity,) is declared to be equal to some known quantity; such as the equation $x^5 + ax^4 - b^2x^3 = 79$. This I take to be the original meaning of the expression an *affected equation*. But, as the language of Vieta has not been adopted by subsequent writers of Algebra, I should think it would be more convenient to call them by some other name. And, perhaps, those of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, that of *multinomial* equations, would be as convenient as any. Thus, $xx + ax = rr$, and $x^3 + ax^2 = r^3$, and $x^3 + a^2x = r^3$, and $x^4 + a^3x = r^4$, and $x^4 + ax^3 = r^4$, might all be called *binomial* equations, because they would be equations in which a *binomial* quantity, or quantity consisting of two terms that involved the unknown quantity x , is declared to be equal to a known quantity; and, for a like reason, the equations $x^3 + ax^2 + b^2x = r^3$, and $x^4 - ax^3 + b^2x^2 = r^4$, and $x^4 - ax^3 + b^3x = r^4$, and $x^5 + ax^4 + b^2x^3 = r^5$, and $x^5 + ax^4 - b^2x^3 = r^5$, and $x^5 + b^2x^3 + c^4x = r^5$, might be called *trinomial* equations. And the like names might be given to equations of a greater number of terms. Dr. Hutton, I observe, in his excellent new Mathematical and Philosophical Dictionary, (published in the year 1795,) calls them *compound* equations; which is likewise a very proper name for them, and less obscure than that of *affected* equations. And Mr. Kersey, in his excellent Treatise of Algebra, published in the year 1673, likewise calls them *compound* equations. But I should think the former appellation of *binomial*, *trinomial*, *quadrinomial*, *quinquinomial*, and, in general, *multinomial* equations would be rather more descriptive of their nature. F. M.

Next, I will suppose the Value of R to be 1,05; and then, the Reserved Quotient 11,49132 being multiplied into 1,05, (the conjectural value of R ,) the Product will be 12,065886.

Again, If $R = 1,05$; it's Logarithm is 0.021,189 } Multiply
And $t = \underline{\quad 9 \quad}$

The Product is the Logarithm 0.190,701 of 1,55132.

Then $12,065886 - 1,55132$ is $= 10,51456$; which is More than 10,49132, the Number called x . Therefore this supposed Value of R , to wit, 1,05, is taken too Little.

Again, For a second Trial, I will suppose $R = 1,07$; and then, if the Reserved Quotient 11,49132 be multiplied into 1,07, the Product will be 12,29571.

And, if $R = 1,07$, it's Logarithm will be 0.029,384 } Multiply
And $t = \underline{\quad 9 \quad}$

The Product is the Logarithm, 0.264,456, of 1,83847.

Then $12,29571 - 1,83847$ will be $= 10,45724$, which is Less than 10,49132, the Number called x . And therefore I conclude that the Last R , to wit, 1,07, was taken too Big, and that the Value of R is some Number between 1,05 and 1,07.

Let us therefore take $R = 1,06$ (*viz. in the Middle between 1,05 and 1,07*). And then, if the Reserved Quotient 11,49132 be multiplied into 1,06, the Product will be 12,180799.

And, if $R = 1,06$, it's Logarithm will be 0.025,306 } Multiply
And $t = \underline{\quad 9 \quad}$

The Product is the Logarithm 0.227,754 of 1,689479.

Then $12,180,799 - 1,689,479$ will be $= 10,49132$, which is just the same with the Number called x .

Whence I conclude that the true Value of R is $= 1,06$, and that 1,06 $- 1$, or 0,06, is the Ratio of the Rate of Interest sought.

Then it will be, As 1 : is to 0,06 :: So is 100 : to 6, the Rate of Interest per Cent. As was required for Yearly Payments.

But if the Payments are to be made either Half-Yearly, or Quarterly, &c.

Then, { *Instead of Multiplying the Reserved Quotient with the supposed Value of R (as above), you must take such a Part of the Logarithm of that R (viz. of the assumed Amount of 11.), as the given Time requires, and with it's Number Multiply the aforesaid Reserved Quotient, and then proceed on as before.*

Example

Example II. In Half-Yearly Payments.

Suppose the same Annuity of 356l. per Annum were to be paid Half-Yearly, (viz. 178l. every Half-Year.) If it be forborn Nine Years, and then Amount to the Sum of 4151l. 7s. 7d. What Rate of Interest per Cent. is allowed, &c.?

Here is given $U = 178$ (viz. the $\frac{1}{2}$ of 356l.); $t = 9$, and $A = 4151,38$; To find R , the Amount of 1l., &c.

First, 178) 4151,38 (23,32236 the Reserved Quotient, and $23,32236 - 1 = 22,32236$, the Number called x . So far is the same as before in whole Years.

Now here I will assume the first Value of $R = 1,06$. Then, since R is $= 1,06$, the $\frac{1}{2}$ of it's Logarithm will be 0.012,653, and the Number to that Half-Logarithm is 1,029563, which mult be taken for the Multiplier, instead of $R = 1,06$, as before: And then, if the Reserved Quotient 23,32236 be multiplied into 1,029563, the Product will be 24,011,839.

Again, since R is $= 1,06$, it's Logarithm will be 0.025,306 } Multiply
And $t = 9$

Their Product is the Logarithm 0.227,754 of 1,689,479.

Then $24,011839 - 1,689479 = 22,32236$, which is just the same with 22,32236, or the Number called x . And therefore the true Value of R is $= 1,06$. And so on for the Rate per Cent, as in the last Example.

Example III. In Quarterly Payments.

An Annuity of 245l. per Annum, to be paid Quarterly (viz. 61l. 5s. every Quarter), is forborn Ten Years and a Half; and then it is said to Amount to the Sum of 3428l. 3s. It is required to find what Rate of Interest per Cent. is allowed, &c.

Now here is given $U = 61,25$ (viz. the $\frac{1}{4}$ of 245l.); $t = 10,5$, and $A = 3428,15$; To find R , the Amount of 1l. &c.

First, 61,25) 3428,15 (55,96979, the Reserved Quotient, and $55,96979 - 1 = 54,96979$, the Number called x .

Then let us here suppose $R = 1,05$; and then the $\frac{1}{4}$ of it's Logarithm will be 0.005,297, whose Number is 1,01227, the Multiplier.

And, if the Reserved Quotient 55,96979 be multiplied into 1,01227, the Product will be 56,65754.

Again, If $R = 1,05$; it's Logarithm is 0.021,189 } Multiply
And $t = 10,5$

The Product is the Logarithm 0.222,484 of 1,66911.

Then will $56,65754 - 1,66911$ be $= 54,98843$; which is a little More than 54,96979, or the Number called x . Therefore the aforefaid Value of R , to wit, 1,05, is taken something too Little.

And

And making a second Trial, by supposing $R = 1,06$, and then proceeding on as before, I find the Result to be about as much Less than 54,96979, or the Number called x , as the last-found Number 54,98843, was greater than the said Number.

I shall therefore assume $R = 1,055$, viz. in the Middle between 1,05 and 1,06.

And then, since R is $= 1,055$, the $\frac{1}{4}$ of it's Logarithm will be 0.005,813; the Number corresponding to such latter, or lesser, Logarithm is 1,01348, which we must take for the Multiplier.

Then, if the Reserved Quotient 55,96979 be multiplied into 1,01348, the Product will be 56,72426.

Again, since $R = 1,055$; it's Logarithm is 0.023,252
And $t = 10,5$ } Multiply

The Product is the Logarithm 0.244,146 of 1,75447.

Then $56,72426 - 1,75447 = 54,96979$, which is exactly the same with the Number called x .

And therefore I conclude that the true Value of R is $= 1,055$, and that $1,055 - 1$, or 0,055, is the Ratio of the Rate. And

Then it will be, As 1 : is to 0,055 :: So is 100 : to 5.5; Viz. Five and a Half is the Rate of Interest *per Cent. per Annum*, that was required.

What hath been here done by Multiplication and Division, may, if you please, be performed by Logarithms, as in the precedent Cases.

As, for Instance, in the last Example;

Wherein $A = 3428,15$; it's Logarithm is 3.535,060
And $U = 61,25$; it's Logarithm is 1.787,108 } Subtract

The Remainder is the Logarithm 1.747,952 of 55,9698, which is here the Reserved Quotient.

And $55,9698 - 1 = 54,9698$, the Number called x .

Then, if $R = 1,055$, the $\frac{1}{4}$ of it's Logarithm is 0.005,813
The Logarithm of the Reserved Quotient is 1.747,952 } Add

The Sum is the Logarithm 1.753,765 of 56,7243.

Again, If $R = 1,055$; it's Logarithm is 0.023,252
And $t = 10,5$ } Multiply

The Product is the Logarithm 0.244,146 of 1,75447.

Then $56,7243 - 1,75447$ is $= 54,96983$; which is just the same with the Number called x . And therefore, &c. as above.

SECT.

SECT. III. To find the Present Worth of Annuities, Pensions, Leases, &c.
at Compound Interest.

I SHALL here make Use of the same Letters to represent the several Parts of the Question, as in Sect. III. Page 679. And it may not be amiss to make Use also of the same Examples as are in that Section; save only that I shall here Compute them for Whole Years, Half-Years, and Quarterly Payments, as in the last Section.

Let $\left\{ \begin{array}{l} U \text{ Denote the Annuity, or Rent,} \\ t \text{ The Time of it's Continuance,} \\ R \text{ The Amount of 1l.} \end{array} \right\}$ As before;

And P The Present Worth of the Annuity, &c.

$$\text{Then will } PR^t = \frac{UR^t - U}{R - 1}.$$

(Vide the Young Mathematician's Guide, Page 269.)

From this Equation are deduced the following Rules, which also admit of Four Cases.

Case 1. Having U , t , and R , given; To find P .

That is, If any proposed Annuity, or Yearly Rent, and the Time of it's Continuance, are given; To find the Present Worth of that Annuity, allowing any Rate of Compound Interest per Cent. to the Purchaser, &c.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Logarithm of } R \text{ (the Amount of 1l.) with the given} \\ \text{Time; and from the Number which belongs to that Product Subtract 1:} \\ \text{Then Multiply the Remainder with the Annuity, and the Product will be} \\ \text{a Dividend.} \\ \text{Then Multiply the Number which belonged to the first Product with} \\ R - 1 \text{ (viz. with the Ratio of the Rate), and make their Product a} \\ \text{Divisor; The Quotient arising from thence will show the Present Worth} \\ \text{for Yearly Payments.} \end{array} \right.$

Example I. In Yearly Rents.

Suppose a Lease of 250l. per Annum, were to be Lett for 21 Years; What may the present Worth of that Lease be, at the Rate of 5 per Cent, &c.?

In this Question there is given $U = 250$, $t = 21$, and $R = 1.05$; To find P , the present Worth for Yearly Rents.

First,

First, R is = 1,05 ; it's Logarithm is 0.021,189 } Multiply
 And t = 21

The Product is the Logarithm 0.444,969 of 2,7859.

Then $2,7859 - 1 = 1,7859$; and $1,7859 \times 250 = 446,475$ for the Dividend.

Again, $R - 1 = 0,05$; and $2,7859 \times 0,05 = 0,139,295$ for the Divisor.

Then 0,139,295) 446,475 (3205,2478 = P .

That is, 3205*l.* 5*s.* will be the present Worth of such a Lease as was required.

Or the same may be otherwise performed by *Logarithms* : Thus,

First, R is = 1,05 ; it's Logarithm is 0.021,189 } Multiply as before.
 And t = 21

The Product is 0.444,969 } Add
 Again, U is = 250 ; it's Logarithm is 2.397,940

The Sum is the Logarithm 2.842,909 of 696,48 ; from which
 Subtract the Annuity 250, and there will Remain
 446,48 ; the Logarithm of which is 2.649,802

The Logarithm of R , Multiplied with t , is 0.444,969 } Add
 And $R - 1 = 0,05$; it's Logarithm is 8.698,970

From the upper Logarithm take this Logar. 9.143,939

There Remains the Logarithm 3.505,863 of 3205,25.

That is, 3205*l.* 5*s.* will be the present Worth required by the Question, as before ; supposing the Rents to be paid but once a Year.

But, if the Payments of the Annuity, or Rents, &c. are to be made at any Times Less than a Compleat Year, As $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{12}$, &c, That is, *Half Yearly*, or *Quarterly*, or *Montbly*, &c ; Then the Divisor must be otherwise formed than by $R - 1$.

Thus, { You must take such a Part of the Logarithm of R , the Amount of 11., as the given Time requires, and make Use of the Number answering to that Part, made Less by 1, instead of $R - 1$, the Ratio, &c. As in the following Examples.

Example II. For Half-Yearly Rents.

Suppose the *aforesaid* 250*l.* per Annum, were to be paid Half Yearly (*viz.* 125*l.* every Half-Year), and a Lease of it were to be Lett for 21 Years ; What would the present Worth of that Lease be, at the Rate of Five per Cent, &c. ?

Here

Here is given $U = 125$, $t = 21$, and $R = 1,05$; To find P .

First, $R = 1,05$; it's Logarithm is $0.021,189$
 And $t = 21$ } Multiply

The Product is the Logarithm, $0.444,969$, of $2,7859$.

Then $2,7859 - 1$ is $= 1,7859$, and $1,7859 \times 125 = 223,2375$ is to be taken for a Dividend: Which is only the Half of the Dividend in the last Example.

Next, R is $= 1,05$; the $\frac{1}{2}$ of it's Logarithm is $0.010,594$, whose Number is $1,0247$; and $1,0247 - 1 = 0,0247$ is to be taken for the Multiplier, instead of $0,05 = R - 1$.

That is, $2,7859 \times 0,0247 = 0,06881$ will be the Divisor.

Then $0,06881$ $223,2375$ ($3244.25 = P$).

Viz. $3244l. 5s.$ will be now the present Worth of that Lease, which is more by $39l.$ than that in the last Example for Yearly Payments.

And by the Way, we may here observe the great Difference that is betwixt the present Worth of any Annuity, or Lease, &c. Computed at Simple Interest, and the same computed at Compound Interest. See the Example in Page 680, where the present Worth of the same Lease at Simple Interest is shewn to be $3873l. 9s. 6d.$

The same may also be performed by Logarithms, as in the last Example, having found the Multiplier instead of $R - 1$, as above.

Then R is $= 1,05$; it's Logarithm is $0.021,189$
 And $t = 21$ } Multiply

Again, U is $= 125$; it's Logarithm is $0.444,969$
 $2.096,910$ } Add

The Sum is the Logarithm $2.541,879$ of $348,24$.

From this $348,24$ Subtract the Rent 125 , and there Remains $223,24$; it's Logarithm is $2.348,772$

The Logarithm of R , Multiplied with t , is $0.444,969$
 The New Multiplier is $0,0247$; it's Log. is $8.392,697$ } Add

From the upper Logar. Subtract this Log. $8.837,666$

There Remains the Logarithm, $3.511,106$, of $3244,2$.

Viz. $3244l. 4s.$ is here found to be the present Worth; which is less than that found by the Pen, by only $1s.$, which Difference is very inconsiderable.

Example III. In Quarterly Rents.

What is 80l. per Annum, to continue 31 Years and a Half, Worth in present Money, at the Rate of 7 per Cent. per Annum; If the Rents be paid Quarterly, (viz. 20l. every Quarter of a Year?)

In this Question, we have $U = 20$, $t = 31,5$, and $R = 1,07$; To find P , as before.

First, R is $= 1,07$; it's Logarithm is $0.029,384$
 And $t = \underline{31,5}$ } Multiply

The Product is the Logarithm, $0.925,596$, of $8,4255$.

And $8,4255 - 1$ is $= 7,4255$, and $7,4255 \times 20$ is $= 148,51$ for the Dividend.

Again, R is $= 1,07$; the $\frac{1}{4}$ of it's Logarithm is $0.007,346$, whose Number is $1,01706$; and $1,01706 - 1$ is $= 0,01706$.

Next, $8,4255 \times 0,01706$ is $= 0,143,739$, the Divisor.

Then $0,143,739) 148,510000$ ($1033,185 = P$.

That is, $1033l. 3s. 8\frac{1}{2}d.$ will be the present Worth, as was required.

Or, having found the Multiplier $0,01706$, as above; Then the Rest of the Work may be performed by *Logarithms*.

Thus, R is $= 1,07$; it's Logarithm is $0.029,384$
 And $t = \underline{31,5}$ } Multiply

Again, U is $= 20$; it's Logarithm is $1.301,030$
 $\frac{0.925,596}{1.301,030}$ } Add

The Sum is the Logarithm, $2.226,626$, of $168,51$.

From this $168,51$ Subtract the Quarterly Rent 20 ,

Remains $148,51$; it's Logarithm is $2.171,756$

The Logarithm of R , Multiplied with t , is $0.925,596$
 The Multiplier is $0,01706$; it's Logarithm is $8.231,979$ } Add

From the 1st Logarithm Subtract this Logarithm $\underline{9.157,575}$

There Remains the Logarithm, $3.014,181$, of $1033,18$.

Viz. $1033l. 3s. 7d.$ is the present Worth found by the Logarithms.

Case 2. When P , t , and R , are given; To find U .

That is, To find what Annuity, or Lease, &c. may be Purchased for any proposed Sum; to continue any assigned Time, at any given Rate of Interest.

Rule.

Rule. $\left\{ \begin{array}{l} \text{First Multiply the Logarithm of } R \text{ with the Time, and find the Number} \\ \text{which belongs to the Product, and call that Number } x, \text{ (as in the last} \\ \text{Section.)} \\ \text{Again, Multiply the same Logarithm of } R \text{ with the Time more } 1, \text{ and} \\ \text{find the Number which belongs to that Product. Then Multiply the Dif-} \\ \text{ference of those Two Numbers with the proposed present Worth, and their} \\ \text{Product will be a Dividend.} \\ \text{And the first Number called } x, \text{ being made Less by } 1, \text{ will be the Divisor,} \\ \text{by which if the aforesaid Dividend be Divided, the Quotient will shew} \\ \text{the Annuity required for Yearly Payments.} \end{array} \right.$

Example I. For Yearly Rents.

What Annuity, or Yearly Rent, to continue Twenty-one Years, may be purchased for 3205l. 5s. at the Rate of Five per Cent. per Annum?

Here is given $P = 3205,25$, $t = 21$, and $R = 1,05$; To find U .

First, $R = 1,05$; it's Logarithm is $0.021,189$
And $t = 21$ } Multiply

The Product is the Logarithm, $0.444,969$, of $2,7859$, or of the Number called x .

Again, $R = 1,05$; it's Logarithm is $0.021,189$
And $t + 1 = 22$ } Multiply

Their Product is the Logarithm, $0.466,158$, of $2,9252$.

Then $2,9252 - 2,7859 = 0,1393$, the Multiplier.

Next, $P = 3205,25 \times 0,1393$ gives $446,4913$ for the Dividend.

And the Number called x , $2,7859 - 1 = 1,7859$, is the Divisor.

Then $1,7859 \mid 446,4913$ ($250\% = U$, the Annuity required for Yearly Payments.

Or, having once found the Two first Numbers, and by them formed the Multiplier and Divisor, as above, the Multiplication and Division may be performed by Logarithms.

Thus, $P = 3205,25$; it's Logarithm is $3.505,861$
Multiplier is $0,1393$; it's Logarithm is $9.143,951$ } Add

Sum $2.649,812$
The Divisor $1,7859$; it's Logarithm is $0.251,857$ } Subtract

Remains the Logarithm, $2.397,955$, of $250 = U$.

Viz. 250% will be the Yearly Annuity, as before. But this Case is more Universally Resolved by the following Rule.

4 X 2

Rule 2.

Rule 2. $\left\{ \begin{array}{l} \text{First Multiply the Logarithm of } R, \text{ the Amount of 1l., with the} \\ \text{Time, and find the Number which belongs to that Product, which call } x, \\ \text{(As before.)} \\ \text{Next, Multiply the proposed present Worth with } R - 1, \text{ viz. with} \\ \text{the Ratio of the given Rate, and Multiply their Product with the first} \\ \text{Number, called } x. \\ \text{Then Divide the last Product by the Number called } x \text{ made Less by 1,} \\ \text{and the Quotient will shew the Annuity, or Rent for Yearly Payments,} \\ \text{(As before.)} \end{array} \right.$

As, for Instance, in the last Example, wherein $P = 3205,25$, $t = 21$, and $R = 1,05$; To find U .

Then $R = 1,05$; it's Logarithm is $0,021,189$
 And $t = 21$ } Multiply

The Product is the Logarithm, $0,444,969$, of $2,7859$, called x .

Next, $R - 1 = 0,05$; And $P = 3205,25 \times 0,05 = 160,2625$.

Again, $160,2625 \times 2,7859 = 446,4753$, the Dividend.

And $2,7859 - 1 = 1,7859$ is the Divisor.

Then $1,7859 \mid 446,4753$ ($250 = U = 250\text{l.}$, the Annuity, or Rent for Yearly Payments.

The same may be performed by *Logarithms*.

Thus, If $R = 1,05$, and $t = 21$; Then $2,7859$, the Number called x , will be found as before.

And, If $P = 3205,25$; it's Logarithm is $3,505,861$
 $R - 1 = 0,05$; it's Logarithm is $8,698,970$ } Add
 Number called x , $2,7859$; it's Logarithm is $0,444,969$

Sum $2,649,800$
 Number $x - 1 = 1,7859$; it's Logarithm is $0,251,857$ } Subtract

There Remains the Logarithm, $2,397,943$, of 250 .

That is, 250l. will be the Rent for Yearly Payments, (*As before.*)

But, if the Annuity, or Rent, &c. is to be paid either *Half-Yearly*, or *Quarterly*, &c.

Then, $\left\{ \begin{array}{l} \text{Instead of Multiplying the proposed present Worth of the Annuity with} \\ R - 1, \text{ (viz. with the Ratio of the Rate, as in Rule 2,) you must take} \\ \text{such a Part of the Logarithm of } R, \text{ as the Time requires (viz. the } \frac{1}{2}, \\ \text{the } \frac{1}{4}, \text{ or } \frac{1}{12}, \text{ \&c.) and it's Number made less by 1, must be the Multi-} \\ \text{plicator; and then proceed by Rule 2; as in the following Examples.} \end{array} \right.$

Example

Example II. In Half-Yearly Payments.

What Annuity, or Rent, to be paid every Half-Year, and to continue 21 Years, may be purchased for the Sum of 3244l. 5s. at the Rate of 5 per Cent. per Annum?

In this Question there is given $P = 3244,25$, $t = 21$, and $R = 1,05$; To find U , the Half-Yearly Rent.

Thus, R is $= 1,05$; it's Logarithm is $0,021,189$
 And $t = 21$ } Multiply

The Product is the Logarithm, $0,444,969$, of $2,7859$, called x .

Then $2,7859 - 1 = 1,7859$, is the Divisor. *So far the Work is the same as for Yearly Payments.*

Next, $R = 1,05$; the $\frac{1}{2}$ of it's Logarithm is $0,010,594$, whose Number is $1,0247$, and $1,0247 - 1 = 0,0247$, is the Multiplier, instead of $0,05 = R - 1$.

Then $P = 3244,25$ is to be multiplied into $0,0247$; the Product is $80,133$. And $80,133$ is to be multiplied into $2,7859$, the Number called x ; the Product is $223,2432$, which is to be taken for the Dividend.

Then $x - 1 = 1,7859$ $223,2432$ ($125 = U$).

That is, 125% will be the Half-Yearly Annuity, or Rent; as was required by the Question.

Or, having once found the Number called x , and taken 1 from it for the Divisor; and having also found the Multiplier which is to be used instead of $R - 1$, as before: Then the Multiplication and Division may be performed by Logarithms; Thus,

$R = 1,05$; it's Logarithm, Multiplied with t , is $0,444,969$
 And P is $= 3244,25$; it's Logarithm is $3,511,114$
 The Multiplier is $0,0247$; it's Logarithm is $8,392,697$ } Add

Sum $2,348,780$
 The Divisor is $1,7859$; it's Logarithm is $0,251,857$ } Subtract

There Remains the Logarithm, $2,096,923$, of $125 = U$, &c.

Example III. In Quarterly Payments.

What Annuity, or Rent, to be paid every Quarter of a Year, and to continue 31 Years and a Half, may be purchased for 1033l. 3s. 8d. at the Rate of 7 per Cent. per Annum?

Here is given $P = 1033,185$, $t = 31,5$, and $R = 1,07$; To find U , the Quarterly Payment.

First,

First, $R = 1,07$; it's Logarithm is $0.029,384$
 And $t = \underline{\quad 31,5 \quad}$ } Multiply

The Product is the Logarithm $0.925,596$ of $8,4255$, called x .

And $8,4255 - 1 = 7,4255$ will be the Divisor.

Again, $R = 1,07$; the $\frac{1}{4}$ of it's Logarithm is $0.007,346$, whose Number is $1,01706$; and we shall have $1,01706 - 1 = 0,01706$ for the Multiplier, instead of $0,05 = R - 1$.

Then $P = 1033,185$ is to be multiplied into $0,01706$; the Product is $17,6261$. And $17,6261$ is to be multiplied into $8,4255$, the Number called x ; the Product will be $148,5087$, the Dividend.

Then $7,4255 \mid 148,5087$ ($20 = U$; That is, 20% will be the Quarterly Annuity, or Rent; as was required.

The same may be found by *Logarithms*, as in the last Example of Half-Yearly Rents; which I suppose is needless to be set down.

Case 3. When U , P , and R , are given; To find t .

That is, When any Annuity, with it's proposed Value or Purchase, and the Rate of Interest *per Cent*, &c. are given; Thence to find how long the Purchaser ought to enjoy it.

Rule. $\left\{ \begin{array}{l} \text{First, Multiply the proposed Value, or present Worth of the Annuity,} \\ \text{with } R - 1 \text{ (Viz. with the Ratio of the given Rate of Interest); and} \\ \text{Subtract the Product from the Annuity, and reserve the Remainder for a Divisor.} \\ \text{Next, Divide the Annuity by that Divisor, and find the Logarithm of} \\ \text{the Quotient; Then, if that Logarithm be Divided by the Logarithm} \\ \text{of } R, \text{ the Amount of } 11., \text{ the Quotient will shew the Time for Yearly} \\ \text{Payments.} \end{array} \right.$

Example I. For Yearly Rents.

Suppose $3205\text{ l. } 5\text{ s.}$ were Lent upon an Annuity of 250 l. per Annum; In what Time would it pay-off that Debt, allowing the Creditor 5 per Cent. per Annum?

In this Question, there is given $P = 3205,25$, $U = 250$, and $R = 1,05$; To find t .

First, P is $= 3205,25$
 And $R - 1$ is $= \underline{\quad 0,05 \quad}$ } Multiply

And $U = 250 - 160,2625$ leaves $89,7375$, the Divisor.

Next, $89,7375 \mid 250,0000$ ($2,7859$, whose Logarithm is $0.444,969$).

Then $R = 1,05$; it's Logarithm is $0.021,189$ $0.444,969$ ($21 = t$).

That is, 21 Years is the Time required in the Question.

The

The same may be performed by *Logarithms*; Thus,

If $P = 3205,25$; it's Logarithm is $3.505,863$ } Add
 And $R - 1 = 0,05$; it's Logarithm is $8.698,970$ }

The Sum is the Logarithm, $2.204,833$, of $160,2625$.

Next, If $U = 250$; it's Logarithm is $2.397,940$ } Subtract
 $250 - 160,2625 = 89,7375$; it's Logarithm is $1.952,972$ }

$R = 1,05$; it's Logarithm is $0.021,189$ $0.444,968$ ($21 = t$).

Viz. 21 Years is the Time, if the Payments are Yearly.

But if the Annuity, or Rent, is to be paid, either by Half-Yearly, or Quarterly Payments, &c.

Then, { *Instead of Multiplying the proposed present Worth of the Annuity with $R - 1$ (as before), you must take such a Part of the Logarithm of R , the Amount of 1l., as the Time requires; and the Number answering to that Pct., being made Less by 1, will be the Multiplier: and then you may proceed-on as before, according to the Rule.*

Example II. For Half-Yearly Rents.

Admit that a Man should give 3244l. 5s. for an Annuity, or clear Rent of 250l. per Annum; to be paid Half-Yearly (viz. 125l. every Half-Year); How long must he Enjoy that Annuity, to be allowed Five per Cent, &c.?

Here is given $U = 125$, $P = 3244,25$, and $R = 1,05$; To find t , the Time, or Number of Years.

First, $R = 1,05$; the $\frac{1}{2}$ of it's Logarithm is $0.010,594$, whose Number is $1,0247$; and $1,0247 - 1 = 0,0247$, which is to be the Multiplier, instead of $0,05 = R - 1$.

Then $P = 3244,25 \times 0,0247 = 80,1329$.

And $U = 125 - 80,1329$ leaves $44,8671$, the Divisor.

Next, $44,8671 \mid 125,0000$ ($2,786$, whose Logarithm is $0,444,981$, and is to be taken for the Dividend.

Then $R = 1,05$; it's Logarithm is $0.021,189$ $0.444,981$ ($21 = t$).

That is, 21 Years is the Time that is required in the Question.

The same performed by *Logarithms*.

Suppose $R = 1,05$; then the $\frac{1}{2}$ of it's Logarithm is $0.010,594$, whose Number is $1,0247$; and $1,0247 - 1 = 0,0247$, will be the Multiplier, instead of $R - 1$; as before.

Then

Then P is = 3244,25 ; it's Logarithm is $\frac{3.511,114}{8.392,697}$ } Add
 The Multiplier is 0,0247 ; it's Logarithm is

Their Sum is the Logarithm, 1.903,811, of 80,133.

Next, U is = 125 ; it's Logarithm is 2 096,910 } Subtract
 And 125 — 80,133 is = 44,867 ; it's Log. is $\frac{1.651,927}{0.444,983}$ } $(21 = t.$

Then $R = 1,05$; it's Logar. is 0 021,189) 0.444,983 $(21 = t.$

Viz. 21 Years, &c. as before.

Example III. In Quarterly Payments.

Suppose one should give 1033l. 3s. 8d. for an Annuity of 80l. per Annum, to be paid Quarterly (viz. to have 20l. every Quarter of a Year); How long may he Enjoy it, to be allowed Seven per Cent. per Annum?

In this Question, there are given $U = 20$, $P = 1033,185$, and $R = 1,07$; To find t , the Time, or Number of Years.

Thus, $R = 1,07$; the $\frac{1}{4}$ of it's Logarithm is 0.007,346, whose Number is 1,01706 ; and $1,01706 - 1 = 0,01706$ will be the Multiplier, to be used instead of $0,07 = R - 1$.

Then $P = 1033,185 \times 0,01706 = 17,6261$; which, being taken from $U = 20$, leaves 2,3739, for the Divisor.

Then 2.3739 (20,0000 (8,425 ; it's Logarithm is 0.925,570.

Lastly, $R = 1,07$; it's Logarithm is 0.029,384) 0.925,570 $(31,5 = t.$

Viz. the Time required will be 31 Years and a Half.

Which may also be found by *Logarithms*.

Thus, If $R = 1,07$; the $\frac{1}{4}$ of it's Logarithm is 0 007,346, whose Number is 1,01706 ; and $1,01706 - 1$ leaves 0,01706 for the Multiplier, which is to be used instead of $R - 1$.

Then $P = 1033,185$; it's Logarithm is $\frac{3.014,181}{8.231,979}$ } Add
 The Multiplier is 0,01706 ; it's Logarithm is

Their Sum is the Logarithm 1.246,160 of 17,6262.

And, if $U = 20$, then $20 - 17,6262 = 2,3738$, the Divisor.

Again, $U = 20$; it's Logarithm is 1.301,030 } Subtract
 Divisor 2,3738 ; it's Logarithm is $\frac{0.375,444}{0.925,586}$ } $(31,5 = t, \&c ;$

$R = 1,07$; it's Logarithm is 0.029,384) 0.925,586 $(31,5 = t, \&c ;$
 That is, the Time required will be 31 Years and a Half, as before.

Cafe

Case 4. By having U , t , and P , given; To find R .

That is, When any Annuity, or Rent *per Annum*, with the Time of it's Continuance, and it's proposed present Worth for that Time, are given; Thence to find what Rate of Interest *per Cent*, &c. is allowed the Purchaser.

This Case is so difficult, that the Ingenious *Michael Dary*, in his *Treatise of Interest Epitomized*, Page 7, sets it down as very troublesome to be Resolved, in regard (*says he*) that R (*the Thing sought*) cannot be brought to one Side of the Equation solely, and the other Side clear. But I rather think that the true Cause which rendered a perfect Solution of this Case so troublesome to him (*or indeed impossible at that Time,*) was because the Resolving of such high Equations as this Case requires, was not in his Time so well known as now it is.

For the Equation, by which the true Value of R may be found, is this:

$$\text{Viz. } \frac{U}{P} = \frac{U}{P}R^t + R^t - R^{t+1}.$$

Which I reduce to the following Equation, as being much better for our present Purpose,

$$\text{Viz. } U = UR^t - PR^t \times \overline{R - 1}.$$

Either of these Equations is more Adfected than that in Page 702, and requires an Algebraical Solution to discover the true Value of R ; and the said true Value can be no otherwise found here than by the same manner of Approachment which was used before in the 4th Case of the last Section: However, in order to render that Approximation as easy and short as possibly I can, I will set down the following Directions.

Viz.

First, Assume such a Number for the Value of R , as you guess may be the the Amount of 11. at the Rate of the Interest required, (as before in Page 702, &c.) Then Multiply the Logarithm of that supposed R with the Time, and find the Number which belongs to that Product, and Multiply it with the Annuity; Call the Product z .

Next, Multiply the same Number with the proposed present Worth, and Multiply their Product with $R - 1$, (*viz.* with the Ratio of the supposed Rate.)

Then Subtract the last Product from the Product called z . If the Remainder be Equal to (*viz.* be the same with) the Annuity, you have hit right upon the true Value of R : But if it be Less than the Annuity, then is the Value of R taken too Big: And on the contrary, if it be more than the Annuity, we may conclude that the Value of R is too Little; and another Trial must be made accordingly.

Example I. In Yearly Rents.

Suppose there were paid 3205l. 5s. for an Annuity, or a Lease, of 250l. per Annum, to continue 21 Years; What Rate of Interest per Cent, &c. is allowed to the Purchaser?

In this Question, there is given $U = 250$, $t = 21$, and $P = 3205,25$; To find R , the Amount of $1l.$ in one Year.

First, I will suppose R to be 1,06; That is, the Amount of $1l.$ in one Year. at the Rate of 6 per Cent.

Then, if $R = 1,06$; it's Logarithm is 0.025,306
And $t = 21$ } Multiply

The Product is the Logarithm, 0.531,426, of 3,3996.

Then $U = 250$ is to be multiplied into 3,3996; the Product is 849,9, which is the Number called z .

Next, $P = 3205,25$ is to be Multiplied into 3,996; the Product is 10896,5679; And 10896,5679 is to be Multiplied into 0,06 (viz. into $R - 1$); the Product will be 653,7941. Then, if this Number be Subtracted from 849,9, the Number called z , there will Remain 196,1059, which is Less than 250, the Annuity. And therefore I conclude that the supposed R , to wit, 1,06, is too Big.

Again, for a Second Trial, I will suppose $R = 1,05$.

Then, if $R = 1,05$; it's Logarithm is 0.021,189
And $t = 21$ } Multiply

The Product will be the Logarithm, 0.444,969 of 2,7859.

And $250 \times 2,7859 = 696,475$, the Number called z .

Next, $3205,25 \times 2,7859 = 8929,5$, which, being Multiplied with 0,05 = $R - 1$, the Product is 446,475.

Then z , $696,475 - 446,475 = 250,000$, which is the very same with the Annuity. Whence I conclude that the true Value of R is 1,05; And $R - 1 = 0,05$ will be the Ratio of the Rate of Interest sought.

Then it will be, As 1 : is to 0,05 :: So is 100 : to 5, the Rate of Interest per Cent, as was required for Yearly Payments.

The same Trials performed by Logarithms.

Now these Trials, for finding the true Value of R , may be performed by *Logarithms* with a very little Trouble. As, for Instance, in this last Example,

Wherein $U = 250$, $P = 3205,25$, and $t = 21$; To find R .

Suppose

Suppose $R = 1,06$; it's Logarithm is $0.025,306$
 And $t = 21$ } Multiply

Sum $0.531,426$
 Next, U is $= 250$; it's Logarithm is $2.397,940$ } Add

The Sum is the Logarithm, $2.929,366$, of $849,9$, the Number called z .

Again, $R = 1,06$; it's Logarithm, Multiplied with t , is $0.531,426$
 And $P = 3205,25$; it's Logarithm is $3.505,863$ } Add
 $R - 1 = 0,06$; it's Logarithm is $8.778,151$

The Sum is the Logarithm, $2.815,440$, of $653,79$.

Then $z - 653,7$, or $849,9 - 653,79$, will be $= 196,11$, which is Less than 250 , the Annuity. Whence it appears that $R = 1,06$ is too Big.

Secondly, Let $R = 1,05$; it's Logarithm is $0.021,189$
 And $t = 21$ } Multiply

Sum $0.444,969$
 Next, U is $= 250$; it's Logarithm is $2.397,940$ } Add

The Sum is the Logarithm, $2.842,909$, of $696,48$, called z .

Again, P is $= 3205,25$; it's Logarithm is $3.505,863$
 $R = 1,05$; it's Logarithm, Multiplied with t , is $0.444,969$ } Add
 And $R - 1 = 0,05$; it's Logarithm is $8.698,970$

Their Sum is the Logarithm, $2.649,802$, of $446,48$.

Then $z - 446,48$ will be $= 696,48 - 446,48 = 250$, which is just the same with the Annuity; Therefore the true Value of R is $1,05$, &c. As before.

But if the Annuity, or Rent, is to be paid, either Half-Yearly, or Quarterly, &c.

Then, { *Instead of Multiplying with $R - 1$, viz. with the Ratio of the supposed Rate of Interest (as above), you must take such a Part of the Logarithm of R (the assumed Amount of $1l.$), as the Time requires, and make it's Number the Multiplier, &c. As in the following Examples.*

Example II. In Half-Yearly Payments.

Admit 3244l. 5s. were given for an Annuity, or Lease, of 250l. per Annum, to be paid Half-Yearly, (viz. 125l. every Half-Year,) and to continue 21 Years; What Rate of Interest per Cent. is allowed the Purchaser?

4 Y 2

In

In this Example, there is given $U = 125$, $t = 21$, and $P = 3244,25$; To find R , the Amount of 1*l*.

First, I suppose $R = 1,06$; it's Logarithm is $0.025,306$
And $t = 21$ } Multiply

Their Product is the Logarithm, $0.531,426$, of $3,3996$.

Then $U = 125$ is to be Multiplied into $3,3996$; the Product is $424,95$, called z .

Next, $P = 3244,25$ is to be Multiplied into $3,3996$; the Product is $11029,1523$.

Again, If $R = 1,06$; the $\frac{1}{2}$ of it's Logarithm is $0.012,653$, whose Number is $1,02956$; and $1,02956 - 1 = 0,02956$, the Multiplier, instead of $0,06 = R - 1$.

Then $11029,1523 \times 0,02956 = 326,0217$, which, being Subtracted from $424,95$, called z , there will Remain $98,9283$; which is Less than 125 , the Half-Yearly Annuity. Whence I conclude that the supposed $R = 1,06$ is taken too Big.

And therefore, for a Second Trial, I will suppose $R = 1,05$, and proceed by *Logarithms* only.

Then, if $R = 1,05$; it's Logarithm is $0.021,189$
And $t = 21$ } Multiply

Sum $0.444,969$
Next, $U = 125$; it's Logarithm is $2.096,910$ } Add

Their Sum is the Logarithm, $2.541,879$ of $348,24$, called z .

Again, if $R = 1,05$; the $\frac{1}{2}$ of it's Logarithm is $0.010,594$, whose Number is $1,0247$; from which if we Subtract 1 , the Remainder 0.0247 will be the Multiplier.

Then $P = 3244,25$; it's Logarithm is $3.511,114$
And $R = 1,05$; it's Logarithm, Multiplied with t , is $0.444,969$
Multiplier $0,0247$; it's Logarithm is $8.392,697$ } Add

Their Sum is the Logarithm, $2.348,780$, of $223,24$.

Then $z - 223,24$ will be $= 348,24 - 223,24 = 125$, which is the very same with 125 , the Half-Yearly Annuity. And therefore I conclude that $R = 1,05$, and $R - 1 = 0,05$, is the Ratio of the Rate of Interest sought.

Consequently, As $1 : 1$ is to $0,05 ::$ So is $100 : 5$, the true Rate of Interest per Cent, &c.

I shall close up this Case with an *Example* of finding the Rate of Interest, that is allowed to the Purchasers of the *Parliamentary* Annuities; and Resolve it by *Logarithms* only.

Example

Example III. Of Quarterly Payments.

There is 1600l. paid for an Annuity of 100l. per Annum, to continue 99 Years, and to be paid Quarterly, viz. 25l. every Quarter of a Year during that Time: It is required to find what Rate of Interest per Cent. is allowed the Purchaser.

In this Question, there is given $U = 25$, $t = 99$, and $P = 1600$; To find R , &c.

First, I will here suppose the Rate of Interest sought to be 6 per Cent, for the Reasons mentioned in Page 686.

Then, if $R = 1,06$; it's Logarithm is 0.025,306
And $t = 99$ } Multiply

The Product is 2.505,294
Next, U is = 25; it's Logarithm is 1.397,940 } Add

The Sum is the Logarithm, 3.903,234, of 8002,6, called z .

Again, if $R = 1,06$; the $\frac{1}{4}$ of it's Logarithm is 0.006,326, whose Number is 1,01467; from which if we take 1, the Remainder 0.01467 will be the Multiplier.

Then $P = 1600$; it's Logarithm is 3.204,120
The Logarithm of R , Multiplied with t , is 2.505,294 } Add
The Multiplier is 0.01467; it's Logarithm is 8.166,430

Their Sum is the Logarithm, 3.875,844, of 7513,5;
Which, being taken from 8002,6 = z , leaves 489,1, which is More than 25 = U . And therefore the Value of R is taken too Little.

And, for a Second Trial, I do assume $R = 1,064$.

Then, if $R = 1,064$; it's Logarithm is 0.026,942
And $t = 99$ } Multiply

The Product is 2.667,258
Next, U is = 25; it's Logarithm is 1.397,940 } Add

The Sum is the Logarithm, 4.065,198 of 11619,8 = z .

Again, the $\frac{1}{4}$ of the Logarithm of R is 0.006,735; it's Number is 1,01563. And 1,01563 - 1 is = 0.01563, the Multiplier.

Then P is = 1600; it's Logarithm is 3.204,120
The Logarithm of R , Multiplied with t , is 2.667,258 } Add
The Multiplier is 0.01563; it's Logarithm is 8.193,959

Their Sum is the Logarithm, 4.065,337, of 11623,5, which is more than 11619,8 = z ; and therefore the Value of this $R = 1,064$ is some small

small matter too Big. Consequently R is some Number between 1,06 and 1,064, and is but a very little Less than 1,064.

Let us therefore suppose R to be = 1,0638 ;

And then it's Logarithm will be 0.026,860
And $t = 99$ } Multiply

Product 2.659,140
Next, $U = 25$; it's Logarithm is 1.397,940 } Add

The Sum is the Logarithm, 4.057,080, of 11414,6 = z .

Again, since R is supposed to be = 1,0638 ; the $\frac{1}{4}$ of it's Logarithm is 0.006,715, whose Number is 1,01558 ; from which if we Subtract 1, the Remainder 0.01558 will be the *Multiplicator*.

Then P is = 1600 ; it's Logarithm is 3.204,120
The Logarithm of R , Multiplied with t , is 2.659,140
The *Multiplicator* is 0,01558 ; it's Log. is 8.192,567 } Add

The Sum is the Logarithm, 4.055,827, of 11377.

Then, if we Subtract this Number 11377 from the Number called z , to wit, 11414,6, the Remainder, or 11414,6 — 11377, will be = 37,6 ; which is now a small matter More than 25 = U .

Consequently this last Value of R , to wit, 1,0638, is a very small matter too Little, or Less than it's true Value, which I judge must be betwixt 1,0638 and 1,0639 ; and accordingly I make one other Trial by assuming the Value of $R = 1,063855$, and find it pretty near to the Truth.

Consequently $R - 1 = 0,063855$, the Ratio of the Rate. And then it will be, As 1 : 0,063855 :: 100 : 6,3855. That is, 6l. 7s. 8½d. will be very near the true Rate of Interest *per Cent* ; as was required.

And, for a Confirmation of the Truth thereof, let us take it for granted that $R = 1,063855$, as above ; and let there be given the Quarterly Annuity, or $U = 25$ l., the Time, or $t = 99$ Years ; To find P , the present Worth (*per Case 1.*) which we already know must be 1600l., if the Value of R be truly found. Then,

If $R = 1,063855$; it's Logarithm is 0.026,882
And $t = 99$ } Multiply

Their Product is 2.661,318
Next, U is = 25 ; it's Logarithm is 1.397,940 } Add

The Sum is the Logarithm, 4.059,258, of 11461,94.

From this 11461,94 Subtract the Quarterly Rent 25, there Remains 11436,94, the Dividend.

Again,

Again, $R = 1,063855$; the $\frac{1}{4}$ of it's Logarithm is $0.006,720$, whose Number is $1,015592$, and $1,015592 - 1 = 0.015,592$, will be the Multiplier, instead of $0,063855 = R - 1$.

The Dividend $11436,94$; it's Logarithm is $4.058,312$

Logarithm of R , Multiplied with t , is	$2.661,318$	} Add
The Multiplier is $0,015592$; it's Logarithm is	$8.192,902$	

From the upper Logarithm Subtract this Logarithm $0.854,220$ Sum

There Remains the Logarithm, $3.204,092$, of $1599,9 = P$.

That is, the present Worth thus found is $1599\text{ l. }18\text{ s.}$, which wants but 2 s. of 1600 l. , the true present Worth: Whence we may be assured, that the Value of R , and consequently the Rate of Interest *per Cent*, found as above, is very near the Truth, for Quarterly Payments. But, supposing those Annuities were to be paid but once a Year, then the Rate of Interest will be only $6\text{ l. }5\text{ s. per Cent}$, or very near it, as may be easily tried, either by finding the present Worth, as in *Example I, Page 707*; or by finding the Annuity, as in *Example I, Page 711, &c.*

Thus you have a full Solution at large to the Eight Cases, which include all the Varieties that can be proposed, either relating to the Arrears, or about the Purchasing of Annuities, and the Taking of such Leases as are either in present Possession when the Contract is made, or are to be immediately Entered upon. But for such Questions, as relate to Annuities, and Taking such Leases, &c. as are in Reversion, they require a further Consideration.

SECT. IV. To find the Present Worth of Annuities, or Leases, &c. in Reversion, at Compound Interest.

I HAVE already touched upon this Part of the Work, *Page 685, &c.* as it may be performed at *Simple Interest*; but shall here handle it more fully in all it's useful Cases. For, although they only depend upon the due Application of the Rules already laid down; yet it will be convenient to shew how those Rules are to be truly Applied, according as the particular Cases require.

Case 1. When the Annuity, or Yearly Rent, with the Time of it's Continuance in Possession, and the Time of the Reversion, (*viz. the Time it is not to be in Possession,*) are given; To find it's Present Worth, at any assigned Rate of Interest *per Cent*.

The

The Solution of this Case requires Two distinct Operations, which must be thus performed.

- First, $\left\{ \begin{array}{l} \text{Find the Value, or present Worth, of the proposed Annuity, or Rent,} \\ \text{for the given Time of it's Continuance, as if it were immediately to be} \\ \text{Entered-upon.} \end{array} \right.$
- Then $\left\{ \begin{array}{l} \text{Find what Principal, or Sum, being forborn at Interest, during the} \\ \text{Time of the Reversion, would Amount to, or Raise, the aforesaid Value;} \\ \text{and that Principal will be the Present Worth of the proposed Annuity,} \\ \text{\&c. in Reversion.} \end{array} \right.$

Example in Yearly Rents.

There is the Reversion of a Lease of 175l. per Annum, to be Lett for Eleven Years, which are not to commence until after Nine Years are expired: It is required to find the Present Worth of that Lease, allowing the Rate of Six per Cent. Interest to the Purchaser.

Or thus:

There is One has Nine Years to come in a Lease of 175l. per Annum, and he is desirous to enlarge his Time Eleven Years more (viz. to Enjoy his Lease Twenty Years yet to come); What Sum must he give in Ready Money for that Purchase, to be allowed the Rate of 6 per Cent, &c.?

In the first Work of this Example, there is given $U = 175$, $t = 11$, and $R = 1,06$; To find P , the Present Worth, as in Case 1, Page 707.

Thus, $R = 1,06$; it's Logarithm is $0.025,306$
 And $t = 11$ } Multiply

The Product is $0.278,366$
 Again, U is $= 175$; it's Logarithm is $2.243,038$ } Add

The Sum is the Logarithm $2.521,404$ of $332,203$.

Then from $332,203$ Subtract the Rent 175 , and there

Remains $157,203$; it's Logarithm is $2.196,459$

The 1st Logarithm, Multiplied with $t = 11$, is $0.278,366$
 And $R - 1 = 0,06$; it's Logarithm is $8.778,151$ } Add

From the upper Logarithm Subtract this Log. $9.056,517$

There Remains the Logarithm, $3.139,942$, of $1380,2 = P$, the Present Worth, supposing the Rent were to be immediately Entered upon; but because it is not so, therefore, for the Second Part of the Work there is given $A = 1380,2$, $t = 9$, and $R = 1,06$; To find P , the Principal, as in Case 2, Page 690.

Thus,

Thus, A is $= 1380,2$; it's Logarithm is $3.139,943$ }
 And R is $= 1,06$; it's Logarithm is $0.025,306$; } $0.227,754$ } Subtract
 which being Multiplied into 9 , or t , is

And there Remains the Logarithm, $2.912,189$, of $816,937 = P$.

Viz. $816l. 18s. 9d.$ will be the true Present Worth of that Lease in Reversion, as was required, *viz.* for Yearly Rents.

But, if the Rents are to be paid, either *Half-Yearly*, or *Quarterly*, &c; Then the first Part of the Work must be found, as in their respective Examples, *viz.* For Half-Yearly Payments it must be found as in the Example, Page 708. And for Quarterly Payments, as in the Example, Page 709. And then proceed-on, as above.

Case 2. To find what Annuity, or Yearly Rent, in Reversion, may be purchased for any proposed Sum, at any assigned Rate of Interest *per Cent*; when the Time during which the Annuity is not to be Entered-upon, and the Time of it's Continuance, are both given.

The Solution of this Case is but the Reverse of the Last, and doth also require Two distinct Operations.

First, { Find what the Sum proposed to be laid-out in the Reversion, would Amount to, supposing it were forborn at the given Rate of Interest, during the Time that the Annuity is not to be in Possession.

Then { Find what Annuity, or Yearly Rent, that Amount will Purchase: and that will be the Answer required.

Example in Yearly Rents.

What Annuity, or Yearly Rent, to commence Nine Years hence, and then to continue Eleven Years after, may be Purchased for $816l. 18s. 9d.$ Ready Money; at the Rate of 6 per Cent, &c.

In the first Work of this Question, there are given $P = 816,937$, $R = 1,06$, and $t = 9$ (the Time during which the Annuity is not to be Entered-upon); To find A ; as in *Case 1*, Page 689.

Thus, $R = 1,06$; it's Logarithm is $0.025,306$ }
 And $t = 9$ } Multiply

The Product is $0.227,754$ }
 Next, P is $= 816,937$; it's Logarithm is $2.912,189$ } Add

The Sum is the Logarithm, $3.139,943$, of $1380,2 = A$, the Amount of $816l. 18s. 9d.$, which we must now call P , for the Second Part of the Work; wherein there is given $P = 1380,2$, $R = 1,06$, and $t = 11$ (the Time during which the Annuity is to be Enjoyed); To find U ; as in *Case 2*, Page 710.

Thus, R is = 1,06 ; it's Logarithm is 0.025,306 } Multiply
 And t = 11

The Product is the Logarithm, 0.278,366, of 1,8983.

And 1,8983 — 1 = 0,8983 is the Divisor.

Again, R is = 1,06 ; it's Logarithm is 0.025,306 } Multiply
 And $t + 1$ = 12

Their Product is the Logarithm, 0.303,672, of 2,0122.

And 2,0122 — 1,8983 = 0.1139, the Multiplier.

Then P is = 1380,2 ; it's Logarithm is 3.139,943 } Add
 The Multiplier is 0,1139 ; it's Logarithm is 9.056,524

Sum 2.196,467 } Subtract
 The Divisor 0,8983 ; it's Logarithm is 9.953,421

There Remains the Logarithm, 2.243,046, of 175 = U .

That is, 175*l. per Annum* is the Rent, or Annuity, required by the Question.

But if the Rents are to be paid every *Half-Year*, or every *Quarter*, &c. Then the Second Part of the Work must be performed as in their respective Examples ; *Viz.* For Half-Yearly Payments, U must be found, as in the Example, Page 713. And for Quarterly Payments, as in the Example, Page 713, &c.

Case 3. Any Annuity, or Yearly Rent, and the Sum that is proposed to be paid for a Lease of it, to commence after a certain Term, or Number of Years, shall have been past, being given ; Thence to find how Long that Lease ought to continue in the Possession of the Purchaser, allowing him any assigned Rate of Interest *per Cent.*

This Case does also require Two separate Operations.

Viz. { First, To find what Sum the proposed Money to be laid-out in the Reversion, would Amount to, &c. As in the first Part of the Last Case.

Then, { Having the Annuity, with it's present Worth, or Value, and the Rate of Interest, given ; To find the Time, &c. As in Case 3, Page 714. •

Example in Yearly Rents.

Suppose a Debt of 816*l.* 18*s.* 9*d.* were proposed to be paid-off, by making over a Lease of 175*l. per Annum* in Reversion, that is not to be Entered-upon, by the Creditor, until Nine Years are past ; The Question is, How many Years the Creditor must

must Enjoy that Lease, in order to have his Debt Cleared; and to be allowed the Rate of 6 per Cent. per Annum?

For the First Work in this Question, there are given $P = 816,937$, $R = 1,06$, and $t = 9$ (the Time before the Lease is to commence); To find A , which will be found $= 1380,2$, as in the first Part of the Last Case; which we must here also call P , for the Second Part of the Work.

Then there will be given new $P = 1380,2$, $U = 175$, and $R = 1,06$; To find the $t =$ to the Time during which the Lease is to be Enjoyed; as in Example I, Page 714.

Thus, P is $= 1380,2$; it's Logarithm is $3.139,942$ } Add
 $R - 1 = 0,06$; it's Logarithm is $8.778,151$ }

The Sum is the Logarithm, $1.918,093$, of $82,812$.

Next, U is $= 175$; it's Logarithm is $2.243,038$ } Subtract
 $175 - 82,812$ is $= 92,188$; it's Logarithm is $1.964,674$ }

R is $= 1,06$; it's Logarithm is $0.025,306$ $0.278,364$ ($11 = t$).

Viz. 11 Years will be the Time required by the Question; supposing the Rents to be paid but once a Year.

But if the Rents are to be paid every Half-Year, Then t , the Time the Lease is to be Enjoyed, may be found as in Example, Page 715. And if Quarterly, then as in Example, Page 716, &c.

Now these Three Cases include all the usual (and useful) Questions that relate to Purchasing Annuities, or Taking of Leases in Reversion; and, although I have thought it best to shew how to Resolve each of them by Two separate and distinct Operations, (being, as I presume, the easiest Way to make them understood,) yet they may be Resolved at One Operation by a due Reduction of the following Equation. For,

Putting $\begin{cases} t = \text{the Time the Lease, \&c. is to be Enjoyed;} \\ T = \text{the Time of it's Reversion;} \end{cases}$

We may thence derive the Equation $UR^t - U = PR^T \times R^t \times R - 1$.

And from hence the Value of R , viz. the Rate of Interest, may also be found. But this I take to be rather a Work of Curiosity, than of any great Use: For the Rate of Interest is always supposed to be known, it being one of the main or chief Considerations in all Bargains about Taking, or Letting, of Leases, &c. And therefore the finding of R , in the Case of Reversions, (or, indeed, in the Two 4th Cases of the Two last Sections,) is rather for a Proof of the Work, and to Compleat the Rounds, than for any Thing else. How-

ever, if any one will (out of his Curiosity) give himself the Trouble of doing it, he may proceed by such Approximation as in the aforesaid Sections, having a due Regard to the different Values of t , and T . But, for Brevity's Sake, I shall omit giving Examples thereof.

C H A P. V.

Of Purchasing Freehold, or Real Estates, and how to Estimate the Value of Annuities, and Leases, for Lives, &c.

ALL Freehold or Real Estates are supposed to be Purchased (or Bought) to continue for Ever; *Viz.* without the Consideration of any limited Time. Therefore the Business of Computing the true Value, or Worth, of such Estates (*according to Art*) is grounded upon a Rank, or Series, of Geometrical Proportionals, continually Decreasing *ad Infinitum*; (*As may be seen in the aforesaid Young Mathematician's Guide, Page 276.*) from whence the following Rules are deduced.

SECT. I. Of Purchasing Freehold Estates.

I SHALL here make Use of the same Letters to denote the several Parts of the Question, as in the last Chapter, save only that t , the Time to limit the Number of Payments, is not concerned.

Viz. $\begin{cases} U = \text{The proposed Rent of an Estate.} \\ P = \text{The Purchase, or Worth of that Estate.} \\ R = \text{The Amount of 1l. for one Year, at any Rate of Interest.} \end{cases}$

Then it will be $PR - P = U$.

This Equation admits of Three different Cases.

Case 1. The Annual Rent of any Freehold Estate being known; To find what is it's Worth in Ready Money, allowing the Purchaser any assigned Rate of Interest *per Cent.*

Rule. $\begin{cases} \text{Divide the proposed Rent by the Ratio of the given Rate of Interest} \\ \text{(viz. by } R - 1), \text{ and the Quotient will shew what the Estate is} \\ \text{Worth at that Rate of Interest.} \end{cases}$

Example

Example I.

Suppose a Freehold Estate of 250l. Yearly Rent, were to be Sold; What is it Worth, allowing the Buyer the Rate of 6 per Cent. Compound Interest for his Money?

Here is given $U = 250$, $R = 1,06$; To find P .

Thus, $R - 1 = 0,06$ $250,00$ ($4166,6667 = P$).

That is, $4166l. 13s. 4d.$ is the Present Worth of that Estate; which is 16 Years and 8 Months Value, supposing the Rents to be paid but once a Year.

The same may be found by *Logarithms*.

Thus, $U = 250$; it's Logarithm is $2.397,940$
And $R - 1 = 0,06$; it's Logarithm is $8.778,151$ } Subtract

There Remains the Logarithm, $3.619,789$, of $4166,66 = P$.

Viz. $4166l. 13s. 4d.$ &c. As above.

Note, Here it is supposed that the Purchaser is to Enter immediately into Possession of the Estate: But, if it be an Estate in Reversion, that is, not to be Entered-upon until after some Term of Years shall be past; Then you must Compute it's present Worth, as in Case 1, Page 723. And so on in the following Cases, compared with their respective Cases in Sect. IV. of the last Chapter.

If the Rents are to be paid either Half-Yearly, or Quarterly, as most generally they are;

Then, { Instead of Dividing the proposed Rent by $R - 1$, as above, you must take such a Part of the Logarithm of R , (the Amount of 1l. for a Year,) as the Times of Payment require; and the Number answering to that Part, being made Less by 1, must be made the Divisor, instead of $R - 1$.

Example in Half-Yearly Rents.

Suppose the aforesaid 250l. per Annum, were to be paid by Half-Yearly Rents (viz. 125l. every Half-Year); What would that Estate be then Worth, at the same Rate of Interest, viz. 6 per Cent, &c.

In this Example, there are given $U = 125$, and $R = 1,06$; To find P . Thus,

First, R is $= 1,06$; the $\frac{1}{2}$ of it's Logarithm is $0.012,653$, whose Number is $1,029563$; and $1,029563 - 1 = 0,029563$ is to be taken for the New Divisor, instead of $R - 1$.

Then

Then U is = 125 ; it's Logarithm is 2.096,910 } Subtract
 The Divisor is 0,029563 ; it's Logarithm is 8.470,748

There Remains the Logarithm, 3.626,162, of 4228,26 = P .

So that 4228*l.* 5*s.* is here the present Worth, or Value, of the same Estate, which is 61*l.* 11*s.* 4*d.* More than it's Worth when the Rents are to be paid Yearly.

Example III. For Quarterly Rents.

*Let it be required to find what the same Estate (viz. 250*l.* per Annum) is Worth, when the Rents are to be paid Quarterly (viz. 62*l.* 10*s.* every Quarter of a Year), at 6 per Cent. Interest, &c. As before.*

Here are given $U = 62,5$, and $R = 1,06$; To find P .

Thus, $R = 1,06$; the $\frac{1}{4}$ of it's Logarithm is 0,006326, whose Number is 1,014674 ; and $1,014674 - 1 = 0,014674$, is to be the Divisor, instead of $R - 1$.

Then U is = 62,5 ; it's Logarithm is 1.795,880 } Subtract
 The Divisor is 0,014674 ; it's Logarithm is 8.166,548

There Remains the Logarithm, 3.629,332, of 4259,24 = P .

Viz. 4259*l.* 4*s.* 9½*d.* is now the Value, or present Worth, of the same Estate ; which is something above 17 Years' Value.

Now from hence it plainly appears, that, although there is no such Thing as a limited Time considered in the Purchasing of Freehold Estates ; yet there is, and ought to be, a due Respect had to the Times of their Rents being paid ; for, according to that, their Values are more, or less, Worth.

Case 2. When any Sum of Money is proposed to be laid-out in a Freehold Estate ; To find what Annual Rent that Sum will Purchase, at any given Rate of Interest *per Cent.*, &c.

Rule. { *Multiply the proposed Sum to be laid-out in the Estate, with $R - 1$; viz. with the Ratio of the given Rate of Interest per Cent ; and the Product will shew the Yearly Rent required.*

Example in Yearly Rents.

*Suppose 3560*l.* were proposed to be laid-out in the Purchase of a Freehold Estate ; What Annual Rent would it Buy, allowing the Purchaser but Four per Cent. per Annum Interest ?*

In this Example, there is given $P = 3560$, $R = 1,04$; To find U , the Yearly Rent.

Thus,

Thus, P is = 3560 } Multiply.
 And $R - 1$ is = 0,04 }

The Product is $142,4 = 142l. 8s. = U$, the Yearly Rent required.

Or the same may be found by *Logarithms*.

Thus, P is = 3560 ; it's Logarithm is $3.551,450$ } Add
 And $R - 1$ is = 0,04 ; it's Logarithm is $8.602,060$ }

The Sum is the Logarithm, $2.153,510$, of $142,4 = U$.

Viz. $142l. 8s.$ will be the Rent required, if it be paid but once a Year ; which is just 25 Years' Value.

But, if the Rent must be paid Half-Yearly, or Quarterly, &c.

Then, { *Instead of Multiplying the Sum proposed to be Laid-out in the Purchase, into $R - 1$ (as above), you must take such a Part of the Logarithm of R (viz. of the Amount of $1l.$), as the Times of Payment require ; and the Number answering to that Part, being made Less by 1, must be the Multiplier, instead of $R - 1$.*

Take the same Example in Half-Yearly Rents.

That is, Let there be given $P = 3560$, and $R = 1,04$; To find U , the Half-Yearly Rent.

First, $R = 1,04$; the $\frac{1}{2}$ of it's Logarithm is $0.008,516$, whose Number is $1,0198$; which, being made Less by 1, is 0.0198 , the Multiplier.

Then $3560 \times 0,0198 = 70,488$, *viz.* $70l. 9s. 9d. = U$, the Half-Yearly Rent.

Or the same may be found by Logarithms, as follows ;

Having found the New Multiplier, 0.0198 , as above ;

Then $P = 3560$; it's Logarithm is $3.551,450$ } Add
 The Multiplier is $0,0198$; it's Logarithm is $8.296,665$ }

The Sum is the Logarithm, $1.848,115$, of $70,488 = U$;

Viz. $70l. 9s. 9d.$ is the Half-Yearly Rent, as before.

Example III. For Quarterly Rents.

Suppose the same Sum of 3560l. were proposed to be Laid-out in the Purchase of a Freehold Estate ; and it were required to find what Quarterly Rent it would Buy, at the Rate of Four per Cent, &c.

Here are also given $P = 3560$, and $R = 1,04$; To find U , the Quarterly Rent. Thus,

First,

First, R is $= 1,04$; the $\frac{1}{4}$ of it's Logarithm is $0,004,258$, whose Number is $1,00986$; and $1,00986 - 1 = 0,00986$, is the New Multiplier.

Then $3560 \times 0,00986$ gives $35,1$, viz. $35^l. 2s. = U$, the Rent to be paid every Quarter; which is Less than the $\frac{1}{4}$ of the Annual Rent, or $142^l. 8s.$, by $10s. \text{ per Quarter}$.

Whence it plainly appears, that the Less the Intervals of the Times are, betwixt the Payments of the Rents, the More valuable the Purchase is; & *vice versa*.

Case 3. The Annual Rent of any Freehold Estate, and the Sum that is paid for it, being known; To find what Rate of Interest *per Cent.* is allowed to the Purchaser.

Rule. $\left\{ \begin{array}{l} \text{Divide the Annual Rent by the Sum that is paid for the Purchase,} \\ \text{and the Quotient will shew the Ratio of the Rate of Interest per Cent.} \end{array} \right.$

Example.

Suppose a Freehold Estate of $250^l.$ per Annum, Cost $4166^l. 13s. 4d.$ What Rate of Interest per Cent. is allowed the Purchaser?

In this Example, there are given $U = 250$, and $P = 4166,667$; To find R , or rather, $R - 1$, the Ratio of the Rate.

Thus, $4166,667 \div 250,000 (0,06 = R - 1)$.

Then it will be, As $1 : 1$ is to $0,06 ::$ So is $100 : 6$, the Rate of Interest *per Cent.* As was required.

Or, by Logarithms; Thus,

First, U is $= 250$; it's Logarithm is $2,397,940$
And P is $= 4166,667$; it's Logarithm is $3,619,789$ } Subtract

The Difference is the Logarithm, $8,778,151$, of $0,06 = R - 1$.
Or to the last Logarithm Add this, $2,000,000$, of 100 .

The Sum will be the Logarithm, $0,778,151$, of 6 , the Rate of Interest *per Cent.*; As before.

Thus far concerning such Questions relating to Interest, Annuities, or Leases, &c. as are limited by any assigned Time, and Real Estates that are Purchased to Continue for Ever; And it is only such Questions that can be Resolved by any Certain Rules. However, it may not (I presume) be unacceptable to the Reader if I proceed a little further, and insert a brief Account of such Helps to Estimate the Values of Annuities, or Leases, for Lives, as have been proposed by Two very Ingenious Persons.

SECT. II. How to *Estimate the Values of Annuities, or Leases, for Lives.*

THE Way generally used in Buying of Annuities, or Letting of Leases, for Lives; is only by an imaginary Valuation, grounded upon Custom, and not upon any Consideration that is had to the Age of the Persons whose Lives are to be inserted in the Lease, &c. It is true, indeed, that there can be no certain Rules prescribed for their true Valuation, because the Lives of all Mankind are uncertain; and it is possible, and daily seen, that a Young Man may Die before one of a greater Age. But yet there is a greater Probability of a Young Man's Living long than of an Old one: and not only so, but there is a Proportional Likelihood of the Length of Men's Lives, according to their different Degrees of Age; The which being duly considered, must needs be found of good Use in Estimating the Values of Annuities, or Leases, for Lives, much better than by a mere Guessing only, as usual. And that such a Proportional Likelihood is worth the Consideration, will appear from what follows.

1. Sir *William Petty*, in a Discourse of his, made before the Royal Society (*Anno 1674*), concerning the Use of *Duplicate Proportion*, doth, amongst other Things, apply it to the Life of Man, and it's Duration; Thus,

"It is found by Experience (*saitb be*), that there are more Persons Living of between 16 and 26 Years old, than of any other Age, or Decad of Years, in the whole Life of Man (*which David and Experience say to be between 70 and 80 Years*). The Reasons whereof are not abstruse; *viz.* because those of 16 Years of Age have passed the Danger of *Teeth, Convulsions, Worms, Rickets, Meazles*, and *Small-Pox*, for the most Part; and those of 26 are scarce come to the *Gout, Stone, Dropsy, Palsies, Lethargies, Apoplexies*, and other Infirmities of Old Age. Now, whether these be sufficient Reasons, is not the present Enquiry; but, taking the afore-mentioned Assertions to be true, I say, that the Root of every Number of Men's Ages under 16 (*whose Root is 4*), compared with the said Number 4, doth shew the Proportion of the Likelihood of such Men's reaching to 70 (*or 80*) Years of Age.

As, for Example; It is Four Times more likely, that One of 16 Years old should live to 70, than a new-born Babe. It is Three Times more likely, that One of 9 Years old should attain the said Age of 70, than the said Infant. Moreover, it is Twice as likely, that One of 16 should reach that Age, as that One of 4 Years old should do it; and One Third more likely, than for One of 9.

On the other Hand, It is Five to Four, that One of 25 Years old will Die before One of 16; and Six to Five, that One of 36 will Die before One of 25; and Six to Four, or Three to Two, that the same Person of 36 shall Die before him of 16: And so forward, according to the Root of any other Year of the declining Age, compared with a Number between 4 and 5 (*viz. with 4,58*),

which is the Root of 21, the most hopeful Year for Longevity, as the Mean between 16 and 26; and is the Year of Perfection, according to the Sense of *Our Law*, and the Age for whose Life a *Lease* is most valuable.

To prove all which, I can produce the Accompts of every Man, Woman, and Child, within a certain Parish of 330 Souls; All which particular Ages being Added together, and the Sum Divided by the whole Number of Souls, made the *Quotient* between 15 and 16; which I call the Age of that Parish, or Numerous *Index* of Longevity there."

Thus you have a Learned Gentleman's Opinion concerning the Likelihood of the Length of Men's Lives, according to the Rules of *Duplicate Proportion*; which was a very ingenious Thought of His. But I must beg Pardon, that I cannot agree with Him in that Part of it, which asserts, That 21 Years is the Age for whose Life a Lease is the most valuable: For, although it is true, that, according to our Law, a Man is said to be then at his Perfect, or Full, Age, as to the Enjoyment of an Estate, or Managing his Affairs without a Guardian, &c. Yet I should rather adhere to the Close of His Discourse, wherein He says, That He found, that the Sum of all the Ages of the 330 Souls (*in a certain Parish*), being Divided by their Number, made the *Quotient* between 15 and 16. Whence I take 16 to be the Age, for whose Life a Lease is the most valuable; and, upon that Supposition, I have Calculated the following Table, according to the aforefaid Rules of *Duplicate Proportion*.

Ages.	Year's Purchase.	Ages.	Year's Purchase.	Ages.	Year's Purchase.
1	2,50	26	7,20	51	5.04
6	5,51	31	6,46	56	4,81
11	7,46	36	6,00	61	4,60
16	9,00	41	5,62	66	4,43
21	7,85	46	5,31	71	4,27

This Table shews, by Inspection, the Value of every Five Years of any single Life, from the Birth to 71 Years old. Supposing that any Annuity, or Lease, &c. is really Worth Nine Year's Purchase for One Life; which is according to the Rate that the Annuities, settled by a late Act of Parliament, for Lives, were valued at, and from thence the rest are computed.

As, for Instance; An Annuity, or Lease, of 100*l. per Annum*, is, by this Table, Worth 900*l.* (*viz.* 9×100) for the Life of One that is about 16 Years old; and it is Worth but 600*l.* for the Life of One that is 36 Years old, and but 250*l.* for the Life of a Child of a Year old: And so on for any other proposed Annuity, or Lease, and Age of the Person for whose Life it is to be Purchased: Which is so easy to apprehend, that I need say no more of it's Uses. And therefore I shall proceed to mention what has been advanced on this Subject by the other eminent Person who has treated of it.

2. This Person is that Ingenious and Great Mathematician, Mr. *Edmund Halley*, *Savilian Professor* of *Geometry* in the *University* of *Oxford*; who (*in the Philosophical Transactions*, Num. 196.) hath given us a most Excellent *Essay* to Estimate the Degrees of the *Mortality* of Mankind, which he deduced, with a great deal of Art and Labour, from curious *Tables* of the *Births* and *Funerals* that were in *Breslaw*, the Capital City of the Province of *Silesia* in *Germany*, for Five Years successively, viz. from the 1687, to 1691 inclusive, drawn up Monthly by one *Doctor Newman* of that City, and communicated to the *Royal Society* here; from whence Mr. *Halley* hath formed the following *Table*; which is, without doubt, very carefully and exactly done, and does give a more perfect Account of the State and Condition of Mankind in Respect of their *Chances* of *Mortality* at all *Ages*, and consequently teaches us how to Estimate the *Values* of *Annuities*, or *Leases*, for *Lives*, better than could be done by any other Method ever proposed before it. The *Table* does shew the *Number* of *Persons* that were Living in their respective *Ages* *Current* annexed thereto, as follows.

Age Curr.	Persons Living.	Age Curr.	Persons Living.	Age Curr.	Persons Living.	Age Curr.	Persons Living.
1	1000	22	586	43	417	64	202
2	855	23	579	44	407	65	192
3	798	24	573	45	397	66	182
4	760	25	567	46	387	67	172
5	732	26	560	47	377	68	162
6	710	27	553	48	367	69	152
7	692	28	546	49	357	70	142
8	680	29	539	50	346	71	131
9	670	30	531	51	335	72	120
10	661	31	523	52	324	73	109
11	653	32	515	53	313	74	98
12	646	33	507	54	302	75	88
13	640	34	499	55	292	76	78
14	634	35	490	56	282	77	68
15	628	36	481	57	272	78	58
16	622	37	472	58	262	79	49
17	616	38	463	59	252	80	41
18	610	39	454	60	242	81	34
19	604	40	445	61	232	82	28
20	598	41	436	62	222	83	23
21	592	42	427	63	212	84	20

This *Table* may be Applied to very many Uses; but I shall only insert what may suffice for the present Purpose.

1. The First Use is to shew the Different Degrees of *Mortality*, or rather *Vitality*, in all *Ages*; For, if the Number of Persons of any Age, Remaining after One Year, be *Divided* by the *Difference* between that and the *Number* of the *Age* proposed, the *Quotient* will shew the *Odds* that there are, that a Person of that Age will not *Die* in One Year.

As, for Instance, a Person of 25 Years of Age has the *Odds* of 560 to 7 (*viz.* 80 to 1), that he does not *Die* in a Year: Because that, of 567 Persons Living of 25 Years of Age, there do *Die* no more than 7 in a Year, Leaving 560 of 26 Years Old: Or,

2. If it be required to find the *Odds*, that any Person does not *Die* before he attain to any proposed *Age*. Then,

Take the *Number* of the Remaining Persons of the *Age* proposed, and *Divide* it by the *Difference* between it and the *Number* of those of the *Age* of the Party proposed; and that shews the *Odds* there are between the *Chances* of the Parties Living, or Dying.

As, for Instances; What are the *Odds* that a Man of 40 may Live 7 Years? Take the *Number* of Persons of 47 Years, which in the *Table* is 377, and *Subtract* it from the *Number* of Persons of 40 Years, which is 445, and the *Difference* is 68, *viz.* $445 - 377 = 68$; which shews that the Persons Dying in the 7 Years are 68, and consequently that it is 377 to 68, or $5\frac{1}{2}$ to 1, that a Man of 40 will Live 7 Years: And the like for any other Number of Years.

3. And, if it be required to find at what *Number of Years*, it is an even Lay that a Person of any proposed *Age* shall *Die*, this *Table* readily performs it. For, if the *Number* of Persons *Living* of the *Age* proposed be halved, it will be found by the *Table* at what Year the said *Number* is Reduced to Half by *Mortality*; and that is the *Age*, to which it is an even Wager, that a Person of the *Age* proposed shall arrive before he *Die*.

As, for Instance, a Person of 30 Years of *Age* is proposed; the *Number* of that *Age* is 531, the Half of it is 265; which *Number* I find to be between 57 and 58 Years: So that a Man of 30 may reasonably expect to Live between 27 and 28 Years.

4. By what has been said, the *Price of Insurance* upon *Lives* ought to be Regulated, and the *Difference* is discovered between the *Price of Insuring* the *Life* of a Man of a Young Age and that of insuring the *Life* of a much older Man.

For Example; It is 592 to 598 — 592, or to 6, or nearly 100 to 1, that a Man of 20 Years of Age *Dies* not in a Year; and it is but (335 to 346 — 335, or to 11, or $\frac{335}{11}$ to $\frac{346}{11}$, or) 30.5 to 1, for a Man of 50 Years of Age, that he will not *Die* in a Year.

5. And upon these *Proportions* depends the *Valuation of Annuities* for *Lives*: For it is plain, that the Purchaser ought to pay only for such a Part of the *Annuity*, as he hath Chances that he may Live to Enjoy; and this ought to be Computed Yearly, and the Sum of all those Yearly *Values* being *Added* together, will be the

the *Value* of the *Annuity* for the *Life* of the *Person* proposed. Now the present *Value* of *Money*. payable after any *Term* of *Years*, may be easily *Computed* by *Case 2*, *Page 690*, at any given *Rate* of *Interest*.

Then, $\left\{ \begin{array}{l} \text{As the Number of Persons Living after that Term of Years, to the} \\ \text{Number Dead; So are the Odds that any one Person is Alive or Dead.} \\ \text{And by Consequence, As the sum of both, or the Number of Persons} \\ \text{Living of the Age first proposed : to the Number Remaining after so} \\ \text{many Years (both being given by the Table) :: So is the present Value} \\ \text{of the Yearly Sum payable after the Term proposed : to the Sum which} \\ \text{ought to be paid for the Chance the Person has to Enjoy such an Annuity} \\ \text{so many Years. And this being repeated for every Year of the Person's} \\ \text{Life, the Sum of all the present Values of those Chances is the true} \\ \text{Value of the Annuity.} \end{array} \right.$

Now, because the *Work* of these *Proportions* is somewhat troublesome to perform; the *Ingenious Author* hath been so kind as to take the *Pains* (which was not a little) to Calculate the following *Table*; which shews the *Value* of *Annuities*, or *Leases*, &c. (at the *Rate* of 6 per Cent.) for every *Fifth Year* of *Age* to the *Seventieth*.

Ages.	Year's Purchase.	Ages.	Year's Purchase.	Ages.	Year's Purchase.
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32 *

This *Table* being of the same *Nature* with that in *Page 734*, there needs no other *Explanation*, or *Example*, to shew it's *Use*, than what has been already said about that *Table*: Only here I must again beg leave to give my *Opinion* about the *Difference* of the *Proportions* in the *Two Tables*; which is, that, as the *Table* in *Page 734* may not be thought a sufficient *Guide* to be depended-upon in *Estimating* the *Length* of *Men's Lives*, &c., because it is only deduced from the bare *Rules* of *Art*, viz. that of *Duplicate Proportion*; so, on the other *Hand*, I doubt the *Estimates* of the *Value* of any *Annuity*, taken from this *Table*, will be found too great in this *Country* (viz. in *England*), which I much fear hath not so *Good* or *Salubrious* an *Air*, as that at the *City* of *Breslaw*, from whence these *Calculations* are drawn. But if, in *Imitation* hereof, the *Curious* in other *Cities* and *Large Towns* would attempt something of the same *Nature*; then, without all *Doubt*, this *Method* of *Estimating* the *Probability* of the *Length* of

* By this *Table* it appears that the age of 10 years (and not that of 16, or of 21, year,) is that at which a *Life-annuity* is of the greatest value.

Men's Lives would prove the Best, and become more universally Useful than can be expected from this one single Instance, more especially if such Observations were continued for any considerable Time, as 20, or 30 Years successively. And then it would be well worth the Time and Pains to calculate proper *Tables*, of the *Values* of *Annuities*, or *Leases*, &c. both for Two, and for Three, Lives, according to the following Rules; which are deduced from the former *Table*.

And, First, Two Lives being proposed, their Values may be thus found.

If the *Number* of *Chances* of each single Life, found in the *Table*, be Multiplied together, the *Product* is the *Chances* of those Two Lives; And, after any certain Term of Years, the *Product* of the Two Remaining Sums will be the *Chances* that both the Persons are Living: And the *Product* of the *Differences*, being the *Numbers* of the *Dead* of both *Ages*, will be the *Chances* that both the Persons are *Dead*.

Then, $\left\{ \begin{array}{l} \text{As the Product of the Two Numbers in the Table for the Two Ages} \\ \text{proposed : is to the Difference between that Product and the Product of} \\ \text{the Two Numbers of the Persons Deceased in any given Space of Time} \\ \text{:: So is the Value of a Sum of Money to be paid after that Time : to} \\ \text{the Value thereof under the Contingency of Mortality.} \end{array} \right.$

And, if Three Lives are proposed, to find the Value of an Annuity during the Continuance of those Lives;

Then, $\left\{ \begin{array}{l} \text{As the Product of the continual Multiplication of the Three Numbers} \\ \text{in the Table, answering to the Ages proposed : is to the Difference of} \\ \text{that Product, and the Product of the Three Numbers of the Deceased} \\ \text{of those Ages in any given Term of Years :: So is the present Value of} \\ \text{a Sum of Money, to be paid certainly after so many Years : to the present} \\ \text{Value of the same Sum to be paid, provided One of those Three Persons} \\ \text{be Living at the Expiration of that Term of Years.} \end{array} \right.$

These Proportions being Yearly repeated, the Sum of all those present Values will be the Value of the *Annuity* granted for those *Lives*.

These Rules are Explained at Large by their Author, both *Algebraically* by Letters, and by *Geometrical Figures*: And he also proceeds-on (*by the same Method*) to Compute the present Value of the *Reversion* of any *Annuity*, or *Lease*, either of One Life after another; or after Two Lives, &c. The which being, not only too long a Discourse to be inserted in this small *Treatise*, but also too Difficult a Piece a Work for any Ordinary Arithmetician to undertake, I have therefore omitted it, and refer those that are Curious, and desire further Satisfaction therein, to the afore-mentioned *Philosophical Transactions*, Number 196, and shall only Add this serious Observation, *Viz.* How unjustly we generally Repine at the Shortness of our Lives, and think ourselves Wronged if we attain not to Old Age; Whereas it appears, that the One Half of those that

that are *Born*, do *Die* in *Seventeen Years' Time*. For, by the aforefaid Bills of *Mortality* at *Breslaw*, it was found, that 1238 were in that Time Reduced to 616. So that, instead of *Murmuring* at what we call a *Short Life*, we ought with *Patience* and *Unconcern* to submit to that *Dissolution* which is the necessary Condition of our perishable Materials, and of our nice and frail Structure and Composition; And to account it as a great *Blessing* that we have survived (perhaps by many Years,) that *Period* of *Life*, whereat the One Half of the whole Race of Mankind does not Arrive.

Having now gone through all the General Cases of Interest, and Annuities, &c. I designed to have concluded here; but, because the Business of *Rebate*, or *Discompt* of Money paid before the Time it becomes Due, comes often into Practice upon several Occasions, and being but just touched-upon in Page 667 and 690, although even what is there done, being duly considered, might be sufficient; yet, lest I should be thought too short or remiss in so useful a Part of Interest as that is, it may be convenient to proceed a little further, and lay down particular Rules for that Purpose.

C H A P. VI.

To find the *Rebate*, or *Discompt*, of any Proposed Sum, &c. and the true *Equated Time* of several Payments; either, at *Simple*, or *Compound Interest*.

SECT. I. To find the true *Discompt* of any Sum, at any Rate of *Simple Interest*.

Suppose $\begin{cases} S = \text{the Sum proposed to be Discompted for;} \\ T = \text{the Interval of Time before it becomes Due;} \\ R = \text{the Ratio of the Rate of Interest per Cent.} \end{cases}$

And $D = \text{the Discompt, or Rebate, sought.}$

Then it will be $\left\{ \frac{TRS}{TR + 1} = D; \right.$

Which, in Words, gives this following Rule.

Rule. $\begin{cases} \text{First, Multiply the given Time with the Ratio of the Rate of Interest,} \\ \text{and to their Product Add 1, that Sum will be the Divisor.} \\ \text{Next, Multiply the first Product, and the Sum that is to be Dis-} \\ \text{compted for, and that Product will be the Dividend; The Quotient} \\ \text{arising from thence will shew the Discompt required.} \end{cases}$

Example.

Example.

Let it be required to find what Discompt ought to be allowed for 3560l., if it be paid 273 Days before it becomes Due, at the Rate of Six per Cent. per Annum, Simple Interest?

Here is given $S = 3560$, $T = 0,74794$ (found by the Table in Page 670); and $R = 0,06$; To find D .

First, $0,74794 \times 0,06 = 0,044876$.

And $0,044876 + 1$ is $1,044876$ for the Divisor.

Next, $0,044876 \times 3560 = 159,75856$, the Dividend.

Then $1,044876 \overline{) 159,75856}$ ($152,897 = D$).

That is, $152l. 18s. ferè$, will be the Discompt required.

The same performed by *Logarithms*.

First, $T = 0,7494$; it's Logarithm is $\overline{9} 873,867$
 And $R = 0,06$; it's Logarithm is $\overline{8} 778,151$ } Add

The Sum is the Logarithm, $\overline{8} 652,018$, of $0,044876$.

Next, $S = 3560$; it's Logarithm is $3.551,450$ Add to the last.

Sum $2.203,468$
 $0,044876 + 1 = 1,044876$; it's Logarithm is $0.019,066$ } Subtract

There Remains the Logarithm, $2.184,402$, of $152,897$.

That is, $152l. 18s.$, as above; which being Subtracted from $3560l.$, the proposed Sum, there will Remain $3407l. 2s.$, the Sum to be paid in Ready Money; as may be easily proved by making it a Principal, and then finding what it would Amount to in 273 Days, at 6 per Cent; which, by Case 1, Page 665, will be just $3560l.$ Consequently the Discompt is truly found.

Secondly, To find the true Rebate, or Discompt, of any Sum, at any given Rate of Compound Interest.

Let us suppose S , t , and D , to represent the same Parts of the Question, as before; and R , to denote the Amount of $1l.$ for One Year, &c. as in Page 689.

$$\text{Then will } \left\{ \frac{SR^t - S}{R^t} = D. \right.$$

And from this Equation is deduced the following Rule.

Rule.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Logarithm of } R, \text{ (the Amount of 1l. for a Year,) at the} \\ \text{given Rate of Interest, with the Time; and the Number which belongs} \\ \text{to their Product will be the Divisor.} \end{array} \right.$

From that Number Subtract 1; then Multiply the Remainder with the proposed Sum that is to be Discounted for; and that Product will be the Dividend: the Quotient arising from thence will shew the Discount required.

Example.

Suppose it were required to find what Discount, or Rebate, must be allowed for the Payment of 956l. 10s. Nine Months (viz. $\frac{3}{4}$ of a Year) before it becomes Due, at the Rate of Five and a Half per Cent. Compound Interest.

In this Question, there is given $S = 956,5$, $t = 0,75$, and $R = 1,055$; To find D .

First, R is $= 1,055$; it's Logarithm is $0,023,252$ } Multiply
And t is $= 0,75$ }

The Product is the Logarithm, $0,017,439$, of $1,04097$, the Divisor.

And $1,04097 - 1 = 0,04097$ will be the Multiplier; and $956,5 \times 0,04097 = 39,1878$, the Dividend. Then $1,04097$ $39,1878$ ($37,645 = D$; viz. 37l. 12s. 10 $\frac{1}{4}$ d. is the Discount required.

Or, having first found the Divisor, and by it the Multiplier, as above; Then the rest of the Work is very easily performed by *Logarithms*.

Thus, S is $= 956,5$; it's Logarithm is $2,980,685$ } Add
The Multiplier is $0,4097$; it's Logar. is $8,612,466$ }

Sum $1,593,151$ } Subtract
Divisor is $1,04097$; it's Logarithm is $0,017,439$ }

There Remains the Logarithm, $1,575,712$, of $37,645$.

Viz. 37l. 12s. 10 $\frac{1}{4}$ d. is the Discount required, as above; which, being taken from the 956l. 10s., leaves 918l. 17s. 1 $\frac{1}{4}$ d., the Sum which is to be paid in Ready Money; as may be tried, by making it a Principal, and then finding what it would Amount to in $\frac{3}{4}$ of a Year, at the Rate of Five and a Half per Cent; the which, by *Case 1*, *Page 689*, will be found to be just 956l. 10s. Therefore, &c.

If these Two Rules, and their Examples, be a little considered, I presume it will be very easy to conceive how to find the true Discount, or Rebate, of any One proposed Sum of Money, Due at the End of any given Interval of Time, and any given Rate, either of Simple, or Compound Interest, according as the Question is proposed.

And, if it be required to find the whole Discompt of several Sums, Due at the End of several different Intervals of Time, it is but Computing them at so many several Operations, and then the Sum of all those particular Discompts, being taken from the Total Sum of all the Debts, will leave the true Discompt required.

As, for Instance ;

Suppose A were indebted to B 750l., to be paid at Three several Payments, in this manner ; viz. 250l. at the End of One Year and a Half, 100l. to be paid at the End of Two Years, and 400l. at the End of Four Years ; The Question is, To find how much of the 750l. B ought to Rebate, or Discompt, at Six per Cent. per Annum, Simple Interest, to have his Whole Debt discharged by A in present Money ?

According to the Data in this Question, the particular Discompts (found by the First Rule in this Section,) will be these following :

$$\text{The Discompt of } \left\{ \begin{array}{l} 250l. \text{ for } 1,5 \text{ Year is } 20,6423 \\ 100l. \text{ for } 2 \text{ Years is } 10,7142 \\ 400l. \text{ for } 4 \text{ Years is } 77,4193 \end{array} \right\} \text{ Add}$$

Consequently the whole Discompt is 108,7758.

Then, if from 750l. the whole Debt, there be Subtracted
108,7758, the whole Discompt, there will Remain

641,2242 = 641l. 4s. 5d. which is the true Sum that A must give to B, in present Money, to be Discharged of his Debt.

Again, Suppose the same Things given as above, and let it be required to find the particular Discompts, &c. at Compound Interest.

Then, by the Second Rule in this Section, the Work will stand thus :

$$\text{The Discompts of } \left\{ \begin{array}{l} 250l. \text{ for } 1,5 \text{ Year is } 20,9238 \\ 100l. \text{ for } 2 \text{ Years is } 11,0003 \\ 400l. \text{ for } 4 \text{ Years is } 83,1633 \end{array} \right\} \text{ Add}$$

Sum is 115,0874

Then 115,0874, taken from 750l., leaves 634,9126.

Viz. 634l. 18s. 3d. is now the Sum which A is to give B, in present Money, to be clear of his Debt.

This is so plain to be understood, that I presume it is needless to say any more in Proof of it, than what has been already said in Page 741.

And from hence Naturally flows the following Method of finding the true Equated Time, wherein several Sums, Due at several Intervals of Time, may be

be paid at one entire Payment, without any Loss, either to the Creditor or Debtor.

SECT. II. The Equated Time of Payments truly Determined.

THE usual Rule laid down in divers Treatises of Arithmetick and Interest, &c. for finding of an Equated Time, for the Payment of several Sums of Money Due at the End of unequal Intervals of Time, is to this Effect.

Rule. $\left\{ \begin{array}{l} \text{Multiply every single Sum of Money with the Time it becomes Due;} \\ \text{and Divide the Sum of those Products by the Total Debt; and the} \\ \text{Quotient will be the true Time (say they) at which all the Money ought} \\ \text{to be paid.} \end{array} \right.$

I shall pass over all the Arguments that are made Use of *pro* and *con*, by Mr. John Kersey, in his Book of Arithmetick, and by Sir Samuel Morland, in his Doctrine of Interest, and other Authors, about the Erroneousness of this Rule; As also, the Rules they lay down instead of it; and shall only proceed to shew how the true Equated Time may be found, from what hath been already done and proved.

When any Number of Payments are proposed to be paid-off at one entire Sum; Then, in order to determine the Equated Time for the Payment of that Sum, you must first find the particular Discompts of all the proposed Payments, whose Times are to be Equated, according to the given Intervals of those Payments, at any Rate, either of Simple, or Compound Interest, as shall be agreed on by the Parties concerned; and by those Discompts find the present Sum that would Clear the Debt, if it were immediately paid; (*As above.*)

Then, $\left\{ \begin{array}{l} \text{If the Sum of all those particular Discompts be Divided by the Product} \\ \text{of the present Sum Multiplied with the Ratio of the same Rate of Interest} \\ \text{by which those Discompts were computed, the Quotient will shew the} \\ \text{true Equated Time required at Simple Interest.} \end{array} \right.$

For an Example of this Rule;

Let it be required to find the true Equated Time, wherein the aforesaid three Sums, Due from A to B (as in the last Instance), may be safely paid without Loss to either, &c.

There the Sum of all the particular Discompts at Six per Cent, Simple Interest, is found to be 108,7758; and $750 - 108,7758 = 641,2242$, the present Sum, if it were to be immediately paid.

5 B 2

Then,

Then, by the Rule above, $641,2242 \times 0,06 = 38,47345$ will be the Divisor; and 108,7758, the Dividend.

Then $38,47345 \div 108,7758 = 2,8273$ is the Quotient.

That is, 2 Years and 302 Days will be the Time when *A* may pay unto *B* his whole Debt of 750*l.*, without any Loss to either of them.

The same found by *Logarithms*.

The Sum of *Discompts* is 108,7758; it's Logarithm is 2.036,544

The *Present Worth* is 641,2242; it's Logarithm is 2.807,010
The *Ratio* of the *Rate* is 0,06; it's Logarithm is 8.778,151 } Add

From the 1st Logarithm Subtract this Logarithm, 1.585,161 Sum;

There Remains the Logarithm, 0.451,383, of 2.8273.

Viz. 2 Years and 302 Days will be the true Equated Time required by the Question; as before.

For, if the *Discompts* be truly found, then it cannot be denied, but that the *Present Worth* of the whole Debt may by them be truly found: And, I say, if that *present Worth* be made a *Principal*, and the *Equated Time* (*as here found*) be made the Time of that *Principal's* being forborn at the same Rate of Interest, it will be found (*per Case 1, Page 665.*) to Amount to just the whole Debt. Therefore the *Equated Time* is here truly found.

And, if it be required to find the *Equated Time* at any proposed Rate of Compound Interest, you must proceed, in the same Manner, to find all the particular *Discompts* of the given Sums, and by them the *present Sum* that would Clear the Debt, if it were immediately paid; as in Page 741.

Then, { If the Logarithm of the present Sum be taken from the Logarithm of the Total Sum of all the Debts, and the Remainder be Divided by the Logarithm of *R*, (the Amount of 1*l.* for a Year,) at the same Rate of Interest by which those *Discompts* were Computed; The Quotient will shew the *Equated Time* required.

As, for Example.

Suppose it were required to find the *Equated Time* at Compound Interest, wherein the 750*l.* Due from *A* to *B*, in the same Manner as before, may be paid at one entire Payment, without Loss, &c.

Then the Sum of all the particular *Discompts*, found at 6 per Cent, Compound Interest, is 115,0874.

And $750 - 115,0874 = 634,9126$ is the present Sum that would Discharge the Debt, if it were immediately paid, as in Page 742.

Then, by the Rule above, it will be

The Present Sum is 634,9126 ; it's Logar. is $\underline{2.802,714}$ } Subtract
 750 ; it's Logarithm is $2.875,061$

R is $= 1,06$; it's Logarithm is $0.025,306$) $0.072,347$ (2,8588.

That is, 2 Years, and 313 Days, is here found to be the Equated Time that A must Pay the 750*l.* unto B , at one entire Payment ; which is but 11 Days more than the Time found at Simple Interest.

And the Truth of the Equated Time, thus found at Compound Interest, may be easily proved by the Help of *Case 1*, Page 690, &c. in the same Manner as that of Simple Interest was in the last Page.

F I N I S.

And the first thing I noticed when I stepped out of the car was the cold. It was a sharp contrast to the warm blanket of the car. I shivered as I walked towards the entrance of the building. The door was open, and I stepped inside. The interior was dimly lit, and I could see the silhouettes of people sitting at tables. I walked towards the bar, and a waiter approached me. He asked if I had a reservation, and I told him I did. He led me to a table, and I sat down. I looked around the room, and I noticed that the atmosphere was quiet and intimate. I took a deep breath and felt a sense of calm.

I looked down at my hands, and I noticed that they were trembling. I took a deep breath and tried to steady myself. I looked up at the waiter, and he smiled at me. He asked if I wanted a drink, and I told him I would like a glass of water. He brought me a glass, and I took a sip. I felt a sense of relief, and I looked up at the waiter. He asked if I wanted anything else, and I told him I would like a glass of wine. He brought me a glass, and I took a sip. I felt a sense of calm, and I looked up at the waiter. He asked if I wanted anything else, and I told him I would like a glass of wine.

I looked down at my hands, and I noticed that they were trembling. I took a deep breath and tried to steady myself. I looked up at the waiter, and he smiled at me. He asked if I wanted a drink, and I told him I would like a glass of water. He brought me a glass, and I took a sip. I felt a sense of relief, and I looked up at the waiter. He asked if I wanted anything else, and I told him I would like a glass of wine. He brought me a glass, and I took a sip. I felt a sense of calm, and I looked up at the waiter. He asked if I wanted anything else, and I told him I would like a glass of wine.

A TABLE OF LOGARITHMS,

For *Numbers* Increasing in their proper Order, from 1 to 10,000.

With their *Differences*.

Num.	Logarithms.	Num.	Logarithms.	Num.	Logarithms.
1	0.000,000	34	1.531,479	67	1.826,075
2	0.301,030	35	1.544,068	68	1.832,509
3	0.477,121	36	1.556,302	69	1.838,849
4	0.602,060	37	1.568,202	70	1.845,098
5	0.698,970	38	1.579,784	71	1.851,258
6	0.778,151	39	1.591,065	72	1.857,332
7	0.845,098	40	1.602,060	73	1.863,323
8	0.903,090	41	1.612,784	74	1.869,232
9	0.954,242	42	1.623,249	75	1.875,061
10	1.000,000	43	1.633,468	76	1.880,814
11	1.041,393	44	1.643,453	77	1.886,491
12	1.079,181	45	1.653,212	78	1.892,095
13	1.113,943	46	1.662,758	79	1.897,627
14	1.146,128	47	1.672,098	80	1.903,090
15	1.176,091	48	1.681,241	81	1.908,485
16	1.205,120	49	1.690,196	82	1.913,814
17	1.230,449	50	1.698,970	83	1.919,078
18	1.255,272	51	1.707,570	84	1.924,279
19	1.278,754	52	1.716,003	85	1.929,419
20	1.301,030	53	1.724,276	86	1.934,498
21	1.422,219	54	1.732,394	87	1.939,519
22	1.342,423	55	1.740,363	88	1.944,483
23	1.361,728	56	1.748,188	89	1.949,390
24	1.380,211	57	1.755,875	90	1.954,242
25	1.397,940	58	1.763,428	91	1.959,041
26	1.414,973	59	1.770,852	92	1.963,788
27	1.431,364	60	1.778,151	93	1.968,483
28	1.447,158	61	1.785,330	94	1.973,128
29	1.462,398	62	1.792,392	95	1.977,724
30	1.477,121	63	1.799,341	96	1.982,271
31	1.491,362	64	1.806,180	97	1.986,772
32	1.505,150	65	1.812,913	98	1.991,226
33	1.518,514	66	1.819,544	99	1.995,635

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
100	000,000	000,434	000,868	001,301	001,734	002,166	002,598	003,029	003,460	003,891	432
101	004,321	004,751	005,181	005,609	006,038	006,466	006,894	007,321	007,748	008,174	428
102	008,600	009,026	009,451	009,876	010,300	010,724	011,147	011,570	011,993	012,415	424
103	012,837	013,259	013,679	014,100	014,520	014,940	015,360	015,779	016,197	016,615	420
104	017,033	017,451	017,868	018,284	018,700	019,116	019,532	019,947	020,361	020,775	416
105	021,189	021,603	022,016	022,428	022,841	023,252	023,664	024,075	024,486	024,896	411
106	025,306	025,715	026,124	026,533	026,942	027,350	027,757	028,164	028,571	028,978	408
107	029,384	029,789	030,195	030,601	031,004	031,408	031,812	032,216	032,619	033,021	404
108	033,424	033,826	034,227	034,628	035,029	035,430	035,830	036,229	036,629	037,028	401
109	037,426	037,825	038,223	038,622	039,017	039,414	039,810	040,207	040,602	040,998	397
110	041,393	041,787	042,182	042,575	042,969	043,362	043,755	044,148	044,540	044,931	393
111	045,323	045,714	046,105	046,495	046,885	047,275	047,664	048,053	048,442	048,830	390
112	049,218	049,606	049,993	050,380	050,766	051,152	051,538	051,924	052,309	052,694	386
113	053,078	053,463	053,846	054,230	054,613	054,996	055,378	055,760	056,142	056,524	383
114	056,905	057,28	057,666	058,046	058,426	058,805	059,185	059,563	059,942	060,320	379
115	060,698	061,075	061,452	061,829	062,206	062,582	062,958	063,333	063,708	064,083	376
116	064,458	064,832	065,206	065,580	065,953	066,326	066,698	067,071	067,443	067,814	373
117	068,186	068,557	068,928	069,298	069,668	070,038	070,407	070,776	071,145	071,514	370
118	071,882	072,250	072,617	072,985	073,352	073,718	074,085	074,451	074,816	075,182	366
119	075,547	075,912	076,276	076,640	077,004	077,368	077,731	078,094	078,457	078,819	364
120	079,181	079,543	079,904	080,266	080,626	080,987	081,347	081,707	082,067	082,426	361
121	082,785	083,144	083,503	083,861	084,219	084,576	084,934	085,291	085,647	086,004	357
122	086,360	086,716	087,071	087,426	087,781	088,136	088,490	088,845	089,199	089,552	355
123	089,905	090,258	090,611	090,963	091,315	091,667	092,018	092,369	092,719	093,071	352
124	093,422	093,772	094,122	094,471	094,820	095,169	095,518	095,866	096,215	096,562	349
125	096,910	097,257	097,604	097,951	098,297	098,644	098,990	99,335	99,680	100,026	347
126	100,371	100,715	101,059	101,403	101,747	102,091	102,434	102,777	103,119	103,462	344
127	103,804	104,145	104,487	104,828	105,169	105,510	105,850	106,191	106,531	106,870	341
128	107,210	107,549	107,888	108,227	108,565	108,903	109,241	109,578	109,916	110,253	338
129	110,590	110,926	111,262	111,598	111,934	112,270	112,605	112,940	113,275	113,609	330
130	113,943	114,277	114,611	114,944	115,278	115,610	115,943	116,276	116,608	116,940	332
131	117,271	117,603	117,934	118,265	118,595	118,926	119,256	119,586	119,915	120,245	330
132	120,574	120,903	121,231	121,560	121,888	122,216	122,543	122,871	123,198	123,525	328
133	123,852	124,178	124,504	124,830	125,156	125,481	125,806	126,131	126,456	126,781	325
134	127,105	127,429	127,752	128,076	128,399	128,722	129,045	129,368	129,690	130,012	323
135	130,334	130,655	130,977	131,298	131,618	131,939	132,260	132,580	132,900	133,219	321
136	133,539	133,858	134,177	134,496	134,814	135,133	135,451	135,768	136,086	136,403	319
137	136,721	137,037	137,354	137,671	137,987	138,303	138,618	138,934	139,249	139,564	316
138	139,879	140,194	140,508	140,822	141,136	141,450	141,763	142,076	142,389	142,701	314
139	143,015	143,327	143,639	143,951	144,263	144,574	144,885	145,196	145,507	145,818	311
140	146,128	146,438	146,748	147,058	147,367	147,676	147,985	148,294	148,603	148,911	309
141	149,219	149,527	149,835	150,142	150,449	150,756	151,063	151,370	151,676	151,982	307

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
142	.152,288	.152,594	.152,900	.153,205	.153,510	.153,815	.154,119	.154,424	.154,728	.155,032	305
143	.155,336	.155,640	.155,943	.156,246	.156,549	.156,852	.157,154	.157,457	.157,759	.158,061	303
144	.158,362	.158,664	.158,965	.159,266	.159,567	.159,868	.160,168	.160,469	.160,769	.161,068	301
145	.161,368	.161,667	.161,967	.162,266	.162,564	.162,863	.163,161	.163,459	.163,758	.164,055	298
146	.164,353	.164,650	.164,947	.165,244	.165,541	.165,838	.166,134	.166,430	.166,726	.167,022	297
147	.167,317	.167,613	.167,908	.168,203	.168,497	.168,792	.169,086	.169,380	.169,674	.169,968	295
148	.170,262	.170,555	.170,848	.171,141	.171,434	.171,726	.172,019	.172,311	.172,603	.172,895	292
149	.173,186	.173,478	.173,769	.174,060	.174,351	.174,641	.174,932	.175,222	.175,512	.175,802	290
150	.176,091	.176,381	.176,670	.176,959	.177,248	.177,536	.177,825	.178,113	.178,401	.178,689	289
151	.178,977	.179,264	.179,552	.179,839	.180,126	.180,413	.180,699	.180,986	.181,272	.181,558	287
152	.181,844	.182,129	.182,415	.182,700	.182,985	.183,270	.183,555	.183,839	.184,123	.184,407	285
153	.184,691	.184,975	.185,259	.185,542	.185,825	.186,108	.186,391	.186,674	.186,956	.187,239	283
154	.187,521	.187,803	.188,084	.188,366	.188,647	.188,928	.189,210	.189,490	.189,771	.190,051	281
155	.190,332	.190,612	.190,892	.191,171	.191,451	.191,730	.192,010	.192,289	.192,567	.192,846	279
156	.193,125	.193,403	.193,681	.193,959	.194,237	.194,514	.194,792	.195,069	.195,346	.195,623	278
157	.195,900	.196,176	.196,453	.196,729	.197,005	.197,281	.197,556	.197,832	.198,107	.198,382	276
158	.198,657	.198,932	.199,206	.199,481	.199,755	.200,029	.200,303	.200,577	.200,850	.201,124	274
159	.201,397	.201,670	.201,943	.202,216	.202,488	.202,761	.203,033	.203,305	.203,577	.203,848	272
160	.204,120	.204,391	.204,663	.204,934	.205,204	.205,475	.205,746	.206,016	.206,287	.206,556	271
161	.206,826	.207,096	.207,365	.207,634	.207,904	.208,173	.208,441	.208,710	.208,978	.209,247	269
162	.209,515	.209,783	.210,051	.210,319	.210,586	.210,853	.211,120	.211,388	.211,654	.211,921	267
163	.212,188	.212,454	.212,720	.212,986	.213,253	.213,518	.213,783	.214,049	.214,314	.214,579	266
164	.214,844	.215,109	.215,373	.215,638	.215,902	.216,166	.216,430	.216,694	.216,957	.217,221	264
165	.217,484	.217,747	.218,010	.218,273	.218,536	.218,798	.219,060	.219,323	.219,585	.219,846	262
166	.220,108	.220,370	.220,631	.220,892	.221,153	.221,414	.221,675	.221,936	.222,196	.222,456	261
167	.222,716	.222,976	.223,236	.223,496	.223,755	.224,015	.224,274	.224,533	.224,792	.225,051	260
168	.225,309	.225,568	.225,826	.226,084	.226,342	.226,600	.226,858	.227,115	.227,372	.227,630	258
169	.227,887	.228,144	.228,400	.228,657	.228,913	.229,170	.229,426	.229,682	.229,938	.230,193	257
170	.230,449	.230,704	.230,960	.231,215	.231,470	.231,724	.231,979	.232,233	.232,488	.232,742	254
171	.232,996	.233,250	.233,504	.233,757	.234,011	.234,264	.234,517	.234,770	.235,023	.235,276	253
172	.235,528	.235,781	.236,033	.236,285	.236,537	.236,789	.237,041	.237,292	.237,544	.237,795	252
173	.238,046	.238,297	.238,548	.238,799	.239,049	.239,299	.239,550	.239,800	.240,050	.240,300	250
174	.240,549	.240,799	.241,048	.241,297	.241,546	.241,795	.242,044	.242,293	.242,541	.242,790	249
175	.243,038	.243,286	.243,534	.243,782	.244,030	.244,277	.244,524	.244,772	.245,019	.245,266	247
176	.245,513	.245,759	.246,006	.246,252	.246,499	.246,745	.246,991	.247,236	.247,482	.247,728	246
177	.247,973	.248,219	.248,464	.248,709	.248,954	.249,198	.249,443	.249,687	.249,932	.250,176	244
178	.250,420	.250,664	.250,908	.251,151	.251,395	.251,638	.251,881	.252,125	.252,367	.252,610	243
179	.252,853	.253,096	.253,338	.253,580	.253,822	.254,064	.254,306	.254,548	.254,790	.255,031	242
180	.255,272	.255,514	.255,755	.255,996	.256,236	.256,477	.256,718	.256,958	.257,198	.257,439	241
181	.257,679	.257,918	.258,158	.258,398	.258,637	.258,877	.259,116	.259,355	.259,594	.259,833	239
182	.260,071	.260,310	.260,548	.260,787	.261,025	.261,263	.261,501	.261,738	.261,976	.262,214	238
183	.262,451	.262,688	.262,925	.263,162	.263,399	.263,636	.263,873	.264,109	.264,345	.264,582	237

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
184	.264,818	.265,054	.265,290	.265,525	.265,761	.265,996	.266,232	.266,467	.266,702	.266,937	235
185	.267,172	.267,406	.267,641	.267,875	.268,110	.268,344	.268,578	.268,812	.269,046	.269,279	234
186	.269,513	.269,746	.269,980	.270,213	.270,446	.270,679	.270,912	.271,144	.271,377	.271,609	233
187	.271,842	.272,074	.272,306	.272,538	.272,770	.273,001	.273,233	.273,464	.273,696	.273,927	231
188	.274,158	.274,389	.274,620	.274,850	.275,081	.275,311	.275,542	.275,772	.276,002	.276,232	230
189	.276,462	.276,691	.276,921	.277,151	.277,380	.277,609	.277,838	.278,067	.278,296	.278,525	229
190	.278,754	.278,982	.279,210	.279,439	.279,667	.279,895	.280,123	.280,351	.280,578	.280,806	228
191	.281,033	.281,261	.281,488	.281,715	.281,942	.282,169	.282,395	.282,622	.282,849	.283,075	227
192	.283,301	.283,527	.283,753	.283,979	.284,205	.284,431	.284,656	.284,882	.285,107	.285,332	226
193	.285,557	.285,782	.286,007	.286,232	.286,456	.286,681	.286,905	.287,130	.287,354	.287,578	225
194	.287,802	.288,025	.288,249	.288,473	.288,696	.288,920	.289,143	.289,366	.289,589	.289,812	224
195	.290,035	.290,257	.290,480	.290,702	.290,925	.291,147	.291,369	.291,591	.291,813	.292,034	222
196	.292,256	.292,478	.292,699	.292,920	.293,141	.293,363	.293,583	.293,804	.294,025	.294,246	221
197	.294,466	.294,687	.294,907	.295,127	.295,347	.295,567	.295,787	.296,007	.296,226	.296,446	220
198	.296,665	.296,884	.297,104	.297,323	.297,542	.297,760	.297,979	.298,198	.298,416	.298,635	219
199	.298,853	.299,071	.299,289	.299,507	.299,725	.299,942	.300,160	.300,378	.300,595	.300,813	218
200	.301,030	.301,247	.301,464	.301,681	.301,898	.302,114	.302,331	.302,547	.302,764	.302,980	217
201	.303,196	.303,412	.303,628	.303,844	.304,059	.304,275	.304,490	.304,706	.304,921	.305,136	216
202	.305,351	.305,566	.305,781	.305,996	.306,210	.306,425	.306,639	.306,854	.307,068	.307,282	215
203	.307,496	.307,710	.307,924	.308,137	.308,351	.308,564	.308,778	.308,991	.309,204	.309,417	213
204	.309,630	.309,843	.310,056	.310,268	.310,481	.310,693	.310,906	.311,118	.311,330	.311,542	212
205	.311,754	.311,966	.312,177	.312,389	.312,600	.312,812	.313,023	.313,234	.313,445	.313,656	211
206	.313,867	.314,078	.314,289	.314,499	.314,710	.314,920	.315,130	.315,340	.315,551	.315,760	210
207	.315,970	.316,180	.316,390	.316,599	.316,809	.317,018	.317,227	.317,436	.317,646	.317,854	209
208	.318,063	.318,272	.318,481	.318,689	.318,898	.319,106	.319,314	.319,522	.319,730	.319,938	209
209	.320,146	.320,354	.320,562	.320,769	.320,977	.321,184	.321,391	.321,598	.321,805	.322,012	207
210	.322,219	.322,426	.322,633	.322,839	.323,046	.323,252	.323,458	.323,665	.323,871	.324,077	206
211	.324,282	.324,488	.324,694	.324,899	.325,105	.325,310	.325,516	.325,721	.325,926	.326,131	205
212	.326,336	.326,541	.326,745	.326,950	.327,155	.327,359	.327,563	.327,767	.327,972	.328,176	204
213	.328,380	.328,583	.328,787	.328,991	.329,194	.329,398	.329,601	.329,805	.330,008	.330,211	203
214	.330,414	.330,617	.330,819	.331,022	.331,225	.331,427	.331,630	.331,832	.332,034	.332,236	202
215	.332,438	.332,640	.332,842	.333,044	.333,246	.333,447	.333,649	.333,850	.334,051	.334,253	202
216	.334,454	.334,655	.334,856	.335,057	.335,257	.335,458	.335,658	.335,859	.336,059	.336,260	201
217	.336,460	.336,660	.336,860	.337,060	.337,260	.337,459	.337,659	.337,858	.338,058	.338,257	200
218	.338,456	.338,656	.338,855	.339,054	.339,253	.339,451	.339,650	.339,849	.340,047	.340,246	199
219	.340,444	.340,642	.340,841	.341,039	.341,237	.341,435	.341,632	.341,830	.342,028	.342,225	198
220	.342,423	.342,620	.342,817	.343,014	.343,212	.343,409	.343,606	.343,802	.343,999	.344,196	197
221	.344,392	.344,589	.344,785	.344,981	.345,178	.345,374	.345,570	.345,766	.345,962	.346,157	196
222	.346,353	.346,549	.346,744	.346,939	.347,135	.347,330	.347,525	.347,720	.347,915	.348,110	195
223	.348,305	.348,500	.348,694	.348,889	.349,083	.349,278	.349,472	.349,666	.349,860	.350,054	194
224	.350,248	.350,442	.350,636	.350,829	.351,023	.351,216	.351,410	.351,603	.351,796	.351,989	193
225	.352,183	.352,375	.352,568	.352,761	.352,954	.353,147	.353,339	.353,532	.353,724	.353,916	193

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
226	.354,108	.354,301	.354,493	.354,685	.354,876	.355,068	.355,260	.355,452	.355,643	.355,834	192
227	.356,026	.356,217	.356,408	.356,599	.356,790	.356,981	.357,172	.357,363	.357,554	.357,744	191
228	.357,935	.358,125	.358,316	.358,506	.358,696	.358,886	.359,076	.359,266	.359,456	.360,646	190
229	.359,835	.360,025	.360,215	.360,404	.360,593	.360,783	.360,972	.361,161	.361,351	.361,539	189
230	.361,728	.361,917	.362,105	.362,294	.362,482	.362,571	.362,859	.363,048	.363,236	.363,424	188
231	.363,612	.363,800	.363,988	.364,176	.364,363	.364,551	.364,739	.364,926	.365,113	.365,301	188
232	.365,488	.365,675	.365,862	.366,049	.366,236	.366,423	.366,610	.366,796	.366,983	.367,169	187
233	.367,356	.367,542	.367,729	.367,915	.368,101	.368,287	.368,473	.368,659	.368,845	.369,030	186
234	.369,216	.369,401	.369,587	.369,772	.369,958	.370,143	.370,328	.370,513	.370,698	.370,883	185
235	.371,068	.371,253	.371,437	.371,622	.371,806	.371,991	.372,175	.372,360	.372,544	.372,728	184
236	.372,912	.373,096	.373,280	.373,464	.373,647	.373,831	.374,015	.374,198	.374,382	.374,565	184
237	.374,748	.374,932	.375,115	.375,298	.375,481	.375,664	.375,846	.376,029	.376,212	.376,394	183
238	.376,577	.376,759	.376,942	.377,124	.377,306	.377,488	.377,670	.377,852	.378,034	.378,216	182
239	.378,398	.378,580	.378,761	.378,953	.379,124	.379,306	.379,487	.379,668	.379,849	.380,030	181
240	.380,211	.380,392	.380,573	.380,754	.380,934	.381,115	.381,296	.381,476	.381,656	.381,837	181
241	.382,017	.382,197	.382,377	.382,557	.382,737	.382,917	.383,097	.383,277	.383,456	.383,636	180
242	.383,815	.383,995	.384,174	.384,353	.384,533	.384,712	.384,891	.385,070	.385,249	.385,427	179
243	.385,606	.385,785	.385,964	.386,142	.386,321	.386,499	.386,677	.386,856	.387,034	.387,212	178
244	.387,390	.387,568	.387,746	.387,923	.388,101	.388,279	.388,456	.388,634	.388,811	.388,989	178
245	.389,166	.389,343	.389,521	.389,697	.389,875	.390,051	.390,228	.390,405	.390,582	.390,758	177
246	.390,935	.391,112	.391,288	.391,464	.391,641	.391,817	.391,993	.392,169	.392,345	.392,521	176
247	.392,697	.392,873	.393,048	.393,224	.393,400	.393,575	.393,751	.393,926	.394,101	.394,276	176
248	.394,452	.394,627	.394,802	.394,977	.395,152	.395,326	.395,501	.395,676	.395,850	.396,025	175
249	.396,199	.396,374	.396,548	.396,722	.396,896	.397,070	.397,245	.397,418	.397,592	.397,766	174
250	.397,940	.398,114	.398,287	.398,461	.398,634	.398,808	.398,981	.399,154	.399,327	.399,501	173
251	.399,674	.399,847	.400,020	.400,192	.400,365	.400,538	.400,711	.400,883	.401,056	.401,228	173
252	.401,400	.401,573	.401,745	.401,917	.402,089	.402,261	.402,433	.402,605	.402,777	.402,949	172
253	.403,120	.403,292	.403,464	.403,635	.403,807	.403,978	.404,149	.404,320	.404,492	.404,663	171
254	.404,834	.405,005	.405,176	.405,346	.405,517	.405,688	.405,858	.406,029	.406,199	.406,370	171
255	.406,540	.406,710	.406,881	.407,051	.407,221	.407,391	.407,561	.407,731	.407,900	.408,070	170
256	.408,240	.408,410	.408,579	.408,749	.408,918	.409,087	.409,257	.409,426	.409,595	.409,764	169
257	.409,933	.410,102	.410,271	.410,440	.410,608	.410,777	.410,946	.411,114	.411,283	.411,451	169
258	.411,620	.411,788	.411,956	.412,124	.412,292	.412,460	.412,628	.412,796	.412,964	.413,132	168
259	.413,300	.413,467	.413,635	.413,802	.413,970	.414,137	.414,305	.414,472	.414,639	.414,806	167
260	.414,973	.415,140	.415,307	.415,474	.415,641	.415,808	.415,974	.416,141	.416,308	.416,474	167
261	.416,640	.416,807	.416,973	.417,139	.417,306	.417,472	.417,638	.417,804	.417,970	.418,135	166
262	.418,301	.418,467	.418,633	.418,798	.418,964	.419,129	.419,295	.419,460	.419,625	.419,791	165
263	.419,956	.420,121	.420,286	.420,451	.420,616	.420,781	.420,945	.421,110	.421,275	.421,439	165
264	.421,604	.421,768	.421,933	.422,097	.422,261	.422,426	.422,590	.422,754	.422,918	.423,082	164
265	.423,246	.423,410	.423,573	.423,737	.423,901	.424,064	.424,228	.424,392	.424,555	.424,718	164
266	.424,882	.425,045	.425,208	.425,371	.425,534	.425,697	.425,860	.426,023	.426,186	.426,349	163
267	.426,511	.426,674	.426,836	.426,999	.427,161	.427,324	.427,486	.427,648	.427,811	.427,973	162

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
268	.428,135	.428,297	.428,459	.428,621	.428,782	.428,944	.429,106	.429,268	.429,429	.429,591	162
269	.429,752	.429,914	.430,075	.430,236	.430,398	.430,559	.430,720	.430,881	.431,042	.431,203	161
270	.431,364	.431,525	.431,685	.431,846	.432,007	.432,167	.432,328	.432,488	.432,649	.432,809	160
271	.432,969	.433,129	.433,290	.433,450	.433,610	.433,770	.433,930	.434,090	.434,249	.434,409	160
272	.434,569	.434,728	.434,888	.435,048	.435,207	.435,366	.435,526	.435,685	.435,844	.436,003	159
273	.436,163	.436,322	.436,481	.436,640	.436,798	.436,957	.437,116	.437,275	.437,433	.437,592	159
274	.437,751	.437,909	.438,067	.438,226	.438,384	.438,542	.438,700	.438,859	.439,017	.439,175	158
275	.439,333	.439,491	.439,648	.439,806	.439,964	.440,122	.440,279	.440,437	.440,594	.440,752	158
276	.440,909	.441,066	.441,224	.441,381	.441,538	.441,695	.441,852	.442,009	.442,166	.442,323	157
277	.442,480	.442,637	.442,793	.442,950	.443,106	.443,263	.443,419	.443,576	.443,732	.443,889	157
278	.444,045	.444,201	.444,357	.444,513	.444,669	.444,825	.444,981	.445,137	.445,293	.445,449	156
279	.445,604	.445,760	.445,915	.446,071	.446,226	.446,382	.446,537	.446,692	.446,848	.447,003	155
280	.447,158	.447,313	.447,468	.447,623	.447,778	.447,933	.448,088	.448,242	.448,397	.448,552	155
281	.448,706	.448,861	.449,015	.449,170	.449,324	.449,478	.449,633	.449,787	.449,941	.450,095	154
282	.450,249	.450,403	.450,557	.450,711	.450,865	.451,018	.451,172	.451,326	.451,479	.451,633	154
283	.451,786	.451,940	.452,093	.452,247	.452,400	.452,553	.452,706	.452,859	.453,012	.453,165	153
284	.453,318	.453,471	.453,624	.453,777	.453,930	.454,082	.454,235	.454,387	.454,540	.454,692	153
285	.454,845	.454,997	.455,149	.455,302	.455,454	.455,606	.455,758	.455,910	.456,062	.456,214	152
286	.456,366	.456,518	.456,670	.456,821	.456,973	.457,125	.457,276	.457,428	.457,579	.457,731	152
287	.457,882	.458,033	.458,184	.458,336	.458,487	.458,638	.458,789	.458,940	.459,091	.459,242	151
288	.459,392	.459,543	.459,694	.459,845	.459,995	.460,146	.460,296	.460,447	.460,597	.460,748	151
289	.460,898	.461,048	.461,198	.461,348	.461,499	.461,649	.461,799	.461,948	.462,098	.462,248	150
290	.462,398	.462,548	.462,697	.462,847	.462,997	.463,146	.463,296	.463,445	.463,594	.463,744	150
291	.463,893	.464,002	.464,191	.464,340	.464,489	.464,639	.464,787	.464,936	.465,085	.465,234	149
292	.465,383	.465,532	.465,680	.465,829	.465,977	.466,126	.466,274	.466,423	.466,571	.466,719	149
293	.466,868	.467,016	.467,164	.467,312	.467,460	.467,608	.467,756	.467,904	.468,052	.468,200	148
294	.468,347	.468,495	.468,643	.468,790	.468,938	.469,085	.469,233	.469,380	.469,527	.469,675	147
295	.469,822	.469,969	.470,116	.470,263	.470,410	.470,557	.470,704	.470,851	.470,998	.471,145	147
296	.471,292	.471,438	.471,585	.471,732	.471,878	.472,025	.472,171	.472,318	.472,464	.472,610	146
297	.472,756	.472,903	.473,049	.473,195	.473,341	.473,487	.473,633	.473,779	.473,925	.474,071	146
298	.474,216	.474,362	.474,508	.474,653	.474,799	.474,944	.475,090	.475,235	.475,381	.475,526	146
299	.475,671	.475,816	.475,962	.476,107	.476,252	.476,397	.476,542	.476,687	.476,832	.476,976	145
300	.477,121	.477,266	.477,411	.477,555	.477,700	.477,844	.477,989	.478,133	.478,278	.478,422	145
301	.478,566	.478,711	.478,855	.478,999	.479,143	.479,287	.479,431	.479,575	.479,719	.479,863	144
302	.480,007	.480,151	.480,294	.480,438	.480,582	.480,725	.480,869	.481,012	.481,156	.481,299	144
303	.481,443	.481,586	.481,729	.481,872	.482,016	.482,159	.482,302	.482,445	.482,588	.482,731	143
304	.482,874	.483,016	.483,159	.483,302	.483,445	.483,587	.483,730	.483,872	.484,015	.484,157	143
305	.484,300	.484,442	.484,585	.484,727	.484,869	.485,011	.485,153	.485,295	.485,437	.485,579	142
306	.485,721	.485,863	.486,005	.486,147	.486,289	.486,430	.486,572	.486,714	.486,855	.486,997	142
307	.487,138	.487,280	.487,421	.487,563	.487,704	.487,845	.487,986	.488,127	.488,269	.488,410	141
308	.488,551	.488,692	.488,833	.488,874	.489,114	.489,255	.489,396	.489,537	.489,677	.489,818	141
309	.489,958	.490,099	.490,239	.490,380	.490,520	.490,661	.490,801	.490,941	.491,081	.491,222	140

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
310	.491,362	.491,502	.491,642	.491,782	.491,922	.492,062	.492,201	.492,341	.492,481	.492,621	140
311	.492,760	.492,900	.493,040	.493,179	.493,319	.493,458	.493,597	.493,737	.493,876	.494,015	139
312	.494,155	.494,294	.494,433	.494,572	.494,711	.494,850	.494,989	.495,128	.495,267	.495,406	139
313	.495,544	.495,683	.495,822	.495,960	.496,099	.496,237	.496,376	.496,514	.496,653	.496,791	139
314	.496,930	.497,068	.497,206	.497,344	.497,482	.497,621	.497,759	.497,897	.498,035	.498,173	138
315	.498,311	.498,448	.498,586	.498,724	.498,862	.498,999	.499,137	.499,275	.499,412	.499,550	138
316	.499,687	.499,824	.499,962	.500,099	.500,236	.500,374	.500,511	.500,648	.500,785	.500,922	137
317	.501,059	.501,196	.501,333	.501,470	.501,607	.501,744	.501,880	.502,017	.502,154	.502,290	137
318	.502,427	.502,564	.502,700	.502,837	.502,973	.503,109	.503,246	.503,382	.503,518	.503,654	136
319	.503,791	.503,927	.504,063	.504,199	.504,335	.504,471	.504,607	.504,743	.504,878	.505,014	136
320	.505,150	.505,286	.505,421	.505,557	.505,692	.505,828	.505,963	.506,099	.506,234	.506,370	136
321	.506,505	.506,640	.506,775	.506,911	.507,046	.507,181	.507,316	.507,451	.507,586	.507,721	135
322	.507,856	.507,991	.508,125	.508,260	.508,395	.508,530	.508,664	.508,799	.508,933	.509,068	135
323	.509,202	.509,337	.509,471	.509,606	.509,740	.509,874	.510,008	.510,143	.510,277	.510,411	134
324	.510,545	.510,679	.510,813	.510,947	.511,081	.511,215	.511,348	.511,482	.511,616	.511,750	134
325	.511,883	.512,017	.512,150	.512,284	.512,417	.512,551	.512,684	.512,818	.512,951	.513,084	134
326	.513,218	.513,351	.513,484	.513,617	.513,750	.513,883	.514,016	.514,149	.514,282	.514,415	133
327	.514,548	.514,681	.514,813	.514,946	.515,079	.515,211	.515,344	.515,476	.515,609	.515,741	132
328	.515,874	.516,006	.516,139	.516,271	.516,403	.516,535	.516,668	.516,800	.516,932	.517,064	132
329	.517,196	.517,328	.517,460	.517,592	.517,724	.517,855	.517,987	.518,119	.518,251	.518,382	131
330	.518,514	.518,645	.518,777	.518,909	.519,040	.519,171	.519,303	.519,434	.519,566	.519,697	131
331	.519,828	.519,959	.520,090	.520,221	.520,352	.520,483	.520,614	.520,745	.520,876	.521,007	131
332	.521,138	.521,269	.521,400	.521,530	.521,661	.521,792	.521,922	.522,053	.522,183	.522,314	131
333	.522,444	.522,575	.522,705	.522,835	.522,966	.523,096	.523,226	.523,356	.523,486	.523,616	130
334	.523,746	.523,876	.524,006	.524,136	.524,266	.524,396	.524,526	.524,656	.524,785	.524,915	130
335	.525,045	.525,174	.525,304	.525,433	.525,563	.525,692	.525,822	.525,951	.526,081	.526,210	129
336	.526,339	.526,468	.526,598	.526,727	.526,856	.526,985	.527,114	.527,243	.527,372	.527,501	129
337	.527,630	.527,759	.527,888	.528,016	.528,145	.528,274	.528,402	.528,531	.528,660	.528,788	129
338	.528,917	.529,045	.529,174	.529,302	.529,430	.529,559	.529,687	.529,815	.529,943	.530,072	128
339	.530,200	.530,328	.530,456	.530,584	.530,712	.530,840	.530,968	.531,095	.531,223	.531,351	128
340	.531,479	.531,607	.531,734	.531,862	.531,989	.532,117	.532,245	.532,372	.532,500	.532,627	128
341	.532,754	.532,882	.533,009	.533,136	.533,263	.533,391	.533,518	.533,645	.533,772	.533,899	128
342	.534,026	.534,153	.534,280	.534,407	.534,534	.534,661	.534,787	.534,914	.535,041	.535,167	127
343	.535,294	.535,421	.535,547	.535,674	.535,800	.535,927	.536,053	.536,179	.536,306	.536,432	127
344	.536,558	.536,685	.536,811	.536,937	.537,063	.537,189	.537,315	.537,441	.537,567	.537,693	127
345	.537,819	.537,945	.538,071	.538,197	.538,322	.538,448	.538,574	.538,699	.538,825	.538,951	126
346	.539,076	.539,202	.539,327	.539,452	.539,578	.539,703	.539,829	.539,954	.540,079	.540,204	125
347	.540,329	.540,455	.540,580	.540,705	.540,830	.540,955	.541,080	.541,205	.541,330	.541,454	125
348	.541,579	.541,704	.541,829	.541,953	.542,078	.542,203	.542,327	.542,452	.542,576	.542,701	125
349	.542,825	.542,950	.543,074	.543,199	.543,323	.543,447	.543,571	.543,696	.543,820	.543,944	124
350	.544,068	.544,192	.544,316	.544,440	.544,564	.544,688	.544,812	.544,936	.545,060	.545,183	124
351	.545,307	.545,431	.545,555	.545,678	.545,802	.545,925	.546,049	.546,172	.546,296	.546,419	124

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
352	.546,543	.546,666	.546,789	.546,913	.547,036	.547,159	.547,282	.547,405	.547,529	.547,652	123
353	.547,775	.547,898	.548,021	.548,144	.548,267	.548,389	.548,512	.548,635	.548,758	.548,881	123
354	.549,003	.549,126	.549,249	.549,371	.549,494	.549,616	.549,739	.549,861	.549,984	.550,106	123
355	.550,228	.550,351	.550,473	.550,595	.550,717	.550,840	.550,962	.551,084	.551,206	.551,328	122
356	.551,450	.551,572	.551,694	.551,816	.551,938	.552,059	.552,181	.552,303	.552,425	.552,547	122
357	.552,668	.552,790	.552,911	.553,033	.553,155	.553,276	.553,398	.553,519	.553,640	.553,762	121
358	.553,883	.554,004	.554,126	.554,247	.554,368	.554,489	.554,610	.554,731	.554,852	.554,973	121
359	.555,094	.555,215	.555,336	.555,457	.555,578	.555,699	.555,820	.555,940	.556,061	.556,182	121
360	.556,303	.556,423	.556,544	.556,664	.556,785	.556,905	.557,026	.557,146	.557,267	.557,387	120
361	.557,507	.557,628	.557,748	.557,868	.557,988	.558,108	.558,228	.558,349	.558,469	.558,589	120
362	.558,709	.558,829	.558,948	.559,068	.559,188	.559,308	.559,428	.559,548	.559,667	.559,787	120
363	.559,907	.560,026	.560,146	.560,265	.560,385	.560,504	.560,624	.560,743	.560,863	.560,982	119
364	.561,101	.561,221	.561,340	.561,459	.561,578	.561,698	.561,817	.561,936	.562,055	.562,174	119
365	.562,293	.562,412	.562,531	.562,650	.562,769	.562,887	.563,006	.563,125	.563,244	.563,362	119
366	.563,481	.563,600	.563,718	.563,837	.563,955	.564,074	.564,192	.564,311	.564,429	.564,548	119
367	.564,666	.564,784	.564,903	.565,021	.565,139	.565,257	.565,376	.565,494	.565,612	.565,730	118
368	.565,848	.565,966	.566,084	.566,202	.566,320	.566,437	.566,555	.566,673	.566,791	.566,909	118
369	.567,026	.567,144	.567,262	.567,379	.567,497	.567,614	.567,732	.567,849	.567,967	.568,084	118
370	.568,202	.568,319	.568,436	.568,554	.568,671	.568,788	.568,905	.569,023	.569,140	.569,257	117
371	.569,374	.569,491	.569,608	.569,725	.569,842	.569,959	.570,076	.570,193	.570,309	.570,426	117
372	.570,543	.570,660	.570,776	.570,893	.571,010	.571,126	.571,243	.571,359	.571,476	.571,592	117
373	.571,709	.571,825	.571,942	.572,058	.572,174	.572,291	.572,407	.572,523	.572,639	.572,755	116
374	.572,872	.572,988	.573,104	.573,220	.573,336	.573,452	.573,568	.573,684	.573,800	.573,915	116
375	.574,031	.574,147	.574,263	.574,379	.574,494	.574,610	.574,726	.574,841	.574,957	.575,072	116
376	.575,188	.575,303	.575,419	.575,534	.575,650	.575,765	.575,880	.575,996	.576,111	.576,226	115
377	.576,341	.576,457	.576,572	.576,687	.576,802	.576,917	.577,032	.577,147	.577,262	.577,377	115
378	.577,492	.577,607	.577,722	.577,836	.577,951	.578,066	.578,181	.578,295	.578,410	.578,525	115
379	.578,639	.578,754	.578,868	.578,983	.579,097	.579,212	.579,326	.579,441	.579,555	.579,669	114
380	.579,784	.579,898	.580,012	.580,126	.580,240	.580,355	.580,469	.580,583	.580,697	.580,811	114
381	.580,925	.581,039	.581,153	.581,267	.581,381	.581,494	.581,608	.581,722	.581,836	.581,950	114
382	.582,063	.582,177	.582,291	.582,404	.582,518	.582,631	.582,745	.582,859	.582,972	.583,085	113
383	.583,199	.583,312	.583,425	.583,539	.583,652	.583,765	.583,879	.583,992	.584,105	.584,218	113
384	.584,331	.584,444	.584,557	.584,670	.584,783	.584,896	.585,009	.585,122	.585,235	.585,348	113
385	.585,461	.585,573	.585,686	.585,799	.585,912	.586,024	.586,137	.586,250	.586,362	.586,475	112
386	.586,587	.586,700	.586,812	.586,925	.587,037	.587,149	.587,262	.587,374	.587,486	.587,599	112
387	.587,711	.587,823	.587,935	.588,047	.588,160	.588,272	.588,384	.588,496	.588,608	.588,720	112
388	.588,832	.588,944	.589,055	.589,167	.589,279	.589,391	.589,503	.589,615	.589,726	.589,838	112
389	.589,950	.590,061	.590,173	.590,284	.590,396	.590,507	.590,619	.590,730	.590,842	.590,953	111
390	.591,065	.591,176	.591,287	.591,398	.591,510	.591,621	.591,732	.591,843	.591,955	.592,066	111
391	.592,177	.592,288	.592,399	.592,510	.592,621	.592,732	.592,843	.592,954	.593,064	.593,175	111
392	.593,286	.593,397	.593,508	.593,618	.593,729	.593,840	.593,950	.594,061	.594,171	.594,282	111
393	.594,393	.594,503	.594,613	.594,724	.594,834	.594,945	.595,055	.595,165	.595,276	.595,386	111

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
394	.595,496	.595,606	.595,717	.595,827	.595,937	.596,047	.596,157	.596,267	.596,377	.596,487	110
395	.596,597	.596,707	.596,817	.596,927	.597,037	.597,146	.597,256	.597,366	.597,476	.597,586	110
396	.597,695	.597,805	.597,914	.598,024	.598,134	.598,243	.598,353	.598,462	.598,572	.598,681	110
397	.598,790	.598,900	.599,009	.599,119	.599,228	.599,337	.599,446	.599,556	.599,665	.599,774	109
398	.599,883	.599,992	.600,101	.600,216	.600,319	.600,428	.600,537	.600,646	.600,755	.600,864	109
399	.600,973	.601,082	.601,190	.601,299	.601,408	.601,517	.601,625	.601,734	.601,843	.601,951	109
400	.602,060	.602,168	.602,277	.602,386	.602,494	.602,603	.602,711	.602,819	.602,928	.603,036	109
401	.603,144	.603,253	.603,361	.603,469	.603,577	.603,685	.603,794	.603,902	.604,010	.604,118	108
402	.604,226	.604,334	.604,442	.604,550	.604,658	.604,766	.604,874	.604,982	.605,089	.605,197	108
403	.605,305	.605,413	.605,521	.605,628	.605,736	.605,843	.605,951	.606,059	.606,166	.606,274	107
404	.606,381	.606,489	.606,596	.606,704	.606,811	.606,918	.607,026	.607,133	.607,240	.607,348	107
405	.607,455	.607,562	.607,669	.607,777	.607,884	.607,991	.608,098	.608,205	.608,312	.608,419	107
406	.608,526	.608,633	.608,740	.608,847	.608,954	.609,060	.609,167	.609,274	.609,381	.609,488	107
407	.609,594	.609,701	.609,808	.609,914	.610,021	.610,128	.610,234	.610,341	.610,447	.610,554	107
408	.610,660	.610,767	.610,873	.610,979	.611,086	.611,192	.611,298	.611,405	.611,511	.611,617	106
409	.611,723	.611,829	.611,936	.612,042	.612,148	.612,254	.612,360	.612,466	.612,572	.612,678	106
410	.612,784	.612,890	.612,996	.613,101	.613,207	.613,313	.613,419	.613,525	.613,630	.613,736	106
411	.613,842	.613,947	.614,053	.614,159	.614,264	.614,370	.614,475	.614,581	.614,686	.614,792	106
412	.614,897	.615,003	.615,108	.615,213	.615,319	.615,424	.615,529	.615,634	.615,740	.615,845	105
413	.615,950	.616,055	.616,160	.616,265	.616,370	.616,475	.616,580	.616,685	.616,790	.616,895	105
414	.617,000	.617,105	.617,210	.617,315	.617,420	.617,524	.617,629	.617,734	.617,839	.617,943	105
415	.618,048	.618,153	.618,257	.618,362	.618,466	.618,571	.618,676	.618,780	.618,884	.618,989	105
416	.619,093	.619,198	.619,302	.619,406	.619,511	.619,615	.619,719	.619,823	.619,928	.620,032	104
417	.620,136	.620,240	.620,344	.620,448	.620,552	.620,656	.620,760	.620,864	.620,968	.621,072	104
418	.621,176	.621,280	.621,384	.621,488	.621,592	.621,695	.621,799	.621,903	.622,007	.622,110	104
419	.622,214	.622,318	.622,421	.622,525	.622,628	.622,732	.622,835	.622,939	.623,042	.623,146	104
420	.623,249	.623,353	.623,456	.623,559	.623,663	.623,766	.623,869	.623,973	.624,076	.624,179	103
421	.624,282	.624,385	.624,488	.624,591	.624,695	.624,798	.624,901	.625,004	.625,107	.625,209	103
422	.625,312	.625,415	.625,518	.625,621	.625,724	.625,827	.625,929	.626,032	.626,135	.626,238	103
423	.626,340	.626,443	.626,546	.626,648	.626,751	.626,853	.626,956	.627,058	.627,161	.627,263	103
424	.627,366	.627,468	.627,571	.627,673	.627,775	.627,878	.627,980	.628,082	.628,185	.628,287	102
425	.628,389	.628,491	.628,593	.628,695	.628,797	.628,900	.629,002	.629,104	.629,206	.629,308	102
426	.629,410	.629,511	.629,613	.629,715	.629,817	.629,919	.630,021	.630,123	.630,224	.630,326	102
427	.630,428	.630,530	.630,631	.630,733	.630,835	.630,936	.631,038	.631,139	.631,241	.631,342	102
428	.631,444	.631,545	.631,647	.631,748	.631,849	.631,951	.632,052	.632,153	.632,255	.632,356	101
429	.632,457	.632,559	.632,660	.632,761	.632,862	.632,963	.633,064	.633,165	.633,266	.633,367	101
430	.633,468	.633,569	.633,670	.633,771	.633,872	.633,973	.634,074	.634,175	.634,276	.634,376	100
431	.634,477	.634,578	.634,679	.634,779	.634,880	.634,981	.635,081	.635,182	.635,283	.635,383	100
432	.635,484	.635,584	.635,685	.635,785	.635,886	.635,986	.636,087	.636,187	.636,287	.636,388	100
433	.636,488	.636,588	.636,688	.636,789	.636,889	.636,989	.637,089	.637,189	.637,289	.637,390	100
434	.637,490	.637,590	.637,690	.637,790	.637,890	.637,990	.638,090	.638,190	.638,289	.638,389	99
435	.638,489	.638,589	.638,689	.638,789	.638,888	.638,988	.639,088	.639,188	.639,287	.639,387	99

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
436	.639,486	.639,586	.639,686	.639,785	.639,885	.639,984	.640,084	.640,183	.640,283	.640,382	99
437	.640,481	.640,581	.640,680	.640,779	.640,879	.640,978	.641,077	.641,177	.641,276	.641,375	99
438	.641,474	.641,573	.641,672	.641,771	.641,871	.641,970	.642,069	.642,168	.642,267	.642,366	99
439	.642,465	.642,563	.642,662	.642,761	.642,860	.642,959	.643,058	.643,156	.643,255	.643,354	99
440	.643,453	.643,551	.643,650	.643,749	.643,847	.643,946	.644,044	.644,143	.644,242	.644,340	98
441	.644,439	.644,537	.644,636	.644,734	.644,832	.644,931	.645,029	.645,127	.645,226	.645,324	98
442	.645,422	.645,521	.645,619	.645,717	.645,815	.645,913	.646,011	.646,109	.646,208	.646,306	98
443	.646,404	.646,502	.646,600	.646,698	.646,796	.646,894	.646,992	.647,089	.647,187	.647,285	98
444	.647,383	.647,481	.647,579	.647,676	.647,774	.647,872	.647,969	.648,067	.648,165	.648,262	98
445	.648,360	.648,458	.648,555	.648,653	.648,750	.648,848	.648,945	.649,043	.649,140	.649,237	97
446	.649,335	.649,432	.649,530	.649,627	.649,724	.649,821	.649,919	.650,016	.650,113	.650,210	97
447	.650,308	.650,405	.650,502	.650,599	.650,696	.650,794	.650,890	.650,987	.651,084	.651,181	97
448	.651,278	.651,375	.651,472	.651,569	.651,666	.651,760	.651,859	.651,956	.652,053	.652,150	97
449	.652,246	.652,343	.652,440	.652,536	.652,633	.652,730	.652,826	.652,923	.653,019	.653,116	97
450	.653,212	.653,309	.653,406	.653,502	.653,598	.653,695	.653,791	.653,888	.653,984	.654,080	97
451	.654,176	.654,273	.654,369	.654,465	.654,562	.654,658	.654,754	.654,850	.654,946	.655,042	96
452	.655,138	.655,234	.655,331	.655,427	.655,523	.655,619	.655,714	.655,810	.655,906	.656,002	96
453	.656,098	.656,194	.656,290	.656,386	.656,481	.656,577	.656,673	.656,769	.656,864	.656,960	96
454	.657,056	.657,151	.657,247	.657,343	.657,438	.657,534	.657,629	.657,725	.657,820	.657,916	96
455	.658,011	.658,107	.658,202	.658,298	.658,393	.658,488	.658,584	.658,679	.658,774	.658,870	95
456	.658,965	.659,060	.659,155	.659,250	.659,346	.659,441	.659,536	.659,631	.659,726	.659,821	95
457	.659,916	.660,011	.660,106	.660,201	.660,296	.660,391	.660,486	.660,581	.660,676	.660,771	95
458	.660,865	.660,960	.661,055	.661,150	.661,245	.661,339	.661,434	.661,529	.661,623	.661,718	94
459	.661,813	.661,907	.662,002	.662,096	.662,191	.662,285	.662,380	.662,474	.662,569	.662,663	94
460	.662,758	.662,852	.662,947	.663,041	.663,135	.663,230	.663,324	.663,418	.663,512	.663,607	94
461	.663,701	.663,795	.663,889	.663,983	.664,078	.664,172	.664,266	.664,360	.664,454	.664,548	94
462	.664,642	.664,736	.664,830	.664,924	.665,018	.665,112	.665,206	.665,299	.665,393	.665,487	94
463	.665,581	.665,675	.665,768	.665,862	.665,956	.666,050	.666,143	.666,237	.666,331	.666,424	94
464	.666,518	.666,612	.666,705	.666,799	.666,892	.666,986	.667,079	.667,173	.667,266	.667,359	94
465	.667,453	.667,546	.667,640	.667,733	.667,826	.667,920	.668,013	.668,106	.668,199	.668,293	94
466	.668,386	.668,479	.668,572	.668,665	.668,758	.668,852	.668,945	.669,038	.669,131	.669,224	94
467	.669,317	.669,410	.669,503	.669,596	.669,689	.669,782	.669,874	.669,967	.670,060	.670,153	93
468	.670,246	.670,339	.670,431	.670,524	.670,617	.670,710	.670,802	.670,895	.670,988	.671,080	93
469	.671,173	.671,265	.671,358	.671,451	.671,543	.671,636	.671,728	.671,821	.671,913	.672,005	93
470	.672,098	.672,190	.672,283	.672,375	.672,467	.672,559	.672,652	.672,744	.672,836	.672,929	93
471	.673,021	.673,113	.673,205	.673,297	.673,390	.673,482	.673,574	.673,666	.673,758	.673,850	92
472	.673,942	.674,034	.674,126	.674,218	.674,310	.674,402	.674,494	.674,586	.674,677	.674,769	92
473	.674,861	.674,953	.675,045	.675,136	.675,228	.675,320	.675,412	.675,503	.675,595	.675,687	92
474	.675,778	.675,870	.675,961	.676,053	.676,145	.676,236	.676,328	.676,419	.676,511	.676,602	91
475	.676,694	.676,785	.676,876	.676,968	.677,059	.677,150	.677,242	.677,333	.677,424	.677,516	91
476	.677,607	.677,698	.677,789	.677,881	.677,972	.678,063	.678,154	.678,245	.678,336	.678,427	91
477	.678,518	.678,609	.678,700	.678,791	.678,882	.678,973	.679,064	.679,155	.679,246	.679,337	91

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
478	.679,428	.679,519	.679,610	.679,700	.679,791	.679,882	.679,973	.680,063	.680,154	.680,245	91
479	.680,335	.680,426	.680,517	.680,607	.680,698	.680,789	.680,879	.680,970	.681,060	.681,151	91
480	.681,241	.681,332	.681,422	.681,513	.681,603	.681,693	.681,784	.681,874	.681,964	.682,055	90
481	.682,145	.682,235	.682,326	.682,416	.682,506	.682,596	.682,686	.682,777	.682,867	.682,957	90
482	.683,047	.683,137	.683,227	.683,317	.683,407	.683,497	.683,587	.683,677	.683,767	.683,857	90
483	.683,947	.684,037	.684,127	.684,217	.684,307	.684,396	.684,486	.684,576	.684,666	.684,756	90
484	.684,845	.684,935	.685,025	.685,114	.685,204	.685,294	.685,383	.685,473	.685,563	.685,652	90
485	.685,742	.685,831	.685,921	.686,010	.686,100	.686,189	.686,279	.686,368	.686,458	.686,547	89
486	.686,536	.686,626	.686,715	.686,804	.686,894	.686,983	.687,072	.687,161	.687,251	.687,340	89
487	.687,529	.687,618	.687,707	.687,796	.687,885	.687,975	.688,064	.688,153	.688,242	.688,331	89
488	.688,420	.688,509	.688,598	.688,687	.688,776	.688,865	.688,953	.689,042	.689,131	.689,220	89
489	.689,309	.689,398	.689,486	.689,575	.689,664	.689,753	.689,841	.689,930	.690,019	.690,107	89
490	.690,196	.690,285	.690,373	.690,462	.690,550	.690,639	.690,727	.690,816	.690,905	.690,993	89
491	.691,081	.691,170	.691,258	.691,347	.691,435	.691,524	.691,612	.691,700	.691,789	.691,877	88
492	.691,965	.692,053	.692,142	.692,230	.692,318	.692,406	.692,494	.692,583	.692,671	.692,759	88
493	.692,847	.692,935	.693,023	.693,111	.693,199	.693,287	.693,375	.693,463	.693,551	.693,639	88
494	.693,727	.693,815	.693,903	.693,991	.694,078	.694,166	.694,254	.694,342	.694,430	.694,517	88
495	.694,605	.694,693	.694,781	.694,868	.694,956	.695,044	.695,131	.695,219	.695,307	.695,394	88
496	.695,482	.695,569	.695,657	.695,744	.695,832	.695,919	.696,007	.696,094	.696,182	.696,269	87
497	.696,356	.696,444	.696,531	.696,618	.696,706	.696,793	.696,880	.696,968	.697,055	.697,142	87
498	.697,229	.697,317	.697,404	.697,491	.697,578	.697,665	.697,752	.697,839	.697,926	.698,014	87
499	.698,101	.698,188	.698,275	.698,362	.698,449	.698,535	.698,622	.698,709	.698,796	.698,883	87
500	.698,970	.699,057	.699,144	.699,231	.699,317	.699,404	.699,491	.699,578	.699,664	.699,751	87
501	.699,838	.699,924	.700,011	.700,098	.700,184	.700,271	.700,358	.700,444	.700,531	.700,617	87
502	.700,704	.700,790	.700,877	.700,963	.701,050	.701,136	.701,222	.701,309	.701,395	.701,482	86
503	.701,568	.701,654	.701,741	.701,827	.701,913	.701,999	.702,086	.702,172	.702,258	.702,344	86
504	.702,430	.702,517	.702,603	.702,689	.702,775	.702,861	.702,947	.703,033	.703,119	.703,205	86
505	.703,291	.703,377	.703,463	.703,549	.703,635	.703,721	.703,807	.703,893	.703,979	.704,065	86
506	.704,151	.704,236	.704,322	.704,408	.704,494	.704,579	.704,665	.704,751	.704,837	.704,922	86
507	.705,008	.705,094	.705,179	.705,265	.705,350	.705,436	.705,522	.705,607	.705,693	.705,778	86
508	.705,864	.705,949	.706,035	.706,120	.706,206	.706,291	.706,376	.706,462	.706,547	.706,632	85
509	.706,718	.706,803	.706,888	.706,974	.707,059	.707,144	.707,229	.707,315	.707,400	.707,485	85
510	.707,570	.707,655	.707,740	.707,826	.707,911	.707,996	.708,081	.708,166	.708,251	.708,336	85
511	.708,421	.708,506	.708,591	.708,676	.708,761	.708,846	.708,931	.709,015	.709,100	.709,185	85
512	.709,270	.709,355	.709,440	.709,524	.709,609	.709,694	.709,779	.709,863	.709,948	.710,033	85
513	.710,117	.710,202	.710,287	.710,371	.710,456	.710,540	.710,625	.710,710	.710,794	.710,879	85
514	.710,963	.711,048	.711,132	.711,217	.711,301	.711,385	.711,470	.711,554	.711,639	.711,723	84
515	.711,807	.711,892	.711,976	.712,060	.712,144	.712,229	.712,313	.712,397	.712,481	.712,566	84
516	.712,650	.712,734	.712,818	.712,902	.712,986	.713,070	.713,154	.713,238	.713,323	.713,406	84
517	.713,491	.713,575	.713,659	.713,742	.713,826	.713,910	.713,994	.714,078	.714,162	.714,246	84
518	.714,330	.714,414	.714,497	.714,581	.714,665	.714,749	.714,833	.714,916	.715,000	.715,084	84
519	.715,167	.715,251	.715,335	.715,418	.715,502	.715,586	.715,669	.715,753	.715,836	.715,920	84

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
520	.716,003	.716,087	.716,170	.716,254	.716,337	.716,421	.716,504	.716,588	.716,671	.716,754	83
521	.716,838	.716,921	.717,004	.717,088	.717,171	.717,254	.717,338	.717,421	.717,504	.717,587	83
522	.717,671	.717,754	.717,837	.717,920	.718,003	.718,086	.718,169	.718,253	.718,336	.718,419	83
523	.718,502	.718,585	.718,668	.718,751	.718,834	.718,917	.719,000	.719,083	.719,165	.719,248	83
524	.719,331	.719,414	.719,497	.719,580	.719,663	.719,745	.719,828	.719,911	.719,994	.720,077	83
525	.720,159	.720,242	.720,325	.720,407	.720,490	.720,573	.720,655	.720,738	.720,821	.720,903	83
526	.720,986	.721,068	.721,151	.721,233	.721,316	.721,398	.721,481	.721,563	.721,646	.721,728	82
527	.721,811	.721,893	.721,975	.722,058	.722,140	.722,222	.722,305	.722,387	.722,469	.722,552	82
528	.722,634	.722,716	.722,798	.722,881	.722,963	.723,045	.723,127	.723,209	.723,291	.723,374	82
529	.723,456	.723,538	.723,620	.723,702	.723,784	.723,866	.723,948	.724,030	.724,112	.724,194	82
530	.724,276	.724,358	.724,440	.724,522	.724,603	.724,685	.724,767	.724,849	.724,931	.725,013	82
531	.725,094	.725,176	.725,258	.725,340	.725,422	.725,503	.725,585	.725,667	.725,748	.725,830	82
532	.725,912	.725,993	.726,075	.726,156	.726,238	.726,320	.726,401	.726,483	.726,564	.726,646	82
533	.726,727	.726,809	.726,890	.726,972	.727,053	.727,134	.727,216	.727,297	.727,379	.727,460	81
534	.727,541	.727,623	.727,704	.727,785	.727,866	.727,948	.728,029	.728,110	.728,191	.728,273	81
535	.728,354	.728,435	.728,516	.728,597	.728,678	.728,759	.728,841	.728,922	.729,003	.729,084	81
536	.729,165	.729,246	.729,327	.729,408	.729,489	.729,570	.729,651	.729,732	.729,813	.729,893	81
537	.729,974	.730,055	.730,136	.730,217	.730,298	.730,378	.730,459	.730,540	.730,621	.730,702	81
538	.730,782	.730,863	.730,944	.731,024	.731,105	.731,186	.731,266	.731,347	.731,428	.731,508	81
539	.731,589	.731,669	.731,750	.731,830	.731,911	.731,991	.732,072	.732,152	.732,233	.732,313	81
540	.732,394	.732,474	.732,555	.732,635	.732,715	.732,796	.732,876	.732,956	.733,037	.733,117	80
541	.733,197	.733,278	.733,358	.733,438	.733,518	.733,598	.733,679	.733,759	.733,839	.733,919	80
542	.733,999	.734,079	.734,159	.734,240	.734,320	.734,400	.734,480	.734,560	.734,640	.734,720	80
543	.734,800	.734,880	.734,960	.735,040	.735,120	.735,199	.735,279	.735,359	.735,439	.735,519	80
544	.735,599	.735,679	.735,759	.735,838	.735,918	.735,998	.736,078	.736,157	.736,237	.736,317	80
545	.736,397	.736,476	.736,556	.736,635	.736,715	.736,795	.736,874	.736,954	.737,034	.737,113	80
546	.737,193	.737,272	.737,352	.737,431	.737,511	.737,590	.737,670	.737,749	.737,829	.737,908	79
547	.737,987	.738,067	.738,146	.738,225	.738,305	.738,384	.738,463	.738,543	.738,622	.738,701	79
548	.738,781	.738,860	.738,939	.739,018	.739,097	.739,177	.739,256	.739,335	.739,414	.739,493	79
549	.739,572	.739,651	.739,731	.739,810	.739,889	.739,968	.740,047	.740,126	.740,205	.740,284	79
550	.740,363	.740,442	.740,521	.740,599	.740,678	.740,757	.740,836	.740,915	.740,994	.741,073	79
551	.741,152	.741,230	.741,309	.741,388	.741,467	.741,546	.741,624	.741,703	.741,782	.741,860	79
552	.741,939	.742,018	.742,096	.742,175	.742,254	.742,332	.742,411	.742,489	.742,568	.742,647	79
553	.742,725	.742,804	.742,882	.742,961	.743,039	.743,118	.743,196	.743,275	.743,353	.743,431	78
554	.743,510	.743,588	.743,667	.743,745	.743,823	.743,902	.743,980	.744,058	.744,136	.744,215	78
555	.744,293	.744,371	.744,449	.744,528	.744,606	.744,684	.744,762	.744,840	.744,919	.744,997	78
556	.745,075	.745,153	.745,231	.745,309	.745,387	.745,465	.745,543	.745,621	.745,699	.745,777	78
557	.745,855	.745,933	.746,011	.746,089	.746,167	.746,245	.746,323	.746,401	.746,479	.746,556	78
558	.746,634	.746,712	.746,790	.746,868	.746,945	.747,023	.747,101	.747,179	.747,256	.747,334	78
559	.747,412	.747,489	.747,567	.747,645	.747,722	.747,800	.747,878	.747,955	.748,033	.748,110	78
560	.748,188	.748,266	.748,343	.748,421	.748,498	.748,576	.748,653	.748,731	.748,808	.748,885	77
561	.748,963	.749,040	.749,118	.749,195	.749,272	.749,350	.749,427	.749,504	.749,582	.749,659	77

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
562	.749,736	.749,814	.749,891	.749,968	.750,045	.750,123	.750,200	.750,277	.750,354	.750,431	77
563	.750,508	.750,586	.750,663	.750,740	.750,817	.750,894	.750,971	.751,048	.751,125	.751,202	77
564	.751,279	.751,356	.751,433	.751,510	.751,587	.751,664	.751,741	.751,818	.751,895	.751,972	77
565	.752,048	.752,125	.752,202	.752,279	.752,356	.752,433	.752,509	.752,586	.752,663	.752,740	77
566	.752,816	.752,893	.752,970	.753,047	.753,123	.753,200	.753,277	.753,353	.753,430	.753,500	77
567	.753,583	.753,660	.753,736	.753,813	.753,889	.753,966	.754,042	.754,119	.754,195	.754,272	77
568	.754,348	.754,425	.754,501	.754,578	.754,654	.754,730	.754,807	.754,883	.754,960	.755,036	76
569	.755,112	.755,189	.755,265	.755,341	.755,417	.755,494	.755,570	.755,646	.755,722	.755,799	76
570	.755,875	.755,951	.756,027	.756,103	.756,179	.756,256	.756,332	.756,408	.756,484	.756,560	76
571	.756,636	.756,712	.756,788	.756,864	.756,940	.757,016	.757,092	.757,168	.757,244	.757,320	76
572	.757,396	.757,472	.757,548	.757,624	.757,700	.757,775	.757,851	.757,927	.758,003	.758,079	76
573	.758,155	.758,230	.758,306	.758,382	.758,458	.758,533	.758,609	.758,685	.758,761	.758,836	76
574	.758,912	.758,988	.759,063	.759,139	.759,214	.759,290	.759,366	.759,441	.759,517	.759,592	76
575	.759,668	.759,743	.759,819	.759,894	.759,970	.760,045	.760,121	.760,196	.760,272	.760,347	75
576	.760,422	.760,498	.760,573	.760,649	.760,724	.760,799	.760,875	.760,950	.761,025	.761,101	75
577	.761,176	.761,251	.761,326	.761,402	.761,477	.761,552	.761,627	.761,702	.761,778	.761,853	75
578	.761,928	.762,003	.762,078	.762,153	.762,228	.762,303	.762,378	.762,453	.762,529	.762,604	75
579	.762,679	.762,754	.762,829	.762,904	.762,978	.763,053	.763,128	.763,203	.763,278	.763,353	75
580	.763,428	.763,503	.763,578	.763,653	.763,727	.763,802	.763,877	.763,952	.764,027	.764,101	75
581	.764,176	.764,251	.764,326	.764,400	.764,475	.764,550	.764,624	.764,699	.764,774	.764,848	75
582	.764,923	.764,998	.765,072	.765,147	.765,221	.765,296	.765,370	.765,445	.765,519	.765,594	75
583	.765,669	.765,743	.765,818	.765,892	.765,966	.766,041	.766,115	.766,190	.766,264	.766,338	74
584	.766,413	.766,487	.766,562	.766,636	.766,710	.766,784	.766,859	.766,933	.767,007	.767,082	74
585	.767,156	.767,230	.767,304	.767,379	.767,453	.767,527	.767,601	.767,675	.767,749	.767,823	74
586	.767,898	.767,972	.768,046	.768,120	.768,194	.768,268	.768,342	.768,416	.768,490	.768,564	74
587	.768,638	.768,712	.768,786	.768,860	.768,934	.769,008	.769,082	.769,156	.769,230	.769,303	74
588	.769,377	.769,451	.769,525	.769,599	.769,673	.769,746	.769,820	.769,894	.769,968	.770,042	74
589	.770,115	.770,189	.770,263	.770,336	.770,410	.770,484	.770,557	.770,631	.770,705	.770,778	74
590	.770,825	.770,926	.770,999	.771,073	.771,146	.771,220	.771,293	.771,367	.771,440	.771,514	74
591	.771,587	.771,661	.771,734	.771,808	.771,881	.771,955	.772,028	.772,102	.772,175	.772,248	73
592	.772,322	.772,395	.772,468	.772,542	.772,615	.772,688	.772,762	.772,835	.772,908	.772,981	73
593	.773,055	.773,128	.773,201	.773,274	.773,348	.773,421	.773,494	.773,567	.773,640	.773,713	73
594	.773,786	.773,860	.773,933	.774,006	.774,079	.774,152	.774,225	.774,298	.774,371	.774,444	73
595	.774,517	.774,590	.774,663	.774,736	.774,809	.774,882	.774,955	.775,028	.775,100	.775,173	73
596	.775,246	.775,319	.775,392	.775,465	.775,538	.775,610	.775,683	.775,756	.775,829	.775,902	73
597	.775,974	.776,047	.776,120	.776,193	.776,265	.776,338	.776,411	.776,483	.776,556	.776,629	73
598	.776,701	.776,774	.776,846	.776,919	.776,992	.777,064	.777,137	.777,209	.777,282	.777,354	73
599	.777,427	.777,499	.777,572	.777,644	.777,717	.777,789	.777,862	.777,934	.778,006	.778,079	72
600	.778,151	.778,224	.778,296	.778,368	.778,441	.778,513	.778,585	.778,658	.778,730	.778,802	72
601	.778,874	.778,947	.779,019	.779,091	.779,163	.779,236	.779,308	.779,380	.779,452	.779,524	72
602	.779,596	.779,669	.779,741	.779,813	.779,885	.779,957	.780,029	.780,101	.780,173	.780,245	72
603	.780,317	.780,389	.780,461	.780,533	.780,605	.780,677	.780,749	.780,821	.780,893	.780,965	72

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
604	.781,037	.781,109	.781,181	.781,253	.781,324	.781,396	.781,468	.781,540	.781,612	.781,684	72
605	.781,755	.781,827	.781,899	.781,971	.782,042	.782,114	.782,186	.782,258	.782,329	.782,401	72
606	.782,473	.782,544	.782,616	.782,688	.782,759	.782,831	.782,902	.782,974	.783,046	.783,117	72
607	.783,189	.783,260	.783,332	.783,403	.783,475	.783,546	.783,618	.783,689	.783,761	.783,832	71
608	.783,904	.783,975	.784,046	.784,118	.784,189	.784,261	.784,332	.784,403	.784,475	.784,546	71
609	.784,617	.784,689	.784,760	.784,831	.784,902	.784,974	.785,045	.785,116	.785,187	.785,259	71
610	.785,330	.785,401	.785,472	.785,543	.785,615	.785,686	.785,757	.785,828	.785,899	.785,970	71
611	.786,041	.786,112	.786,183	.786,254	.786,325	.786,396	.786,467	.786,538	.786,609	.786,680	71
612	.786,751	.786,822	.786,893	.786,964	.787,035	.787,106	.787,177	.787,248	.787,319	.787,390	71
613	.787,460	.787,531	.787,602	.787,673	.787,744	.787,815	.787,885	.787,956	.788,027	.788,098	71
614	.788,168	.788,239	.788,310	.788,381	.788,451	.788,522	.788,593	.788,663	.788,734	.788,804	71
615	.788,875	.788,946	.789,016	.789,087	.789,157	.789,228	.789,299	.789,369	.789,440	.789,510	71
616	.789,581	.789,651	.789,722	.789,792	.789,863	.789,933	.790,004	.790,074	.790,144	.790,215	70
617	.790,285	.790,356	.790,426	.790,496	.790,567	.790,637	.790,707	.790,778	.790,848	.790,918	70
618	.790,988	.791,059	.791,129	.791,199	.791,269	.791,340	.791,410	.791,480	.791,550	.791,620	70
619	.791,691	.791,761	.791,831	.791,901	.791,971	.792,041	.792,111	.792,181	.792,252	.792,322	70
620	.792,392	.792,462	.792,532	.792,602	.792,672	.792,742	.792,812	.792,882	.792,952	.793,022	70
621	.793,092	.793,162	.793,231	.793,301	.793,371	.793,441	.793,511	.793,581	.793,651	.793,721	70
622	.793,790	.793,860	.793,930	.794,000	.794,070	.794,139	.794,209	.794,279	.794,349	.794,418	70
623	.794,488	.794,558	.794,627	.794,697	.794,767	.794,836	.794,906	.794,976	.795,045	.795,115	70
624	.795,185	.795,254	.795,324	.795,393	.795,463	.795,532	.795,602	.795,672	.795,741	.795,810	70
625	.795,880	.795,949	.796,019	.796,088	.796,158	.796,227	.796,297	.796,366	.796,436	.796,505	69
626	.796,574	.796,644	.796,713	.796,782	.796,852	.796,921	.796,990	.797,060	.797,129	.797,198	69
627	.797,268	.797,337	.797,406	.797,475	.797,545	.797,614	.797,683	.797,752	.797,821	.797,890	69
628	.797,960	.798,029	.798,098	.798,167	.798,236	.798,305	.798,374	.798,443	.798,513	.798,582	69
629	.798,651	.798,720	.798,789	.798,858	.798,927	.798,996	.799,065	.799,134	.799,203	.799,272	69
630	.799,341	.799,409	.799,478	.799,547	.799,616	.799,685	.799,754	.799,823	.799,892	.799,961	69
631	.800,029	.800,098	.800,167	.800,236	.800,305	.800,373	.800,442	.800,511	.800,580	.800,648	69
632	.800,717	.800,786	.800,854	.800,923	.800,992	.801,061	.801,129	.801,198	.801,266	.801,335	69
633	.801,404	.801,472	.801,541	.801,610	.801,678	.801,747	.801,815	.801,884	.801,952	.802,021	69
634	.802,089	.802,158	.802,226	.802,295	.802,363	.802,432	.802,500	.802,568	.802,637	.802,705	68
635	.802,774	.802,842	.802,910	.802,979	.803,047	.803,116	.803,184	.803,252	.803,321	.803,389	68
636	.803,457	.803,525	.803,594	.803,662	.803,730	.803,798	.803,867	.803,935	.804,003	.804,071	68
637	.804,139	.804,208	.804,276	.804,344	.804,412	.804,480	.804,548	.804,616	.804,685	.804,753	68
638	.804,821	.804,889	.804,957	.805,025	.805,093	.805,161	.805,229	.805,297	.805,365	.805,433	68
639	.805,501	.805,569	.805,637	.805,705	.805,773	.805,840	.805,908	.805,976	.806,044	.806,112	68
640	.806,180	.806,248	.806,316	.806,384	.806,451	.806,519	.806,587	.806,655	.806,722	.806,790	68
641	.806,858	.806,926	.806,993	.807,061	.807,129	.807,197	.807,264	.807,332	.807,400	.807,467	68
642	.807,535	.807,603	.807,670	.807,738	.807,805	.807,873	.807,941	.808,008	.808,076	.808,143	68
643	.808,211	.808,279	.808,346	.808,414	.808,481	.808,548	.808,616	.808,684	.808,751	.808,818	67
644	.808,886	.808,953	.809,021	.809,088	.809,156	.809,223	.809,290	.809,358	.809,425	.809,492	67
645	.809,560	.809,627	.809,694	.809,762	.809,829	.809,896	.809,964	.810,031	.810,098	.810,165	67

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
646	.810,233	.810,300	.810,367	.810,434	.810,501	.810,569	.810,636	.810,703	.810,770	.810,837	67
647	.810,904	.810,971	.811,039	.811,106	.811,173	.811,240	.811,307	.811,374	.811,441	.811,508	67
648	.811,575	.811,642	.811,709	.811,776	.811,843	.811,910	.811,977	.812,044	.812,111	.812,178	67
649	.812,245	.812,312	.812,379	.812,445	.812,512	.812,579	.812,646	.812,713	.812,780	.812,847	67
650	.812,913	.812,980	.813,047	.813,114	.813,181	.813,247	.813,314	.813,381	.813,448	.813,514	66
651	.813,581	.813,648	.813,714	.813,781	.813,848	.813,914	.813,981	.814,048	.814,114	.814,181	66
652	.814,248	.814,314	.814,381	.814,447	.814,514	.814,581	.814,647	.814,714	.814,780	.814,847	66
653	.814,913	.814,980	.815,046	.815,113	.815,179	.815,246	.815,312	.815,379	.815,445	.815,511	66
654	.815,578	.815,644	.815,711	.815,777	.815,843	.815,910	.815,976	.816,042	.816,109	.816,175	66
655	.816,241	.816,308	.816,374	.816,440	.816,506	.816,573	.816,639	.816,705	.816,771	.816,838	66
656	.816,904	.816,970	.817,036	.817,102	.817,169	.817,235	.817,301	.817,367	.817,433	.817,499	66
657	.817,565	.817,631	.817,698	.817,764	.817,830	.817,896	.817,962	.818,028	.818,094	.818,160	66
658	.818,226	.818,292	.818,358	.818,424	.818,490	.818,556	.818,622	.818,688	.818,754	.818,819	66
659	.818,885	.818,951	.819,017	.819,083	.819,149	.819,215	.819,281	.819,347	.819,412	.819,478	66
660	.819,544	.819,610	.819,676	.819,741	.819,807	.819,873	.819,939	.820,004	.820,070	.820,136	66
661	.820,202	.820,267	.820,333	.820,399	.820,464	.820,530	.820,596	.820,661	.820,727	.820,792	66
662	.820,858	.820,924	.820,989	.821,055	.821,120	.821,186	.821,251	.821,317	.821,382	.821,448	66
663	.821,514	.821,579	.821,645	.821,710	.821,776	.821,841	.821,906	.821,972	.822,037	.822,103	65
664	.822,168	.822,233	.822,299	.822,364	.822,430	.822,495	.822,560	.822,626	.822,691	.822,756	65
665	.822,822	.822,887	.822,952	.823,018	.823,083	.823,148	.823,213	.823,279	.823,344	.823,409	65
666	.823,474	.823,539	.823,605	.823,670	.823,735	.823,800	.823,865	.823,930	.823,996	.824,061	65
667	.824,126	.824,191	.824,256	.824,321	.824,386	.824,451	.824,516	.824,581	.824,646	.824,711	65
668	.824,777	.824,842	.824,907	.824,972	.825,036	.825,101	.825,166	.825,231	.825,296	.825,361	65
669	.825,426	.825,491	.825,556	.825,621	.825,686	.825,751	.825,815	.825,880	.825,945	.826,010	65
670	.826,075	.826,140	.826,204	.826,269	.826,334	.826,399	.826,464	.826,528	.826,593	.826,658	65
671	.826,723	.826,787	.826,852	.826,917	.826,981	.827,046	.827,111	.827,175	.827,240	.827,305	65
672	.827,369	.827,434	.827,499	.827,563	.827,628	.827,692	.827,757	.827,821	.827,886	.827,951	65
673	.828,015	.828,080	.828,144	.828,209	.828,273	.828,338	.828,402	.828,466	.828,531	.828,596	64
674	.828,660	.828,724	.828,789	.828,853	.828,918	.828,982	.829,046	.829,111	.829,175	.829,240	64
675	.829,304	.829,368	.829,432	.829,497	.829,561	.829,625	.829,690	.829,754	.829,818	.829,882	64
676	.829,947	.830,011	.830,075	.830,139	.830,204	.830,268	.830,332	.830,396	.830,460	.830,525	64
677	.830,589	.830,653	.830,717	.830,781	.830,845	.830,909	.830,973	.831,037	.831,102	.831,166	64
678	.831,230	.831,294	.831,358	.831,422	.831,486	.831,550	.831,614	.831,678	.831,742	.831,806	64
679	.831,870	.831,934	.831,998	.832,062	.832,126	.832,189	.832,253	.832,317	.832,381	.832,445	64
680	.832,509	.832,573	.832,637	.832,700	.832,764	.832,828	.832,892	.832,956	.833,019	.833,083	64
681	.833,147	.833,211	.833,275	.833,338	.833,402	.833,466	.833,530	.833,593	.833,657	.833,721	64
682	.833,784	.833,848	.833,912	.833,975	.834,039	.834,103	.834,166	.834,230	.834,294	.834,357	64
683	.834,421	.834,484	.834,548	.834,611	.834,675	.834,738	.834,802	.834,866	.834,929	.834,993	64
684	.835,056	.835,120	.835,183	.835,247	.835,310	.835,373	.835,437	.835,500	.835,564	.835,627	63
685	.835,691	.835,754	.835,817	.835,881	.835,944	.836,007	.836,070	.836,134	.836,197	.836,261	63
686	.836,324	.836,387	.836,451	.836,514	.836,577	.836,641	.836,704	.836,767	.836,830	.836,894	63
687	.836,957	.837,020	.837,083	.837,146	.837,209	.837,273	.837,336	.837,399	.837,462	.837,525	63

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
688	.837,588	.837,652	.837,715	.837,778	.837,841	.837,904	.837,969	.838,030	.838,093	.838,156	63
689	.838,210	.838,282	.838,345	.838,408	.838,471	.838,534	.838,597	.838,660	.838,723	.838,786	63
690	.838,849	.838,912	.838,975	.839,038	.839,101	.839,164	.839,227	.839,289	.839,352	.839,415	63
691	.839,478	.839,541	.839,604	.839,667	.839,729	.839,792	.839,855	.839,918	.839,981	.840,043	63
692	.840,106	.840,169	.840,232	.840,294	.840,357	.840,420	.840,482	.840,545	.840,608	.840,671	63
693	.840,733	.840,796	.840,859	.840,921	.840,984	.841,046	.841,109	.841,172	.841,234	.841,297	63
694	.841,359	.841,422	.841,485	.841,547	.841,610	.841,672	.841,735	.841,797	.841,860	.841,922	63
695	.841,985	.842,047	.842,110	.842,172	.842,235	.842,297	.842,360	.842,422	.842,484	.842,547	62
696	.842,600	.842,672	.842,734	.842,796	.842,859	.842,921	.842,983	.843,046	.843,108	.843,170	62
697	.843,233	.843,295	.843,357	.843,420	.843,482	.843,544	.843,606	.843,669	.843,731	.843,793	62
698	.843,855	.843,918	.843,980	.844,042	.844,104	.844,166	.844,229	.844,291	.844,353	.844,415	62
699	.844,477	.844,539	.844,601	.844,664	.844,726	.844,788	.844,850	.844,912	.844,974	.845,036	62
700	.845,098	.845,160	.845,222	.845,284	.845,346	.845,408	.845,470	.845,532	.845,594	.845,656	62
701	.845,718	.845,780	.845,842	.845,904	.845,966	.846,028	.846,090	.846,151	.846,213	.846,275	62
702	.846,337	.846,399	.846,461	.846,523	.846,585	.846,646	.846,708	.846,770	.846,832	.846,894	62
703	.846,955	.847,017	.847,079	.847,141	.847,202	.847,264	.847,326	.847,388	.847,449	.847,511	62
704	.847,573	.847,634	.847,696	.847,758	.847,819	.847,881	.847,943	.848,004	.848,066	.848,127	62
705	.848,189	.848,251	.848,312	.848,374	.848,435	.848,497	.848,559	.848,620	.848,682	.848,743	62
706	.848,805	.848,866	.848,928	.848,989	.849,051	.849,112	.849,174	.849,235	.849,297	.849,358	61
707	.849,419	.849,481	.849,542	.849,604	.849,665	.849,726	.849,788	.849,849	.849,911	.849,972	61
708	.850,033	.850,095	.850,156	.850,217	.850,279	.850,340	.850,401	.850,462	.850,524	.850,585	61
709	.850,646	.850,707	.850,769	.850,830	.850,891	.850,952	.851,014	.851,075	.851,136	.851,197	61
710	.851,258	.851,319	.851,381	.851,442	.851,503	.851,564	.851,625	.851,686	.851,747	.851,809	61
711	.851,869	.851,931	.851,992	.852,053	.852,114	.852,175	.852,236	.852,297	.852,358	.852,419	61
712	.852,480	.852,541	.852,602	.852,663	.852,724	.852,785	.852,846	.852,907	.852,968	.853,029	61
713	.853,089	.853,150	.853,211	.853,272	.853,333	.853,394	.853,455	.853,516	.853,576	.853,637	61
714	.853,698	.853,759	.853,820	.853,881	.853,941	.854,002	.854,063	.854,124	.854,184	.854,245	61
715	.854,306	.854,367	.854,428	.854,488	.854,549	.854,610	.854,670	.854,731	.854,792	.854,852	61
716	.854,913	.854,974	.855,034	.855,095	.855,156	.855,216	.855,277	.855,337	.855,398	.855,459	61
717	.855,519	.855,580	.855,640	.855,701	.855,761	.855,822	.855,882	.855,943	.856,003	.856,064	61
718	.856,124	.856,185	.856,245	.856,306	.856,366	.856,427	.856,487	.856,548	.856,608	.856,668	60
719	.856,729	.856,789	.856,850	.856,910	.856,970	.857,031	.857,091	.857,151	.857,212	.857,272	60
720	.857,333	.857,393	.857,453	.857,513	.857,574	.857,634	.857,694	.857,754	.857,815	.857,875	60
721	.857,935	.857,995	.858,056	.858,116	.858,176	.858,236	.858,296	.858,357	.858,417	.858,477	60
722	.858,537	.858,597	.858,657	.858,718	.858,778	.858,838	.858,898	.858,958	.859,018	.859,078	60
723	.859,138	.859,198	.859,258	.859,318	.859,378	.859,439	.859,499	.859,559	.859,619	.859,679	60
724	.859,739	.859,799	.859,859	.859,918	.859,978	.860,038	.860,098	.860,158	.860,218	.860,278	60
725	.860,338	.860,398	.860,458	.860,518	.860,578	.860,637	.860,697	.860,757	.860,817	.860,877	60
726	.860,937	.860,996	.861,056	.861,116	.861,176	.861,236	.861,295	.861,355	.861,415	.861,475	60
727	.861,534	.861,594	.861,654	.861,714	.861,773	.861,833	.861,893	.861,952	.862,012	.862,072	60
728	.862,131	.862,191	.862,251	.862,310	.862,370	.862,430	.862,489	.862,549	.862,608	.862,668	60
729	.862,727	.862,787	.862,847	.862,906	.862,966	.863,025	.863,085	.863,144	.863,204	.863,263	60

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
730	.863,323	.863,382	.863,442	.863,501	.863,561	.863,620	.863,680	.863,739	.863,798	.863,858	59
731	.863,917	.863,977	.864,036	.864,096	.864,155	.864,214	.864,274	.864,333	.864,392	.864,452	59
732	.864,511	.864,570	.864,630	.864,689	.864,748	.864,808	.864,867	.864,926	.864,985	.865,045	59
733	.865,104	.865,163	.865,223	.865,282	.865,341	.865,400	.865,459	.865,518	.865,578	.865,637	59
734	.865,696	.865,755	.865,814	.865,873	.865,933	.865,992	.866,051	.866,110	.866,169	.866,228	59
735	.866,287	.866,346	.866,405	.866,465	.866,524	.866,583	.866,642	.866,701	.866,760	.866,819	59
736	.866,878	.866,937	.866,996	.867,055	.867,114	.867,173	.867,232	.867,291	.867,350	.867,409	59
737	.867,467	.867,526	.867,585	.867,644	.867,703	.867,762	.867,821	.867,880	.867,939	.867,997	59
738	.868,056	.868,115	.868,174	.868,233	.868,292	.868,350	.868,409	.868,468	.868,527	.868,586	59
739	.868,644	.868,703	.868,762	.868,821	.868,879	.868,938	.868,997	.869,056	.869,114	.869,173	59
740	.869,232	.869,290	.869,349	.869,408	.869,466	.869,525	.869,584	.869,642	.869,701	.869,760	59
741	.869,818	.869,877	.869,935	.869,994	.870,053	.870,111	.870,170	.870,228	.870,287	.870,345	59
742	.870,404	.870,462	.870,521	.870,579	.870,638	.870,696	.870,755	.870,813	.870,872	.870,930	58
743	.870,989	.871,047	.871,106	.871,164	.871,223	.871,281	.871,339	.871,398	.871,456	.871,515	58
744	.871,573	.871,631	.871,690	.871,748	.871,806	.871,865	.871,923	.871,981	.872,040	.872,098	58
745	.872,156	.872,215	.872,273	.872,331	.872,389	.872,448	.872,506	.872,564	.872,622	.872,681	58
746	.872,739	.872,797	.872,855	.872,913	.872,972	.873,030	.873,088	.873,146	.873,204	.873,262	58
747	.873,321	.873,379	.873,437	.873,495	.873,553	.873,611	.873,669	.873,727	.873,785	.873,843	58
748	.873,902	.873,960	.874,018	.874,076	.874,134	.874,192	.874,250	.874,308	.874,366	.874,424	58
749	.874,482	.874,540	.874,598	.874,656	.874,714	.874,772	.874,830	.874,887	.874,945	.875,003	58
750	.875,061	.875,119	.875,177	.875,235	.875,293	.875,351	.875,409	.875,466	.875,524	.875,582	58
751	.875,640	.875,698	.875,756	.875,813	.875,871	.875,929	.875,987	.876,044	.876,102	.876,160	58
752	.876,218	.876,276	.876,333	.876,391	.876,449	.876,506	.876,564	.876,622	.876,680	.876,737	58
753	.876,795	.876,853	.876,910	.876,968	.877,026	.877,083	.877,141	.877,198	.877,256	.877,314	58
754	.877,371	.877,429	.877,487	.877,544	.877,602	.877,659	.877,717	.877,774	.877,832	.877,889	58
755	.877,947	.878,004	.878,062	.878,119	.878,177	.878,234	.878,292	.878,349	.878,407	.878,464	57
756	.878,522	.878,579	.878,637	.878,694	.878,751	.878,809	.878,866	.878,924	.878,981	.879,038	57
757	.879,096	.879,153	.879,211	.879,268	.879,325	.879,383	.879,440	.879,497	.879,555	.879,612	57
758	.879,669	.879,726	.879,784	.879,841	.879,898	.879,956	.880,013	.880,070	.880,127	.880,185	57
759	.880,242	.880,299	.880,356	.880,413	.880,471	.880,528	.880,585	.880,642	.880,699	.880,756	57
760	.880,814	.880,871	.880,928	.880,985	.881,042	.881,099	.881,156	.881,213	.881,270	.881,328	57
761	.881,385	.881,442	.881,499	.881,556	.881,613	.881,670	.881,727	.881,784	.881,841	.881,898	57
762	.881,955	.882,012	.882,069	.882,126	.882,183	.882,240	.882,297	.882,354	.882,411	.882,468	57
763	.882,524	.882,581	.882,638	.882,695	.882,752	.882,809	.882,866	.882,923	.882,980	.883,036	57
764	.883,093	.883,150	.883,207	.883,264	.883,321	.883,377	.883,434	.883,491	.883,548	.883,605	57
765	.883,661	.883,718	.883,775	.883,832	.883,888	.883,945	.884,002	.884,059	.884,115	.884,172	57
766	.884,229	.884,285	.884,342	.884,399	.884,455	.884,512	.884,569	.884,625	.884,682	.884,739	57
767	.884,795	.884,852	.884,909	.884,965	.885,022	.885,078	.885,135	.885,191	.885,248	.885,305	57
768	.885,361	.885,418	.885,474	.885,531	.885,587	.885,644	.885,700	.885,757	.885,813	.885,870	57
769	.885,926	.885,983	.886,039	.886,096	.886,152	.886,209	.886,265	.886,321	.886,378	.886,434	56
770	.886,491	.886,547	.886,604	.886,660	.886,716	.886,773	.886,829	.886,885	.886,942	.886,998	56
771	.887,054	.887,111	.887,167	.887,223	.887,280	.887,336	.887,392	.887,449	.887,505	.887,561	56

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
772	.887,617	.887,674	.887,730	.887,786	.887,842	.887,898	.887,955	.888,011	.888,067	.888,123	56
773	.888,179	.888,236	.888,292	.888,348	.888,404	.888,460	.888,516	.888,573	.888,629	.888,685	56
774	.888,741	.888,797	.888,853	.888,909	.888,965	.889,021	.889,077	.889,133	.889,190	.889,246	56
775	.889,302	.889,358	.889,414	.889,470	.889,526	.889,582	.889,638	.889,694	.889,750	.889,806	56
776	.889,862	.889,918	.889,974	.890,030	.890,086	.890,141	.890,197	.890,253	.890,309	.890,365	56
777	.890,421	.890,477	.890,533	.890,589	.890,644	.890,700	.890,756	.890,812	.890,868	.890,924	56
778	.890,980	.891,035	.891,091	.891,147	.891,203	.891,259	.891,314	.891,370	.891,426	.891,482	56
779	.891,537	.891,593	.891,649	.891,705	.891,760	.891,816	.891,872	.891,928	.891,983	.892,039	56
780	.892,095	.892,150	.892,206	.892,262	.892,317	.892,373	.892,428	.892,484	.892,540	.892,595	56
781	.892,651	.892,707	.892,762	.892,818	.892,873	.892,929	.892,985	.893,040	.893,096	.893,151	56
782	.893,207	.893,262	.893,318	.893,373	.893,429	.893,484	.893,540	.893,595	.893,651	.893,706	56
783	.893,762	.893,817	.893,873	.893,928	.893,984	.894,039	.894,094	.894,150	.894,205	.894,261	55
784	.894,316	.894,371	.894,427	.894,482	.894,538	.894,593	.894,648	.894,704	.894,759	.894,814	55
785	.894,870	.894,925	.894,980	.895,036	.895,091	.895,146	.895,201	.895,257	.895,312	.895,367	55
786	.895,422	.895,478	.895,533	.895,588	.895,643	.895,699	.895,754	.895,809	.895,864	.895,919	55
787	.895,975	.896,030	.896,085	.896,140	.896,195	.896,251	.896,306	.896,361	.896,416	.896,471	55
788	.896,526	.896,581	.896,636	.896,691	.896,747	.896,802	.896,857	.896,912	.896,967	.897,022	55
789	.897,077	.897,132	.897,187	.897,242	.897,297	.897,352	.897,407	.897,462	.897,517	.897,572	55
790	.897,627	.897,682	.897,737	.897,792	.897,847	.897,902	.897,957	.898,012	.898,067	.898,122	55
791	.898,176	.898,231	.898,286	.898,341	.898,396	.898,451	.898,506	.898,561	.898,615	.898,670	55
792	.898,725	.898,780	.898,835	.898,890	.898,944	.898,999	.899,054	.899,109	.899,164	.899,218	55
793	.899,273	.899,328	.899,383	.899,437	.899,492	.899,547	.899,602	.899,656	.899,711	.899,766	55
794	.899,820	.899,875	.899,930	.899,985	.900,039	.900,094	.900,149	.900,203	.900,258	.900,312	55
795	.900,367	.900,422	.900,476	.900,531	.900,586	.900,640	.900,695	.900,749	.900,804	.900,859	55
796	.900,913	.900,968	.901,022	.901,077	.901,131	.901,186	.901,240	.901,295	.901,349	.901,404	55
797	.901,458	.901,513	.901,567	.901,622	.901,676	.901,731	.901,785	.901,840	.901,894	.901,948	54
798	.902,003	.902,057	.902,112	.902,166	.902,220	.902,275	.902,329	.902,384	.902,438	.902,492	54
799	.902,547	.902,601	.902,655	.902,710	.902,764	.902,818	.902,873	.902,927	.902,981	.903,036	54
800	.903,090	.903,144	.903,198	.903,253	.903,307	.903,361	.903,416	.903,470	.903,524	.903,578	54
801	.903,632	.903,687	.903,741	.903,795	.903,849	.903,903	.903,958	.904,012	.904,066	.904,120	54
802	.904,174	.904,228	.904,283	.904,337	.904,391	.904,445	.904,499	.904,553	.904,607	.904,661	54
803	.904,715	.904,770	.904,824	.904,878	.904,932	.904,986	.905,040	.905,094	.905,148	.905,202	54
804	.905,276	.905,310	.905,364	.905,418	.905,472	.905,526	.905,580	.905,634	.905,688	.905,742	54
805	.905,796	.905,850	.905,904	.905,958	.906,012	.906,065	.906,119	.906,173	.906,227	.906,281	54
806	.906,335	.906,389	.906,443	.906,497	.906,550	.906,604	.906,658	.906,712	.906,766	.906,820	54
807	.906,873	.906,927	.906,981	.907,035	.907,089	.907,142	.907,196	.907,250	.907,304	.907,358	54
808	.907,411	.907,465	.907,519	.907,573	.907,626	.907,680	.907,734	.907,787	.907,841	.907,895	54
809	.907,948	.908,002	.908,056	.908,109	.908,163	.908,217	.908,270	.908,324	.908,378	.908,431	54
810	.908,485	.908,539	.908,592	.908,646	.908,699	.908,753	.908,807	.908,860	.908,914	.908,967	54
811	.909,021	.909,074	.909,128	.909,181	.909,235	.909,288	.909,342	.909,395	.909,449	.909,502	54
812	.909,556	.909,609	.909,663	.909,716	.909,770	.909,823	.909,877	.909,930	.909,984	.910,037	53
813	.910,090	.910,144	.910,197	.910,251	.910,304	.910,358	.910,411	.910,464	.910,518	.910,571	53

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
814	.910,624	.910,678	.910,731	.910,784	.910,838	.910,891	.910,944	.910,998	.911,051	.911,104	53
815	.911,158	.911,211	.911,264	.911,317	.911,371	.911,424	.911,477	.911,530	.911,584	.911,637	53
816	.911,690	.911,743	.911,797	.911,850	.911,903	.911,956	.912,009	.912,063	.912,116	.912,169	53
817	.912,222	.912,275	.912,323	.912,381	.912,435	.912,488	.912,541	.912,594	.912,647	.912,700	53
818	.912,753	.912,806	.912,859	.912,913	.912,966	.913,019	.913,072	.913,125	.913,178	.913,231	53
819	.913,283	.913,377	.913,390	.913,443	.913,496	.913,549	.913,602	.913,655	.913,708	.913,761	53
820	.913,814	.913,867	.913,920	.913,973	.914,026	.914,079	.914,132	.914,184	.914,237	.914,290	53
821	.914,343	.914,396	.914,449	.914,502	.914,555	.914,608	.914,660	.914,713	.914,766	.914,819	53
822	.914,872	.914,925	.914,977	.915,030	.915,083	.915,136	.915,189	.915,241	.915,294	.915,347	53
823	.915,400	.915,453	.915,505	.915,558	.915,611	.915,664	.915,716	.915,769	.915,822	.915,874	53
824	.915,927	.915,980	.916,033	.916,085	.916,138	.916,191	.916,243	.916,296	.916,349	.916,401	53
825	.916,454	.916,507	.916,559	.916,612	.916,664	.916,717	.916,770	.916,822	.916,875	.916,927	53
826	.916,980	.917,033	.917,085	.917,138	.917,190	.917,243	.917,295	.917,348	.917,400	.917,453	53
827	.917,505	.917,558	.917,611	.917,663	.917,715	.917,768	.917,820	.917,873	.917,925	.917,978	53
828	.918,030	.918,083	.918,135	.918,188	.918,240	.918,292	.918,345	.918,397	.918,450	.918,502	53
829	.918,554	.918,607	.918,659	.918,712	.918,764	.918,816	.918,869	.918,921	.918,973	.919,026	53
830	.919,078	.919,130	.919,183	.919,235	.919,287	.919,340	.919,392	.919,444	.919,496	.919,549	52
831	.919,601	.919,653	.919,706	.919,758	.919,810	.919,862	.919,914	.919,967	.920,019	.920,071	52
832	.920,123	.920,175	.920,228	.920,280	.920,332	.920,384	.920,436	.920,489	.920,541	.920,593	52
833	.920,645	.920,697	.920,749	.920,801	.920,853	.920,906	.920,958	.921,010	.921,062	.921,114	52
834	.921,166	.921,218	.921,270	.921,322	.921,374	.921,426	.921,478	.921,530	.921,582	.921,634	52
835	.921,686	.921,738	.921,790	.921,842	.921,894	.921,946	.921,998	.922,050	.922,102	.922,154	52
836	.922,206	.922,258	.922,310	.922,362	.922,414	.922,466	.922,518	.922,570	.922,622	.922,674	52
837	.922,725	.922,777	.922,829	.922,881	.922,933	.922,985	.923,037	.923,088	.923,140	.923,192	52
838	.923,244	.923,296	.923,348	.923,399	.923,451	.923,503	.923,555	.923,607	.923,658	.923,710	52
839	.923,762	.923,814	.923,865	.923,917	.923,969	.924,021	.924,072	.924,124	.924,176	.924,228	52
840	.924,279	.924,331	.924,383	.924,434	.924,486	.924,538	.924,589	.924,641	.924,693	.924,744	52
841	.924,796	.924,848	.924,899	.924,951	.925,002	.925,054	.925,106	.925,157	.925,209	.925,261	52
842	.925,312	.925,364	.925,415	.925,467	.925,518	.925,570	.925,621	.925,673	.925,725	.925,776	52
843	.925,828	.925,879	.925,931	.925,982	.926,034	.926,085	.926,137	.926,188	.926,239	.926,291	51
844	.926,342	.926,394	.926,445	.926,497	.926,548	.926,600	.926,651	.926,702	.926,754	.926,805	51
845	.926,857	.926,908	.926,959	.927,011	.927,062	.927,114	.927,165	.927,216	.927,268	.927,319	51
846	.927,370	.927,422	.927,473	.927,524	.927,576	.927,627	.927,678	.927,730	.927,781	.927,832	51
847	.927,883	.927,935	.927,986	.928,037	.928,088	.928,140	.928,191	.928,242	.928,293	.928,345	51
848	.928,396	.928,447	.928,498	.928,549	.928,601	.928,652	.928,703	.928,754	.928,805	.928,856	51
849	.928,908	.928,959	.929,010	.929,061	.929,112	.929,163	.929,214	.929,266	.929,317	.929,368	51
850	.929,419	.929,470	.929,521	.929,572	.929,623	.929,674	.929,725	.929,776	.929,827	.929,878	51
851	.929,930	.929,981	.930,032	.930,083	.930,134	.930,185	.930,236	.930,287	.930,338	.930,389	51
852	.930,440	.930,491	.930,541	.930,592	.930,643	.930,694	.930,745	.930,796	.930,847	.930,898	51
853	.930,949	.931,000	.931,051	.931,102	.931,153	.931,203	.931,254	.931,305	.931,356	.931,407	51
854	.931,458	.931,509	.931,560	.931,610	.931,661	.931,712	.931,763	.931,814	.931,864	.931,915	51
855	.931,966	.932,017	.932,068	.932,118	.932,169	.932,220	.932,271	.932,322	.932,372	.932,423	51

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
856	.932,474	.932,524	.932,575	.932,626	.932,677	.932,727	.932,778	.932,829	.932,879	.932,930	51
857	.932,981	.933,031	.933,082	.933,133	.933,183	.933,234	.933,285	.933,335	.933,386	.933,437	51
858	.933,487	.933,538	.933,588	.933,639	.933,690	.933,740	.933,791	.933,841	.933,892	.933,943	51
859	.933,993	.934,044	.934,094	.934,145	.934,195	.934,246	.934,296	.934,347	.934,397	.934,448	51
860	.934,498	.934,549	.934,599	.934,650	.934,700	.934,751	.934,801	.934,852	.934,902	.934,953	50
861	.935,003	.935,054	.935,104	.935,154	.935,205	.935,255	.935,306	.935,356	.935,406	.935,457	50
862	.935,507	.935,558	.935,608	.935,658	.935,709	.935,759	.935,809	.935,860	.935,910	.935,960	50
863	.936,011	.936,061	.936,111	.936,162	.936,212	.936,262	.936,313	.936,363	.936,413	.936,463	50
864	.936,514	.936,564	.936,614	.936,665	.936,715	.936,765	.936,815	.936,865	.936,916	.936,966	50
865	.937,016	.937,066	.937,116	.937,167	.937,217	.937,267	.937,317	.937,367	.937,418	.937,468	50
866	.937,518	.937,568	.937,618	.937,668	.937,718	.937,769	.937,819	.937,869	.937,919	.937,969	50
867	.938,019	.938,069	.938,119	.938,169	.938,219	.938,269	.938,319	.938,370	.938,420	.938,470	50
868	.938,520	.938,570	.938,620	.938,670	.938,720	.938,770	.938,820	.938,870	.938,920	.938,970	50
869	.939,020	.939,070	.939,120	.939,170	.939,220	.939,270	.939,320	.939,369	.939,420	.939,470	50
870	.939,519	.939,569	.939,619	.939,669	.939,719	.939,769	.939,819	.939,868	.939,918	.939,968	50
871	.940,018	.940,068	.940,118	.940,168	.940,218	.940,267	.940,317	.940,367	.940,417	.940,467	50
872	.940,516	.940,566	.940,616	.940,666	.940,716	.940,765	.940,815	.940,865	.940,915	.940,964	50
873	.941,014	.941,064	.941,114	.941,163	.941,213	.941,263	.941,313	.941,362	.941,412	.941,462	50
874	.941,511	.941,561	.941,611	.941,660	.941,710	.941,760	.941,809	.941,859	.941,909	.941,958	50
875	.942,008	.942,058	.942,107	.942,157	.942,206	.942,256	.942,306	.942,355	.942,405	.942,454	50
876	.942,504	.942,554	.942,603	.942,653	.942,702	.942,752	.942,801	.942,851	.942,900	.942,950	50
877	.943,000	.943,049	.943,096	.943,148	.943,198	.943,247	.943,297	.943,346	.943,396	.943,445	49
878	.943,494	.943,544	.943,593	.943,643	.943,692	.943,742	.943,791	.943,841	.943,890	.943,939	49
879	.943,989	.944,038	.944,088	.944,137	.944,186	.944,236	.944,285	.944,335	.944,384	.944,433	49
880	.944,483	.944,532	.944,581	.944,631	.944,680	.944,729	.944,779	.944,828	.944,877	.944,927	49
881	.944,976	.945,025	.945,074	.945,124	.945,173	.945,222	.945,272	.945,321	.945,370	.945,419	49
882	.945,469	.945,518	.945,567	.945,616	.945,665	.945,715	.945,764	.945,813	.945,862	.945,911	49
883	.945,961	.946,010	.946,059	.946,108	.946,157	.946,207	.946,256	.946,305	.946,354	.946,403	49
884	.946,452	.946,501	.946,551	.946,600	.946,649	.946,698	.946,747	.946,796	.946,845	.946,894	49
885	.946,943	.946,992	.947,041	.947,090	.947,139	.947,189	.947,238	.947,287	.947,336	.947,385	49
886	.947,434	.947,483	.947,532	.947,581	.947,630	.947,679	.947,728	.947,777	.947,826	.947,875	49
887	.947,924	.947,973	.948,022	.948,070	.948,119	.948,168	.948,217	.948,266	.948,315	.948,364	49
888	.948,413	.948,462	.948,511	.948,560	.948,609	.948,657	.948,706	.948,755	.948,804	.948,853	49
889	.948,902	.948,951	.948,999	.949,048	.949,097	.949,146	.949,195	.949,244	.949,292	.949,341	49
890	.949,390	.949,439	.949,488	.949,536	.949,585	.949,633	.949,683	.949,731	.949,780	.949,829	49
891	.949,878	.949,926	.949,975	.950,024	.950,073	.950,121	.950,170	.950,219	.950,267	.950,316	49
892	.950,365	.950,414	.950,462	.950,511	.950,560	.950,608	.950,657	.950,706	.950,754	.950,803	49
893	.950,851	.950,900	.950,949	.950,997	.951,046	.951,095	.951,143	.951,192	.951,240	.951,289	49
894	.951,337	.951,386	.951,435	.951,483	.951,532	.951,580	.951,629	.951,677	.951,726	.951,775	49
895	.951,823	.951,872	.951,920	.951,969	.952,017	.952,066	.952,114	.952,163	.952,211	.952,259	49
896	.952,308	.952,356	.952,405	.952,453	.952,502	.952,550	.952,599	.952,647	.952,696	.952,744	48
897	.952,792	.952,841	.952,889	.952,938	.952,986	.953,034	.953,083	.953,131	.953,180	.953,228	48

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
898	.953,276	.953,325	.953,373	.953,421	.953,470	.953,518	.953,566	.953,615	.953,663	.953,711	48
899	.953,760	.953,808	.953,856	.953,905	.953,953	.954,001	.954,049	.954,098	.954,146	.954,194	48
900	.954,242	.954,291	.954,339	.954,387	.954,435	.954,484	.954,532	.954,580	.954,628	.954,677	48
901	.954,725	.954,773	.954,821	.954,869	.954,918	.954,966	.955,014	.955,062	.955,110	.955,158	48
902	.955,207	.955,255	.955,303	.955,351	.955,399	.955,447	.955,495	.955,543	.955,591	.955,640	48
903	.955,688	.955,736	.955,784	.955,832	.955,880	.955,928	.955,976	.956,024	.956,072	.956,120	48
904	.956,168	.956,216	.956,264	.956,313	.956,361	.956,409	.956,457	.956,505	.956,553	.956,601	48
905	.956,649	.956,697	.956,745	.956,793	.956,840	.956,888	.956,936	.956,984	.957,032	.957,080	48
906	.957,128	.957,176	.957,224	.957,272	.957,320	.957,368	.957,416	.957,464	.957,512	.957,559	48
907	.957,607	.957,655	.957,703	.957,751	.957,799	.957,847	.957,894	.957,942	.957,990	.958,038	48
908	.958,086	.958,134	.958,181	.958,229	.958,277	.958,325	.958,373	.958,421	.958,468	.958,516	48
909	.958,564	.958,612	.958,659	.958,707	.958,755	.958,803	.958,850	.958,898	.958,946	.958,994	48
910	.959,041	.959,089	.959,137	.959,184	.959,232	.959,280	.959,328	.959,375	.959,423	.959,471	48
911	.959,518	.959,566	.959,614	.959,661	.959,709	.959,757	.959,804	.959,852	.959,900	.959,947	48
912	.959,995	.960,042	.960,090	.960,138	.960,185	.960,233	.960,280	.960,328	.960,376	.960,423	48
913	.960,471	.960,518	.960,566	.960,613	.960,661	.960,709	.960,756	.960,804	.960,851	.960,899	48
914	.960,946	.960,994	.961,041	.961,089	.961,136	.961,184	.961,231	.961,279	.961,326	.961,374	47
915	.961,421	.961,468	.961,516	.961,563	.961,611	.961,658	.961,706	.961,753	.961,801	.961,848	47
916	.961,895	.961,943	.961,990	.962,038	.962,085	.962,132	.962,180	.962,227	.962,275	.962,322	47
917	.962,369	.962,417	.962,464	.962,511	.962,559	.962,606	.962,653	.962,701	.962,748	.962,795	47
918	.962,843	.962,890	.962,937	.962,985	.963,032	.963,079	.963,126	.963,174	.963,221	.963,268	47
919	.963,315	.963,363	.963,410	.963,457	.963,504	.963,552	.963,599	.963,646	.963,693	.963,741	47
920	.963,788	.963,835	.963,882	.963,929	.963,977	.964,024	.964,071	.964,118	.964,165	.964,212	47
921	.964,260	.964,307	.964,354	.964,401	.964,448	.964,495	.964,542	.964,590	.964,637	.964,684	47
922	.964,731	.964,778	.964,825	.964,872	.964,919	.964,966	.965,013	.965,061	.965,108	.965,155	47
923	.965,202	.965,249	.965,296	.965,343	.965,390	.965,437	.965,484	.965,531	.965,578	.965,625	47
924	.965,672	.965,719	.965,766	.965,813	.965,860	.965,907	.965,954	.966,001	.966,048	.966,095	47
925	.966,142	.966,189	.966,236	.966,283	.966,329	.966,376	.966,423	.966,470	.966,517	.966,564	47
926	.966,611	.966,658	.966,705	.966,752	.966,798	.966,845	.966,892	.966,939	.966,986	.967,033	47
927	.967,080	.967,127	.967,173	.967,220	.967,267	.967,314	.967,361	.967,408	.967,454	.967,501	47
928	.967,548	.967,595	.967,642	.967,688	.967,735	.967,782	.967,829	.967,875	.967,922	.967,969	47
929	.968,016	.968,062	.968,109	.968,156	.968,203	.968,249	.968,296	.968,343	.968,389	.968,436	47
930	.968,483	.968,530	.968,576	.968,623	.968,670	.968,716	.968,763	.968,810	.968,856	.968,903	47
931	.968,950	.968,996	.969,043	.969,090	.969,136	.969,183	.969,229	.969,276	.969,323	.969,369	47
932	.969,416	.969,463	.969,509	.969,556	.969,602	.969,649	.969,695	.969,742	.969,789	.969,835	47
933	.969,882	.969,928	.969,975	.970,021	.970,068	.970,114	.970,161	.970,207	.970,254	.970,300	47
934	.970,347	.970,393	.970,440	.970,486	.970,533	.970,579	.970,626	.970,672	.970,719	.970,765	46
935	.970,812	.970,858	.970,904	.970,951	.970,997	.971,044	.971,090	.971,137	.971,183	.971,229	46
936	.971,276	.971,322	.971,369	.971,415	.971,461	.971,508	.971,554	.971,601	.971,647	.971,693	46
937	.971,740	.971,786	.971,832	.971,879	.971,925	.971,971	.972,018	.972,064	.972,110	.972,157	46
938	.972,203	.972,249	.972,295	.972,342	.972,388	.972,434	.972,481	.972,527	.972,573	.972,619	46
939	.972,666	.972,712	.972,758	.972,804	.972,851	.972,897	.972,943	.972,989	.973,035	.973,082	46

Num.	0	1	2	3	4	5	6	7	8	9	Diff.
940	.973,128	.973,174	.973,220	.973,266	.973,313	.973,359	.973,405	.973,451	.973,497	.973,543	46
941	.973,590	.973,636	.973,682	.973,728	.973,774	.973,820	.973,866	.973,913	.973,959	.974,005	46
942	.974,051	.974,097	.974,143	.974,189	.974,235	.974,281	.974,327	.974,374	.974,420	.974,466	46
943	.974,512	.974,558	.974,604	.974,650	.974,696	.974,742	.974,788	.974,834	.974,880	.974,926	46
944	.974,972	.975,018	.975,064	.975,110	.975,156	.975,202	.975,248	.975,294	.975,340	.975,386	46
945	.975,432	.975,478	.975,524	.975,570	.975,616	.975,662	.975,707	.975,753	.975,799	.975,845	46
946	.975,891	.975,937	.975,983	.976,029	.976,075	.976,121	.976,167	.976,212	.976,258	.976,304	46
947	.976,350	.976,396	.976,442	.976,488	.976,533	.976,579	.976,625	.976,671	.976,717	.976,763	46
948	.976,808	.976,854	.976,900	.976,946	.976,992	.977,037	.977,083	.977,129	.977,175	.977,220	46
949	.977,266	.977,312	.977,358	.977,403	.977,449	.977,495	.977,541	.977,586	.977,632	.977,678	46
950	.977,724	.977,769	.977,815	.977,861	.977,906	.977,952	.977,998	.978,043	.978,089	.978,135	46
951	.978,181	.978,226	.978,272	.978,317	.978,363	.978,409	.978,454	.978,500	.978,546	.978,591	46
952	.978,637	.978,683	.978,728	.978,774	.978,819	.978,865	.978,911	.978,956	.979,002	.979,047	45
953	.979,093	.979,138	.979,184	.979,230	.979,275	.979,321	.979,366	.979,412	.979,457	.979,503	45
954	.979,548	.979,594	.979,639	.979,685	.979,730	.979,776	.979,821	.979,867	.979,912	.979,958	45
955	.980,003	.980,049	.980,094	.980,140	.980,185	.980,231	.980,276	.980,322	.980,367	.980,412	45
956	.980,458	.980,503	.980,549	.980,594	.980,640	.980,685	.980,730	.980,776	.980,821	.980,867	45
957	.980,912	.980,957	.981,003	.981,048	.981,093	.981,139	.981,184	.981,229	.981,275	.981,320	45
958	.981,366	.981,411	.981,456	.981,501	.981,547	.981,592	.981,637	.981,683	.981,728	.981,773	45
959	.981,819	.981,864	.981,909	.981,954	.982,000	.982,045	.982,090	.982,135	.982,181	.982,226	45
960	.982,271	.982,316	.982,362	.982,407	.982,452	.982,497	.982,543	.982,588	.982,633	.982,678	45
961	.982,723	.982,769	.982,814	.982,859	.982,904	.982,949	.982,994	.983,040	.983,085	.983,130	45
962	.983,175	.983,220	.983,265	.983,310	.983,356	.983,401	.983,446	.983,491	.983,536	.983,581	45
963	.983,626	.983,671	.983,716	.983,762	.983,807	.983,852	.983,897	.983,942	.983,987	.984,032	45
964	.984,077	.984,122	.984,167	.984,212	.984,257	.984,302	.984,347	.984,392	.984,437	.984,482	45
965	.984,527	.984,572	.984,617	.984,662	.984,707	.984,752	.984,797	.984,842	.984,887	.984,932	45
966	.984,977	.985,022	.985,067	.985,112	.985,157	.985,202	.985,247	.985,292	.985,337	.985,382	45
967	.985,426	.985,471	.985,516	.985,561	.985,606	.985,651	.985,696	.985,741	.985,786	.985,830	45
968	.985,875	.985,920	.985,965	.986,010	.986,055	.986,100	.986,144	.986,189	.986,234	.986,279	45
969	.986,324	.986,369	.986,413	.986,458	.986,503	.986,548	.986,593	.986,637	.986,682	.986,727	45
970	.986,772	.986,816	.986,861	.986,906	.986,951	.986,996	.987,040	.987,085	.987,130	.987,175	45
971	.987,219	.987,264	.987,309	.987,353	.987,398	.987,443	.987,488	.987,532	.987,577	.987,622	45
972	.987,666	.987,711	.987,756	.987,800	.987,845	.987,890	.987,934	.987,979	.988,024	.988,068	45
973	.988,113	.988,157	.988,202	.988,247	.988,291	.988,336	.988,381	.988,425	.988,470	.988,514	45
974	.988,559	.988,604	.988,648	.988,693	.988,737	.988,782	.988,826	.988,871	.988,916	.988,960	45
975	.989,005	.989,049	.989,094	.989,138	.989,183	.989,227	.989,272	.989,316	.989,361	.989,405	45
976	.989,450	.989,494	.989,539	.989,583	.989,628	.989,672	.989,717	.989,761	.989,806	.989,850	44
977	.989,895	.989,939	.989,983	.990,028	.990,072	.990,117	.990,161	.990,206	.990,250	.990,294	44
978	.990,339	.990,383	.990,428	.990,472	.990,516	.990,561	.990,605	.990,650	.990,694	.990,738	44
979	.990,783	.990,827	.990,871	.990,916	.990,960	.991,004	.991,049	.991,093	.991,137	.991,182	44
980	.991,226	.991,270	.991,315	.991,359	.991,403	.991,448	.991,492	.991,536	.991,580	.991,625	44
981	.991,669	.991,713	.991,757	.991,802	.991,846	.991,890	.991,934	.991,979	.992,023	.992,067	44

Num.	0	1	2	3	4	5	6	7	8	9	Dif.
982	.992,112	.992,156	.992,200	.992,244	.992,288	.992,333	.992,377	.992,421	.992,465	.992,509	44
983	.992,554	.992,598	.992,642	.992,686	.992,730	.992,774	.992,819	.992,863	.992,907	.992,951	44
984	.992,995	.993,039	.993,083	.993,127	.993,172	.993,216	.993,260	.993,304	.993,348	.993,392	44
985	.993,436	.993,480	.993,524	.993,568	.993,613	.993,657	.993,701	.993,745	.993,789	.993,833	44
986	.993,877	.993,921	.993,965	.994,009	.994,053	.994,097	.994,141	.994,185	.994,229	.994,273	44
987	.994,317	.994,361	.994,405	.994,449	.994,493	.994,537	.994,581	.994,625	.994,669	.994,713	44
988	.994,757	.994,801	.994,845	.994,889	.994,933	.994,977	.995,021	.995,065	.995,108	.995,152	44
989	.995,196	.995,240	.995,284	.995,328	.995,372	.995,416	.995,460	.995,504	.995,547	.995,591	44
990	.995,635	.995,679	.995,723	.995,767	.995,811	.995,854	.995,898	.995,942	.995,986	.996,030	44
991	.996,074	.996,117	.996,161	.996,205	.996,249	.996,293	.996,337	.996,380	.996,424	.996,468	44
992	.996,512	.996,555	.996,599	.996,643	.996,687	.996,731	.996,774	.996,818	.996,862	.996,906	44
993	.996,949	.996,993	.997,037	.997,080	.997,124	.997,168	.997,212	.997,255	.997,299	.997,343	44
994	.997,386	.997,430	.997,474	.997,517	.997,561	.997,605	.997,648	.997,692	.997,736	.997,779	44
995	.997,823	.997,867	.997,910	.997,954	.997,998	.998,041	.998,085	.998,129	.998,172	.998,216	44
996	.998,259	.998,303	.998,346	.998,390	.998,434	.998,477	.998,521	.998,564	.998,608	.998,652	44
997	.998,695	.998,739	.998,782	.998,826	.998,869	.998,913	.998,956	.999,000	.999,043	.999,087	44
998	.999,130	.999,174	.999,218	.999,261	.999,305	.999,348	.999,392	.999,435	.999,479	.999,522	44
999	.999,565	.999,609	.999,652	.999,696	.999,739	.999,783	.999,826	.999,870	.999,913	.999,956	44

POSTSCRIPT.

IN the *Young Mathematician's Guide*, Page 248, just after *Sett. I.* of *Simple Interest*, there follows a *Scholium**, which accidentally is left out of it's proper Place in this *Treat.* However, rather than that it should be wholly omitted, I have inserted it here, with a brief *Application* of it.

The Scholium is this :

Although it be according to the *Laws* and *Custom* of *England* to Compute *Interest* at the *Proportion* of 6 per Cent. per Annum, *Viz.* 3*l.* per Cent. for 6 Months, and 3*s.* per Cent. for 3 Months, &c.; yet he that takes-up Money at that *Rate* of *Interest*, for any *Time* less than a compleat Year, pays more *Interest* than seems reasonably *Due*, according to the *strict Rules* of *Art.*

As, for Instance; If 100*l.* be forborn at *Interest* 12 Months, it Amounts to 106*l.*, which is just: But (*I say*) it should not Amount to 103*l.* in 6 Months; nor to 101*l.* 10*s.* in 3 Months; as appears from this *Proportion*.

Let a = the Amount Due at the End of 6 Months; then it will be, As 100 : a :: a : 106, the true Amount at the End of 12 Months: Ergo, aa = 10600, and $a = \sqrt{10600} = 102,9563$, &c., which is Less than 103; and if it be paid at the End of 3 Months, then it will be $\sqrt{102,9563} = 101,4673$, &c., which is Less than 101,5. So that the quicker the Payments are, the greater the Error must needs be, which in large Sums, and little Intervals of Time, will be very considerable; as is manifest from this following Example.

Suppose the Crown were indebted Ten Millions, *viz.* 10,000,000*l.*, and were to pay after the Rate of 6 per Cent. per Annum, Simple Interest, for it: Then the Interest of that 10,000,000*l.* would be 600,000*l.* for 12 Months, which is really true: And, according to the usual Proportion of Payments, the Interest would be 300,000*l.* for 6 Months, and 150,000*l.* for 3 Months; whereas, in reality, the Interest of 10,000,000*l.* should be but 295,630*l.* 2*s.* 9*d.* for 6 Months, and but 146,738*l.* 9*s.* 3*d.*, instead of 150,000*l.* for 3 Months, which is 3261*l.* 10*s.* 9*d.* too much.

And this may suffice at this Time (as a Hint) to shew what vast Advantage the Money-Merchants make in any Government that is forced to Take-up Money, and pay the Interest for it at short Intervals of Time, as Quarterly, or Monthly, &c.

* See above, page 603.

AN
APPENDIX
TO
THE FOREGOING TREATISE OF MR. JOHN WARD,

INTITLED,

“ Clavis Usuræ, or A Key to Interest, both Simple and Compound ;”

Containing Resolutions of some of the Questions proposed in that Work and
in his other celebrated Work called *The Young Mathematician's Guide*,
by some of the Methods described above in the Appendix
to Dr. Halley's Discourse on Compound Interest,
in Pages 359, 360, &c to Page 516
of the present Volume.

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF THE COURT OF EXCHEQUER IN ENGLAND.

AN
APPENDIX

THE FIFTEENTH NATIONAL ORAL HISTORY

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AN APPENDIX

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Containing Resolutions of some of the Questions proposed in that Work and in his other celebrated Work called *The Young Mathematician's Guide*, by some of the Methods described above in the Appendix to Dr. Halley's Discourse on Compound Interest, in Pages 359, 360, &c to Page 516 of the present Volume.

Article 1. **I**N the foregoing Treatise of Mr. John Ward on the Interest of Money, and in the Extracts from his *Young Mathematician's Guide* relating to the same subject, there occur a few examples of affected, or multinomial, algebraick equations which it is necessary to resolve in order to obtain the proper answers to the questions from which they arise: and Mr. Ward has pointed-out a very proper method of resolving these equations by repeated processes of approximation. But I conceive that these equations may be resolved with greater ease and expedition by some of the methods described above in the Appendix to Dr. Halley's Discourse on Compound Interest. I shall therefore now proceed to resolve these equations first, by Mr. Ward's method, and afterwards by some of those methods, that the reader may be the better able to compare them with each other, and to determine which of them is the more convenient.

Art. 2. The first of these examples occurs above in page 623 of the present Volume, in the resolution of the fourth question there proposed; which is in these words.

"Question 4. If 30*l.* per annum, being unpaid for nine years, will amount to 344*l.* 14*s.* 9½*d.*, allowing compound interest for every payment as it becomes due; what must the rate of interest be per cent?"

Here

Here is given $u = 30$, $A = 344.7395$, and $t = 9$; to find R by the last of the four equations, to wit, $\frac{A}{u} \times R - R^9 = \frac{A - u}{u}$.

First, $\frac{A}{u}$ is $= \frac{344.7395}{30} = 11.491,317$. And $\frac{A - u}{u}$ is $(= \frac{A}{u} - \frac{u}{u} = \frac{A}{u} - 1 = 11.491,317 - 1) = 10.491,317$.

Hence there is this equation $11.491,317R - R^9 = 10.491,317$." Mr. Ward's method of resolving this equation, if set forth more fully than it is above in page 624, will be as follows.

In this equation (according to Mr. Ward's notation set down in page 609,) R denotes the value of 1 pound together with it's interest for a year, and therefore must be greater than 1 pound, or 1.

Mr. Ward then denotes the unknown excess of R above 1*l.* by the letter e , or supposes R to be equal to $1 + e$, and then proceeds to resolve the equation $11.491,317R - R^9 = 10.491,317$ in the manner following.

Since R is $= 1 + e$, we shall (by the binomial theorem) have $R^9 = \overline{1 + e}^9$
 $(= 1^9 + 9 \times 1^8 \times e + 9 \times \frac{9-1}{2} \times 1^7 \times ee + \&c, \text{ continued to ten terms,})$
 $= 1 + 9 \times e + 9 \times \frac{8}{2} \times ee + \&c, \text{ continued to ten terms,})$
 $= 1 + 9e + 9 \times 4 \times ee + \&c, \text{ continued to ten terms,})$
 $= 1 + 9e + 36ee + \&c, \text{ continued to ten terms.}$

But, as e , or the interest of 1*l.* for a year, is much less than the principal itself, or 1*l.*, the powers of e , to wit, $e, ee, e^3, e^4, e^5, e^6, e^7, e^8, e^9$, will form a swiftly-decreasing progression; and therefore Mr. Ward conceived that all the terms of the series $1 + 9e + 36ee + \&c$, (continued to ten terms,) that came after the first three terms $1 + 9e + 36ee$, and which involved in them the following powers $e^3, e^4, e^5, e^6, e^7, e^8$, and e^9 , would be very small in comparison of the said three first terms, and might therefore be safely neglected. And then the binomial quantity $11.491,317R - R^9$ will become equal to
 $(11.491,317 \times \overline{1 + e} - \overline{1 + 9e + 36ee} = 11.491,317 \times 1 + 11.491,317 \times e - \overline{1 + 9e + 36ee} =$

$\left. \begin{array}{l} 11.491,317 + 11.491,317e \\ - 1.000,000 - 9.000,000e - 36ee \\ 10.491,317 + 2.491,317e - 36ee. \end{array} \right\}$, or to) the trinomial quantity

But the binomial quantity $11.491,317R - R^9$ is $= 10.491,317$.

Therefore

Therefore the trinomial quantity $10.491,317 + 2.491,317e - 36ee$ will also be $= 10.491,317$.

Therefore (adding $36ee$ to both sides,) we shall have $10.491,317 + 2.491,317e = 10.491,317 + 36ee$; and, (subtracting $10.491,317$ from both sides,) we shall have $2.491,317e = 36ee$; and (dividing both sides by e ,) we shall have $2.491,317 = 36e$, and (dividing both sides by 36 ,) we shall have $e (= \frac{2.491,317}{36}) = 0.069,203$, or (neglecting all but the first figure of this fraction,) $e = 0.06$. Therefore R , or $1 + e$, will be $= 1.06$. Q. E. I.

In this resolution Mr. Ward neglects all but the first figure of the quotient $0.069,203$, and thereby obtains 0.06 for the value of e , and consequently 1.06 for the value of $(1 + e)$, or R , which is it's exact value. But this he did, because he knew before-hand that 1.06 was the true value of R , this question being only the reverse of the preceeding question, in which the interest of money had been given, and had been 6 per cent, or R had been supposed to be $= 1.06$. But, if he had not known this, and had not been thereby directed to neglect all the figures of the quotient $0.069,203$ except the first figure 0.06 , it would have required at least two more processes of approximation, to enable him to conclude with certainty that the true value of R was 1.06 . For, if he had used only one such process, he would have found the next near value of R after $1.069,203$, or 1.069 , to be $= 1.062,671$, which is still considerably greater than the true value 1.06 . This second process would be as follows.

Art. 3. The equation $11.491,317R - R^9 = 10.491,317$ has two roots, of which one is evidently $= 1$. For, if R be taken $= 1$, we shall have $R^9 (= 1^9) = 1$, and $11.491,317R (= 11.491,317 \times 1) = 11.491,317$, and consequently $11.491,317R - R^9 (= 11.491,317 - 1) = 10.491,317$, which is the absolute term of the said equation; and therefore 1 is a root of that equation. But this is not the root that we are now seeking, or that will express the value of R in the question that gave rise to the equation; because that value of R is necessarily greater than 1 . Therefore the value of R that we are seeking will be the greater root of the equation $11.491,317R - R^9 = 10.491,317$. Now

in a binomial equation of this general form $Px - x^m = Q$, (under which the present equation falls,) or of the still more general form $Px^m - x^{m+n} = Q$, in which the higher power of the unknown quantity x is subtracted from the term which involves it's lower power, if a quantity somewhat greater than the greater root of the equation be substituted instead of the said greater root in the binomial quantity that forms the first, or left-hand, side of such equation, the value of the said binomial quantity resulting from such substitution will be less than the absolute term, Q , of such equation; and, if a quantity somewhat less than the said greater root be substituted instead of the said greater root in the said

saïd binomial quantity, the value of the saïd binomial quantity resulting from such substitution will be greater than the absolute term of such equation. And therefore, if a quantity that is known to be nearly equal to the saïd greater root of such an equation be substituted instead of the saïd greater root in the saïd binomial quantity, and the value of the saïd binomial quantity resulting from such substitution shall be less than the absolute term of such equation, it will follow that the quantity so substituted instead of the saïd greater root must be greater than the saïd greater root; and, if the value of the saïd binomial quantity resulting from such substitution shall be greater than the absolute term of such equation, it will follow that the quantity so substituted instead of the saïd greater root must be less than the saïd greater root.

Since therefore it has been found, by the foregoing first process of Mr. Ward's approximation, that R , or the greater root of the equation $11.491,317R - R^9 = 10.491,317$, is nearly equal to $1.069,203$, or (neglecting the three last figures 203 , as not likely to be exact,) nearly equal to 1.069 , we will now proceed to substitute this near value of R instead of R in the binomial quantity $11.491,317R - R^9$, in order to discover whether the value of the saïd binomial quantity resulting from such substitution will be less, or greater, than $10.491,317$, or the absolute term of the saïd equation, and consequently whether the saïd near value of R will be greater, or less, than it's true value. This substitution may be made in the following manner.

If R is supposed to be $= 1.069$, we shall have $R^9 = \overline{1.069}^9 =$ (by logarithms,) $1.823,052,9$, and $11.491,317R (= 11.491,317 \times 1.069) = 12.284,217,873$, and consequently $11.491,317R - R^9 (= 12.284,217,873 - 1.823,052,9) = 10.461,164,973$; which is less than $10.491,317$, or the absolute term of the equation $11.491,317R - R^9 = 10.491,317$. Therefore 1.069 must be greater than the true value of R , or the greater root of the saïd equation.

Q. E. I.

Art. 4. Since 1.069 is thus found to be greater than R , let their difference be denoted by the letter f .

Then will R be $= 1.069 - f$; and consequently R^9 will be $= \overline{1.069 - f}^9 =$ (by the binomial theorem,) $\overline{1.069}^9 - 9 \times \overline{1.069}^8 \times f + 36 \times \overline{1.069}^7 \times f^2 - \&c = 1.823,052,9 - 9 \times \frac{1.823,052,9}{1.069} \times f + 36 \times \frac{1.823,052,9}{1.069^2} \times f^2 - \&c = 1.823,052,9 - 9 \times 1.705,381,5 \times f + 36 \times 1.595,305,4 \times f^2 - \&c = 1.823,052,9 - 15.348,433,5 \times f + 57.440,994,4 \times f^2 - \&c$; and $11.491,317R$ will be $(= 11.491,317 \times \overline{1.069 - f} = 11.491,317 \times 1.069 - 11.491,317 \times f) = 12.284,217,873 - 11.491,317 \times f$. Therefore the binomial quantity $11.491,317R - R^9$ will be $=$ the compound quantity

12.284,

$$\left\{ \begin{array}{l} 12.284,217,873 - 11.491,317 \times f + 57.440,994,4 \times f^2 + \&c \\ - 1.823,052,9 + 15.348,433,5 \times f - 57.440,994,4 \times f^2 + \&c \end{array} \right\} \\ = 10.461,164,973 + 3.857,116,5 \times f - 57.440,994,4 \times f^2 + \&c.$$

But the binomial quantity $11.491,317R - R^2$ is $= 10.491,317$.

Therefore the compound quantity $10.461,164,973 + 3.857,116,5 \times f - 57.440,994,4 \times f^2 + \&c$ will also be $= 10.491,317$. And consequently (subtracting $10.461,164,973$ from both sides,) we shall have $3.857,116,5 \times f - 57.440,994,4 \times f^2 + \&c = 0.030,152,027$, and (dividing all the terms by $57.440,994,4$, the co-efficient of f^2 ;) we shall have $0.067,149,1 \times f - f^2 + \&c = 0.000,524,9$. Therefore, if we take half the co-efficient of f , that is, half of $0.067,149,1$, which is $0.033,574,5$, and raise it's square, which will be $0.001,127,247,050,25$, and from this square subtract both sides of the last equation, we shall have $0.033,574,5^2 - 0.067,149,1 \times f + f^2 - \&c (= 0.001,127,247,050,25 - 0.000,524,9) = 0.000,602,347,050,25$; and (by extracting the square-roots of both sides,) we shall have $0.033,574,5 - f = 0.027,246,0$. Therefore (adding f to both sides,) we shall have $0.033,574,5 = 0.027,246,0 + f$, and (subtracting $0.027,246,0$ from both sides,) we shall have $f = 0.006,328,5$. Therefore R , or $1.069 - f$, will be $(= 1.069 - 0.006,328,5) = 1.062,671,5$. Q. E. D.

It appears therefore that this second near value of R , obtained by this second process of Mr. Ward's method of approximation (which has been rather long and laborious,) to wit, $1.062,671,5$, is still considerably greater than the true value of R , which is 1.06 . And it would require at least one more such process of Mr. Ward's method of approximation to obtain the value of R to a sufficient degree of exactness.

A Remark on the foregoing Resolution of the Equation $11.491,317R - R^2 = 10.491,317$ by Mr. Ward's Method of Approximation.

Art. 5. But it may be observed that, in resolving the equation $11.491,317R - R^2 = 10.491,317$ by approximation, Mr. Ward might have retained four terms, instead of three, of the series which is equal to $1 + e^2$, or R^2 , without having any higher equation than a quadratick to resolve at the end of the calculation in order to obtain a near value of e . For $11.491,317R - R^2$ is accurately equal to $11.491,317 \times 1 + e - \sqrt{1 + e^2} (= 11.491,317 + 11.491,317e - \text{the series } 1 + 9e + 9 \times \frac{9-1}{2} \times e^2 + 9 \times \frac{9-1}{2} \times \frac{9-2}{2} \times e^3 + \&c, \text{ continued to ten terms,} = 11.491,317 + 11.491,317e - \text{the series } 1 + 9e + 9 \times \frac{8}{2} \times e^2 + 9 \times \frac{8}{2} \times \frac{7}{3} \times e^3 + \&c, \text{ continued to}$

ten terms, $= 11.491,317 + 11.491,317e - \text{the series } 1 + 9e + 36e^2 + 36 \times \frac{7}{3} \times e^3 + \&c, \text{ continued to ten terms,} = 11.491,317 + 11.491,317e - \text{the series } 1 + 9e + 36e^2 + 84e^3 + \&c, \text{ continued to ten terms} = 11.491,317 + 11.491,317e - 1 - 9e - 36e^2 - 84e^3 - \text{fix more terms,} = 10.491,317 + 2.491,317e - 36e^2 - 84e^3 - \text{fix more terms of the said series.}$

But $11.491,317R - R^9$ is $= 10.491,317$.

Therefore $10.491,317 + 2.491,317e - 36e^2 - 84e^3 - \text{fix more terms of the said series}$ will also be accurately equal to $10.491,317$. And consequently $10.491,317 + 2.491,317e$ will be accurately equal to $10.491,317 + 36e^2 + 84e^3 + \text{fix more terms of the said series, and (subtracting } 10.491,317 \text{ from both sides,)} 2.491,317e$ will be accurately equal to $36e^2 + 84e^3 + \text{fix more terms of the said series.}$ Therefore, if we neglect those fix terms of the said series on account of their smallness, as they will involve the powers e^4, e^5, e^6, e^7, e^8 , and e^9 , which are very small quantities, we shall have $2.491,317e$ nearly equal to, but somewhat greater than, $36e^2 + 84e^3$; and consequently (dividing all the terms by e ,) we shall have $2.491,317$ nearly equal to, but somewhat greater than, $36e + 84e^2$, or $84e^2 + 36e$ nearly equal to, but somewhat less than, $2.491,317$. Therefore $e^2 + \frac{36e}{84}$ will be nearly equal to, but somewhat less than, $\frac{2.491,317}{84}$, or $e^2 + \frac{3e}{7}$ will be nearly equal to, but somewhat less than, $0.029,658,5$. Therefore $e^2 + \frac{3e}{7} + \left[\frac{3}{14}\right]^2$ will be nearly equal to, but somewhat less than, $0.029,658,5 + \frac{9}{196}$, or than $0.029,658,5 + 0.045,918,3$, or than $0.075,576,8$. Therefore $e + \frac{3}{14}$ will be nearly equal to, but somewhat less than, $\sqrt{0.075,576,8}$, or 0.2749 , and consequently e will be nearly equal to, but somewhat less than, $0.2749 - \frac{3}{14}$, or than $0.2749 - 0.2142$, or than 0.0607 . Therefore R , or $1 + e$, will be nearly equal to, but somewhat less than, $1 + 0.0607$, or 1.0607 . Q. E. I.

This value of R is much nearer to its true value 1.0600 , or 1.06 , than even the second near value of it found above by the second process of Mr. Ward's method of approximation, which was $1.062,671,5$. And, if Mr. Ward had resolved the equation $11.491,317R - R^9 = 10.491,317$, or $11.491,317 + 11.491,317e - \sqrt{1 + e^9} = 10.491,317$, or $11.491,317 + 11.491,317e - \text{the series } 1 + 9e + 36e^2 + 84e^3 + \&c, \text{ continued to ten terms,} = 10.491,317$, in this manner, by retaining the fourth term of the said series as well as the three first terms of it, he might reasonably have conjectured, (if he had not known it before-hand,) that the true value of R was 1.06 , because the

near

near value of it that would have been found for it in this way, to wit, 1.0607, (which is known to be somewhat greater than the said true value,) has a cypher in the place of figures immediately following the figures 1.06, so that it's excess above 1.06 is only 0.0007, or little more than a 100th part of 0.06, which may very well be neglected both as inconsiderable in respect of 0.06, and as being probably equal to the real excess of the said value 1.0607 above the true value of e , which is known to be less than the said near value.

Art. 6. This last method of resolving this equation $11.491,317R - R^9 = 10.491,317$ by retaining four terms of the series $1 + 9e + 36e^2 + 84e^3 + \&c$, (which is equal to $\overline{1 + e^9}$), instead of retaining only three terms of it, as Mr. Ward does, is, I believe, the very best method that can be taken for obtaining by a single process of approximation a tolerably near value of $1 + e$, or R , in this equation. And, as the discovery, made by this method of proceeding, that the first near value of R (which is known to be greater than the truth) is 1.0607, leads so naturally to a supposition that it's true value will be 1.06, and consequently to a trial of the said number 1.06 by substituting it, instead of R , in the binomial quantity $11.491,317R - R^9$, (upon which trial it will be found that the value of the said binomial quantity resulting from such substitution will be equal to the number 10.491,317, or the absolute term of the proposed equation $11.491,317R - R^9 = 10.491,317$, and consequently that 1.06 is the true value of R in the said equation,)—it seems needless to have recourse to any other method of resolving this equation. And the same observation will be true of all other numeral examples falling under this 4th Question of Mr. Ward in page 623 of the present Volume, or under the general equation $\frac{A}{u} \times R - R^t = \frac{A - u}{u}$, or $\frac{A}{u} \times R - R^t = \frac{A}{u} - 1$, (in which the absolute term of the equation is less by an unit than the co-efficient of the simple power of R),

when t , or the index of the power of R in the second term R^t of the equation, is not greater, or not much greater, than 9. But, if t should be much greater than 9, as, for instance, if it should be equal to 30, or 40, or any greater number, the near value of R obtained in this manner, (or by retaining the four first terms of the series which is equal to $\overline{1 + e^t}$, or R^t), would be much less exact than in the foregoing example of the equation $11.491,317R - R^9 = 10.491,317$. And therefore in such a case it would be expedient to have recourse to some other method of proceeding, different from that of Mr. Ward, (in which the binomial quantity $1 + e$ was substituted instead of R in the proposed equation,) and to begin our approximation to the value of R from some known quantity that is greater than 1, and that approaches nearer than 1 to the true value of R . And three such methods have been given for this purpose in the foregoing part of this Volume, in the *Appendix to Dr. Halley's Discourse on Compound Interest*, in pages 362, 363, &c to page 406, of which I

recommend the first as the most easy and convenient of the three. This method may be applied to the foregoing equation $11.491,317R - R^9 = 10.491,317$ in the following manner.

The Resolution of the foregoing Equation $11.491,317R - R^9 = 10.491,317$ by the first of the three Methods given for this Purpose in the "Appendix to Dr. Halley's Discourse on Compound Interest," contained above in the foregoing Part of the present Volume.

Art. 7. According to the notation used in this *Appendix to Dr. Halley's Discourse on Compound Interest*, in pages 401, 402, &c to page 406 of this present Volume, we shall have $P = 11.491,317$, and $m = 9$, and $M =$

$$\frac{P}{m} \left| \frac{1}{m-1} \right| = \frac{11.491,317}{9} \left| \frac{1}{9-1} \right| = \frac{11.491,317}{9} \left| \frac{1}{8} \right| = 1.276,813 \left| \frac{1}{8} \right| = (\text{by logarithms,}$$

$1.031,06$. Therefore d , or $M - 1$, will be $(= 1.031,06 - 1) = 0.031,06$, and $M + d$ will be $(= 1.031,06 + 0.031,06) = 1.062,12$. This quantity $M + d$, or $1.062,12$, will be a little greater than the true value of R in the proposed equation $11.491,317R - R^9 = 10.491,317$; but their difference will be but small, and consequently this number $1.062,12$ will be a very good first near value of R in the said equation, or a good ground-work for a further approach to it's true value.

This first near value of R , to wit, $1.062,12$, is nearer to it's true value $1.060,00$, or 1.06 , than the number $1.062,671$, is, though that number was obtained by two processes of Mr. Ward's method of resolving this equation.

Now let y be put $= 1.062,12$, and let the binomial quantity $11.491,317y - y^9$ be computed, and the result be called D . This may be done as follows.

Since y is $= 1.062,12$, we shall have $y^9 = \overline{1.062,12^9} = (\text{by logarithms}) 1.720,106,0$, and $11.491,317y (= 11.491,317 \times 1.062,12) = 12.205,157,6$. Therefore $11.491,317y - y^9$ will be $(= 12.205,157,6 - 1.720,106,0) = 10.485,051,6$; which is less than $10.491,317$, or the absolute term of the proposed equation. Therefore y , or $1.062,12$, must be greater than the true value of R , or the greater root of the proposed equation $11.491,317R - R^9 = 10.491,317$.

Let D be $= 10.485,051,6$, or the result of the substitution of y , or $1.062,12$, for R in the binomial quantity $11.491,317R - R^9$.

From

From y , or 1.062,12, take $\frac{y}{200}$, (or $\frac{1.062,12}{200}$, or $\frac{0.010,621,2}{2}$), or 0.005,310,6; and the remainder will be $= (1.062,12 - 0.005,310,6)$, or 1.056,809,4. Let this remainder 1.056,809,4 be substituted instead of R in the binomial quantity $11.491,317R - R^2$, and let the result be called E . This substitution may be made as follows.

If R is supposed to be $= 1.056,809,4$, we shall have, (by logarithms,) $R^2 = 1.6441$. And $11.491,317R$ will be $(= 11.491,317 \times 1.056,809,4) = 12.144,023,8$. Therefore $11.491,317R - R^2$ will be $(= 12.144,023,8 - 1.6441) = 10.499,923,8$; that is, E will be $= 10.499,923,8$.

This quantity E , or 10.499,923,8, is greater than 10.491,317, or the absolute term of the equation $11.491,317R - R^2 = 10.491,317$. Therefore 1.056,809,4 must be less than the true value of R , or the greater root of that equation.

We must now subtract D from E , that is, 10.485,051,6 from 10.499,923,8; and the remainder will be $= 0.014,872,2$; that is, $E - D$ will be $= 0.014,872,2$. And we must likewise subtract D , or 10.485,051,6, from 10.491,317, or the absolute term of the equation $11.491,317R - R^2 = 10.491,317$; and the remainder will be $= 0.006,265,4$. And we must now make the following proportion; to wit, As $E - D$ is to $10.491,317 - D$,

so (very nearly,) will $y - \left(y - \frac{y}{200}\right)$, (or $y - y + \frac{y}{200}$), or $\frac{y}{200}$, be to $y - R$; that is, as 0.014,872,2 is to 0.006,265,4, so (very nearly,) will 0.005,310,6 be to $y - R$, or to $1.062,12 - R$. Therefore $1.062,12 - R$ will be nearly $= \frac{0.006,265,4 \times 0.005,310,6}{0.014,872,2} = \frac{0.000,033,273,033,24}{0.014,872,2} = 0.002,237,2$;

and consequently 1.062,12 will be $= 0.002,237,2 + R$, and R will be $(= 1.062,12 - 0.002,237,2) = 1.059,882,8$. Therefore the second near value of R in the proposed equation $11.491,317R - R^2 = 10.491,317$ will be 1.059,882,8.

Q. E. I.

This second near value of R (which has been obtained by a single process of the *Differential Method of Approximation*), differs from its true value 1.06 by only the small quantity 0.000,117,2, which is less than the 9044th part of the said true value 1.06. This value of R is near enough for all useful purposes: but, if we wish to obtain a still nearer value of it, we need only employ one process of Mr. Raphson's method of approximation (which requires only the resolution of a simple equation,) to obtain its value to a great degree of exactness. This may be done in the following manner.

Let the last value of R , to wit, 1.059,882,8, be substituted instead of R in the binomial quantity $11.491,317R - R^2$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be

greater, or less, than 10.491,317, or the absolute term of the equation $11.491,317R - R^9 = 10.491,317$, and consequently whether the last near value of R , to wit, 1.059,882,8, is less, or greater, than it's true value.

Now, if R is = 1.059,882,8, we shall (by logarithms,) have $R^9 = 1.687,800,0$; and $11.491,317R$ will be $(= 11.491,317 \times 1.059,882,8) = 12.179,449,2$. Therefore the binomial quantity $11.491,317R - R^9$ will be $(= 12.179,449,2 - 1.687,800,0) = 10.491,649,2$; which is something greater than the absolute term 10.491,317. Therefore 1.059,882,8 must be something less than the true value of R in the proposed equation $11.491,317R - R^9 = 10.491,317$.

Having thus discovered that 1.059,882,8 must be somewhat less than R , let z be put for the unknown difference between them.

Then will R be = 1.059,882,8 + z , and consequently R^9 will be = $\overline{1.059,882,8 + z}^9 =$ (by the binomial theorem,) $\overline{1.059,882,8}^9 + 9 \times \overline{1.059,882,8}^8 \times z + \&c = \overline{1.059,882,8}^9 + 9 \times \frac{\overline{1.059,882,8}^9}{1.059,882,8} \times z + \&c = 1.687,800,0 + 9 \times \frac{1.687,800,0}{1.059,882,8} \times z + \&c (= 1.687,800,0 + 9 \times 1.592,440,2 \times z + \&c) = 1.687,800,0 + 14.331,961,8 \times z + \&c$; and $11.491,317R$ will be $(= 11.491,317 \times \overline{1.059,882,8 + z} = 11.491,317 \times 1.059,882,8 - 11.491,317 \times z) = 12.179,449,2 + 11.491,317 \times z$. And consequently the binomial quantity $11.491,317R - R^9$ will be equal to the compound quantity

$$\left\{ \begin{array}{l} 12.179,449,2 + 11.491,317 \times z \\ - 1.687,800,0 - 14.331,961,8 \times z - \&c \end{array} \right\} \\ = 10.491,649,2 - 2.840,644,8 \times z - \&c.$$

But the binomial quantity $11.491,317R - R^9$ is = 10.491,317.

Therefore the compound quantity $10.491,649,2 - 2.840,644,8 \times z$ will also be = 10.491,317. And consequently $10.491,649,2$ will be = 10.491,317 + $2.840,644,8 \times z$, and (subtracting 10.491,317 from both sides,) $2.840,644,8 \times z$ will be = 0.000,332,2, and z will be $(= \frac{0.000,332,2}{2.840,644,8}) = 0.000,116,9$. Therefore R , or $1.059,882,8 + z$, will be $(= 1.059,882,8 + 0.000,116,9) = 1.059,999,7$; which differs from 1.06, or the true value of R , by only the very small quantity 0.000,000,3, which is less than the 3,533,333d part of the said true value.

Q. E. I.

This method of resolving the equation $\frac{A}{u} \times R - R^t = \frac{A}{u} - 1$, or (putting $P = \frac{A}{u}$,) the equation $P \times R - R^t = P - 1$, will always enable

us to find a very exact value of R , or the greater root of this equation, even when t , or the index of the power of R in the second term of the equation, is a great number; (as 40, or 50, or 60,) as well as when it is a small number, (as 9, or 10, or 12,) as happens in the present example. And of this an Example has been given in the above-mentioned Appendix to Dr. Halley's Discourse on Compound Interest, in pages 377, 378, - - - 383, where the equation $967.648,136,7r - r^{70} = 966.648,136,7$, (in which the index t is = 70) is resolved by this method, and it's greater root is found to be nearly = 1.059,997,3, which differs from it's true value (which is 1.06,) by only 0.000,002,7, or less than the 392,592nd part of the said true value 1.06.

A Second Example of the Resolution of an Affected Equation resulting from a Question concerning Compound Interest proposed by Mr. Ward.

Art. 8. The next Example of an affected equation resulting from a Question concerning Compound Interest, given by Mr. John Ward in the parts of his works that have been here reprinted, occurs above in page 626, in the words following.

" Question 4. Suppose one should give 167*l.* 9*s.* 5*d.* for the purchase of a pension, or annuity, of 30*l.* per annum, to continue seven years. At what rate of interest per cent would that purchase be made, allowing compound interest to the purchaser ?

" In this Question there is given $P = 167.4716$, $u = 30$, and $t = 7$; to find R . By the fourth Theorem, (set down in the preceeding page 625,) expressed in this equation $\frac{u}{P} = \frac{u}{P} \times R^t + R^t - R^{t+1}$, [or $\frac{u}{P} \times R^t + R^t - R^{t+1} = \frac{u}{P}$, or $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$] the said equation being brought into numbers, and it's root extracted, as in the fourth Question of the last Section (in page 623,) the value of R will be found to be 1.06; and then it will be, 1 : 0.06 :: 100 : 6, the rate per cent, as was required."

The numeral equation resulting from the application of the General Theorem expressed by the equation $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$ to the Question here proposed, has not been set down by Mr. Ward : but he has left his reader both to derive it from the said general equation, and afterwards to resolve it by the method of approximation which he had given before in page 624 for the resolution of the former equation $11.491,317R - R^9 = 10.491,317$; which method consisted in substituting $1 + e$ instead of R in the said equation, and retaining only the three first terms of the series that is equal to $1 + e^9$, and supposing them to be nearly equal to the value of the whole of the said series, and resolving the new equation resulting from the said substitution in consequence

quence of that supposition. These directions I will now proceed to follow in order to obtain the answer to the Question above-stated concerning the rate of Interest allowed to the said purchaser of an annuity of 30*l. per annum* that is to continue for seven years.

Art. 9. In applying the general equation $\left(\frac{u}{P} + 1\right) \times R^t - R^{t+1} = \frac{u}{P}$ to the foregoing question proposed by Mr. Ward, we shall have u , (or the annuity purchased,) = 30*l. per annum*, and P , (or the purchase-money, or price paid for it,) = 167*l. 9s. 5d.* = 167.4716*l.*, and t , (or the time, or number of years, during which the annuity is to continue,) = 7 years. Therefore R^t will be = R^7 , and R^{t+1} will be = R^8 , and $\frac{u}{P}$ will be = $\frac{30}{167.4716} = 0.179,134$, and $\frac{u}{P} + 1$ will be = 1.179,134; and consequently the general

equation $\left(\frac{u}{P} + 1\right) \times R^t - R^{t+1} = \frac{u}{P}$ will in this case become 1.179,134 $\times R^7 - R^8 = 0.179,134$. The resolution of this equation, performed according to Mr. Ward's method delivered in page 624, will be as follows.

The Resolution of the Equation 1.179,134R⁷ - R⁸ = 0.179,134 performed according to Mr. Ward's Method delivered above in Page 624.

Since R is greater than 1, let us suppose it to be = $1 + e$.

Then will R^7 be = $(1 + e)^7$ = (by the binomial theorem,) to the series $1 + 7e + 7 \times \frac{7-1}{2} \times e^2 + \&c$, continued to eight terms, (= the series $1 + 7e + 7 \times \frac{6}{2} \times e^2 + \&c$, continued to eight terms,) = the series $1 + 7e + 21e^2 + \&c$, continued to eight terms; and R^8 will be = $(1 + e)^8$ = the series $1 + 8e + 8 \times \frac{8-1}{2} \times e^2 + \&c$, continued to nine terms, (= the series $1 + 8e + 8 \times \frac{7}{2} \times e^2 + \&c$, continued to nine terms, = the series $1 + 8e + 4 \times 7 \times e^2 + \&c$, continued to nine terms,) = the series $1 + 8e + 28e^2 + \&c$, continued to nine terms; and 1.179,134 $\times R^7$ will be = 1.179,134 $\times$ the series $1 + 7e + 21e^2 + \&c$, continued to eight terms (= 1.179,134 + 1.179,134 $\times 7e$ + 1.179,134 $\times 21e^2$ + five more terms) = 1.179,134 + 8.253,938*e* + 24.761,814*e*² + five more terms. Therefore the binomial quantity 1.179,134R⁷ - R⁸ will be accurately equal to the compound quantity

$$\left\{ \begin{array}{l} 1.179,134 + 8.253,938e + 24.761,814e^2 + \text{five more terms,} \\ - 1.000,000 - 8.000,000e - 28.000,000e^2 - \text{fix more terms,} \\ = 0.179,134 + 0.253,938e - 3.238,186e^2 + \text{five more terms} \\ - \text{fix more terms.} \end{array} \right\}$$

But

But the binomial quantity $1.179,134R^7 - R^8$ is $= 0.179,134$.

Therefore the compound quantity $0.179,134 + 0.253,938e - 3.238,186e^2$ + five more terms — fix more terms, will also be $= 0.179,134$.

But, because R^8 is less than $1.179,134R^7$ (from which it is subtracted,) it is evident that $\frac{R^8}{R^7}$ must be less than $\frac{1.179,134R^7}{R^7}$, that is, that R must be less than $1.179,134$. Therefore $1 + e$, which is equal to R , must be less than $1.179,134$, and consequently (subtracting 1 from both sides, e must be less than $0.179,134$, and, *à fortiori*, less than 0.2 , or $\frac{2}{10}$, or $\frac{1}{5}$. Therefore the several powers of e involved in the terms of the last equation, to wit, $e, e^2, e^3, e^4, e^5, e^6, e^7$, and e^8 , will form a decreasing progression of terms that will be respectively smaller than the terms of the progression $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \frac{1}{625}, \frac{1}{3125}, \frac{1}{15,625}, \frac{1}{78,125}, \frac{1}{390,625}$, and $\frac{1}{1,953,125}$; and this great decrease of the powers of e will contribute greatly to diminish the several terms in which they are involved, and make the latter terms, which involve the higher powers of e , be much smaller than the former terms, which involve the lower powers of e , and more especially than the two terms $0.253,938e$ and $3.238,186e^2$, which involve it's two lowest powers. And, further, it must be observed that five of the eleven terms which are not set down in the last equation, and which will involve the powers e^3, e^4, e^5, e^6 , and e^7 , are to be marked with the sign $+$, or to be added to the value of the three terms $0.179,134 + 0.253,938e - 3.238,186e^2$, and will therefore tend to increase the said value; but the other fix terms, which will involve the powers of e^3, e^4, e^5, e^6, e^7 , and e^8 , are to be marked with the sign $-$, or to be subtracted from the value of the three first terms $0.179,134 + 0.253,938e - 3.238,186e^2$, and will therefore tend to diminish the said value. Therefore the fix latter terms, if all the terms of the equation were to be retained, would tend to counteract, and, perhaps, nearly counter-balance and destroy the five former terms, and cause the value of the said three first terms to be nearly the same as if all the said eleven terms had been left out of the equation. It seems reasonable therefore to conclude that, partly on account of the quick decrease of the powers of e , and partly on account of this opposition between the first five terms involving the powers e^3, e^4, e^5, e^6 , and e^7 , and the last six terms involving the powers e^3, e^4, e^5, e^6, e^7 , and e^8 , the value of the said three first terms $0.179,134 + 0.253,938e - 3.238,186e^2$ will be nearly equal to the value of the whole fourteen terms that form the left-hand side of the last equation, and consequently to the absolute term of the equation, or to $0.179,134$. And, upon this supposition, we shall have the equation $0.179,134 + 0.253,938e - 3.238,186e^2$ nearly $= 0.179,134$; and consequently (adding $3.238,186e^2$ to both sides,) $0.179,134 + 0.253,938e$ will be nearly $= 0.179,134 + 3.238,$

$3.238,186e^2$, and (subtracting $0.179,134$ from both sides,) $0.253,938e$ will be nearly $= 3.238,186e^2$, and (dividing both sides by e ,) $0.253,938$ will be nearly $= 3.238,186e$, and consequently e will be nearly $(= \frac{0.253,938}{3.238,186}) = 0.078$. Therefore R , or $1 + e$, will be nearly $= 1.078$, or the greater root of the proposed equation $1.179,134R^7 - R^8 = 0.179,134$ will be nearly $= 1.078$.

Q. E. I:

This value of R , (which has been obtained by Mr. Ward's method of proceeding, as delivered in page 624,) is much greater than it's true value, which is 1.06 . And, if we were to prosecute the resolution of the equation $1.179,134R^7 - R^8 = 0.179,134$ further by Mr. Ward's method, or by substituting the binomial quantity $1.078 - f$ instead of R in the terms of the said equation, and resolving the transformed equation resulting from such substitution, as if it were a quadratick equation, by dropping all the terms that involved any higher power of the unknown quantity f than f^2 , or it's square, it would be necessary to go through two more processes of this mode of approximation, before we should obtain a value of R that would be so little greater than it's true value, 1.06 , that their difference should be less than 0.001 , or a 60th part of $.06$, or that the near value of R should be less than 1.061 ; which yet would not be any great degree of exactness. It would therefore be better, as I conceive, to resolve this equation by one of the methods of approximation given above for this purpose in the Appendix to Dr. Halley's Discourse on Compound Interest. But I will first try the effect of retaining four terms, instead of three, of the two serieses that are equal to R^7 and R^8 , or to $\overline{1 + e^7}$ and $\overline{1 + e^8}$, in the transformed equation obtained above, in Mr. Ward's method of resolution, by substituting the binomial quantity $1 + e$ instead of R in the proposed equation $1.179,134R^7 - R^8 = 0.179,134$. This may be done in the following manner.

The Resolution of the same Equation $1.179,134R^7 - R^8 = 0.179,134$ by a Method somewhat different from Mr. Ward's Method, but suggested by it.

Art. 10. If R be put $= 1 + e$, we shall have $R^7 = \overline{1 + e^7}$ ($= 1 + 7e + 7 \times \frac{7-1}{2} \times e^2 + 7 \times \frac{7-1}{2} \times \frac{7-2}{3} \times e^3 + \&c$ continued to eight terms, $= 1 + 7e + 7 \times \frac{6}{2} \times e^2 + 7 \times \frac{6}{2} \times \frac{5}{3} \times e^3 + \&c$, continued to eight terms, $= 1 + 7e + 7 \times 3 \times e^2 + 7 \times 3 \times \frac{5}{3} \times e^3 + \&c$, continued to eight terms,) $= 1 + 7e + 21e^2 + 35e^3 + \&c$, continued to eight terms; and we shall have $R^8 = \overline{1 + e^8}$ ($= 1 + 8e + 8 \times \frac{8-1}{2} \times e^2 + 8$

X

$\times \frac{8-1}{2} \times \frac{8-2}{3} \times e^3 + \&c$, continued to nine terms, $= 1 + 8e + 8 \times \frac{7}{2} \times e^2 + 8 \times \frac{7}{2} \times \frac{6}{3} \times e^3 + \&c$, continued to nine terms,) $= 1 + 8e + 28e^2 + 56e^3 + \&c$, continued to nine terms; and $1.179,134 \times R^7$ will be $= 1.179,134 \times$ the series $1 + 7e + 21e^2 + 35e^3 + \&c$, continued to eight terms, $= 1.179,134 \times 1 + 1.179,134 \times 7e + 1.179,134 \times 21e^2 + 1.179,134 \times 35e^3 +$ four more terms, $= 1.179,134 + 8.253,938e + 24.761,814e^2 + 41.269,690e^3 +$ four terms more. Therefore the binomial quantity $1.179,134R^7 - R^8$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1.179,134 + 8.253,938e + 24.761,814e^2 + 41.269,690e^3 \\ - 1.000,000 - 8.000,000e - 28.000,000e^2 \quad + \text{four terms more} \\ \quad \quad \quad - 56.000,000e^3 \\ \quad \quad \quad \quad \quad - \text{five terms more,} \end{array} \right\}$$

$$= 0.179,134 + 0.253,938 \times e - 3.238,186 \times e^2 - 14.730,310 \times e^3$$

+ four terms more
- five terms more.

But the binomial quantity $1.179,134R^7 - R^8$ is $= 0.179,134$.

Therefore the compound quantity $0.179,134 + 0.253,938 \times e - 3.238,186 \times e^2 - 14.730,310 \times e^3$, together with nine more terms, (of which four will be marked with the sign $+$, or added to the value of the four first terms here set-down, and the other five will be marked with the sign $-$, or subtracted from the said value of the four first terms,) will also be $= 0.179,134$. And we may observe that the sum of the five subtracted terms will be greater than the sum of the other four terms, which are added to the value of the four first terms. Therefore the value of the four first terms $0.179,134 + 0.253,938 \times e - 3.238,186 \times e^2 - 14.730,310 \times e^3$ will be greater than the whole of the said compound quantity, and consequently greater than it's equal, $0.179,134$. Therefore, if we add the two terms $3.238,186 \times e^2$ and $14.730,310 \times e^3$ to both sides, the binomial quantity $0.179,134 + 0.253,938e$ will be greater than the trinomial quantity $0.179,134 + 3.238,186e^2 + 14.730,310e^3$; and (subtracting $0.179,134$ from both sides,) the single quantity $0.253,938e$ will be greater than the binomial quantity $3.238,186e^2 + 14.730,310 \times e^3$, and (dividing all the terms by e ,) the single quantity $0.253,938$ will be greater than the binomial quantity $3.238,186e + 14.730,310e^2$, and consequently the binomial quantity $14.730,310e^2 + 3.238,186 \times e$ will be less than the single quantity $0.253,938$. Therefore the binomial quantity $e^2 + \frac{3.238,186}{14.730,310} \times e$ will be less than the single quantity $\frac{0.253,938}{14.730,310}$, that is, the binomial quantity $e^2 + 0.219,831 \times e$ will be less the single quantity $0.017,231$. Therefore (adding the square of $\frac{0.219,831}{2}$, or the square of $0.109,915$, to wit, $0.012,081,307,225$, to both sides,) the trinomial quantity

quantity $e^2 + 0.219,831 \times e + \overline{0.109,915}^2$ will be (less than $0.017,231 + 0.012,081,307,225$, or) less than $0.029,312,307,225$, and $e + 0.109,915$ will be less than ($\sqrt{0.029,312,307,225}$, or) $0.171,208$; and consequently e will be less than ($0.171,208 - 0.109,915$, or) $0.061,293$. Therefore R , or $1 + e$, will be less than $1.061,293$, or (neglecting the last three figures,) less than 1.061 . Q. E. I.

This near value of R , or the greater root of the proposed equation $1.179,134R^7 - R^8 = 0.179,134$, is near enough to it's true value 1.06 , to give us good ground for conjecturing that the true value of R is 1.06 ; which, if it be tried by being substituted instead of R in the binomial quantity $1.179,134R^7 - R^8$, will be found to make the said quantity become equal to the absolute term $0.179,134$, and therefore to be the true value of the greater root of the said equation.

A Remark on the foregoing Method of resolving the Equation $1.179,134R^7 - R^8 = 0.179,134$.

It appears therefore that this manner of resolving the equation $1.179,134R^7 - R^8 = 0.179,134$ by retaining four terms of the two serieses that are equal to R^7 and R^8 , or $\overline{1 + e}^7$ and $\overline{1 + e}^8$, is much more exact than Mr. Ward's method of resolving it by retaining only three terms of the said serieses. And indeed it has given us so good a first near value of R in the present equation that there seems to be no occasion to have recourse to any other method of resolving this equation in order to obtain a better first near value of it. And, if it should be required, in resolving this, or any other equation of the same form, in which the index t of the lower power of R was a small number not greater than 7, to find the value of R to a greater degree of exactness than it had been found by this first process, it might be most convenient to investigate a second near value of it by one or two processes of Mr. Raphson's method of approximation. But, when the index t is a pretty large number, (as, for example, 30, or 40, or 50, or some greater number,) this method of resolving the equation $\frac{u}{P} + 1 \times R^t - R^{t+1} = \frac{u}{P}$, by retaining four terms of the two serieses that are equal to R^t and R^{t+1} , or to $\overline{1 + e}^t$ and $\overline{1 + e}^{t+1}$, will be found to be much less exact than in the foregoing equation, because in such a case, the co-efficients of several of the omitted terms will be such high numbers as to make those terms be too great to be omitted without making the value of e obtained by means of their omission be much greater than it's true value. In such cases therefore I should think it would be convenient to have recourse to the method given for this purpose in the Appendix to Dr. Halley's Discourse on Compound Interest, and to make use of the limits of the magnitude of the greater root of the general, binomial, equation $Px^m - x^{m+1} =$

= Q (under which the equation $1.179,134R^7 - R^8 = 0.179,134$ is comprehended,) which are determined in pages 414, 415, 416, &c, - - to 423. To shew how those limits may be usefully applied to the resolution of an equation of this kind, I will now proceed to resolve the foregoing equation $1.179,134R^7 - R^8 = 0.179,134$ over again in the manner described in those pages.

The Resolution of the foregoing Equation $1.179,134R^7 - R^8 = 0.179,134$ in the manner described in the Appendix to Dr. Halley's Discourse on Compound Interest in Pages 414, 415, 416, &c - - to Page 423.

Art. 11. It is shewn in pages 414, 415, and 416, that the greater root of the general equation $Px^m - x^{m+1} = Q$ is always less than P, but greater than $\frac{m}{m+1} \times P$; which limits, when m is a high number, (as 40, or 50, or 60,) are very near to each other, their difference being only $\frac{1}{m+1}$ th part of P, which is $= \frac{m+1}{m+1} \times P$. Therefore, when m is a high number, if we take an arithmetical mean between these limits $\frac{m}{m+1} \times P$ and $\frac{m+1}{m+1} \times P$, to wit, the quantity $\frac{m+\frac{1}{2}}{m+1} \times P$, or $\frac{2m+1}{2m+2} \times P$, the said arithmetical mean, $\frac{2m+1}{2m+2} \times P$, will be a pretty good first near value of the greater root of the said equation.

Further, it is observed in page 417, that, if the lesser root of the equation $Px^m - x^{m+1} = Q$ is known, and the excess of $\frac{m}{m+1} \times P$, or the lower limit of the greater root, above the lesser root be called d , and the said lower limit of the greater root, to wit, $\frac{m}{m+1} \times P$, be called M, the binomial quantity $M + \frac{d}{2}$ will usually be a pretty good near value of the greater root.

And thus, when the lesser root is known, we may easily find two pretty good near values of the greater root of the equation $Px^m - x^{m+1} = Q$, by computing the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$.

Now

Now in the equation $1.179,134R^7 - R^8 = 0.179,134$ we know that the lesser root of the equation is 1. Therefore in this equation we may easily find two pretty good near values of the greater root R by computing the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$; which may be done as follows.

In the equation $1.179,134R^7 - R^8 = 0.179,134$, the co-efficient 1.179,134 answers to the co-efficient P in the general equation $Px^m - x^{m+1} = Q$, and R answers to x , and 7 answers to m , and 0.179,134 answers to Q . Therefore $\frac{2m+1}{2m+2} \times P$ will be $= \frac{2 \times 7 + 1}{2 \times 7 + 2} \times 1.179,134 (= \frac{14+1}{14+2} \times 1.179,134 = \frac{15}{16} \times 1.179,134 = \frac{17.687,010}{16}) = 1.105,438$. Therefore 1.105,438 will be a tolerably good first near value of R , or the greater root of the equation $1.179,134R^7 - R^8 = 0.179,134$.

Further, M , or $\frac{m}{m+1} \times P$, or the lower limit of R , will be $= \frac{7}{8} \times 1.179,134 (= \frac{8.253,931}{8}) = 1.031,742$, and d , or $M - 1$, will be $= 0.031,742$. Therefore $\frac{d}{2}$ will be $(= \frac{0.031,742}{2}) = 0.015,871$, and $M + \frac{d}{2}$ will be $(= 1.031,742 + 0.015,871) = 1.047,613$. Therefore 1.047,613 will also be a tolerably good first near value of R , or the greater root of the equation $1.179,134R^7 - R^8 = 0.179,134$.

Of these two first near values of R , to wit, 1.105,438 and 1.047,613, the latter is nearest to the true value of R , which is 1.06. But the contrary will often happen in equations of this form when m is a high number, as 40, or 50, or 60; for the limits of the value of R , (which are P , or $\frac{m+1}{m+2} \times P$, and $\frac{m}{m+1} \times P$), will be much nearer to each other in that case than in the present case, in which m is only the small number 7. But, in order to determine which of these two first values of R , namely, 1.105,438 and 1.047,613 comes nearest to the true value of R , we must substitute each of them, instead of R , in the binomial quantity $1.179,134R^7 - R^8$, that we may compare the two values of the said binomial quantity resulting from the said substitutions with each other and with 0.179,134, or the absolute term of the equation. This may be done in the following manner.

In making these substitutions of the numbers 1.105,438 and 1.047,613 instead of R in the binomial quantity $1.179,134R^7 - R^8$, I shall neglect the three last figures of each of them, as not likely to be true, and shall suppose these near values to be 1.105 and 1.047. And first we will substitute the number 1.105 instead of R in the said binomial quantity.

Now,

Now, if we suppose R to be $= 1.105$, we shall have $R^7 = \overline{1.105}^7$, and $R^8 = \overline{1.105}^8$, and $1.179,134R^7 - R^8 = 1.179,134 \times \overline{1.105}^7 - \overline{1.105}^8$. We must therefore find the values of $\overline{1.105}^7$ and $\overline{1.105}^8$.

Now the logarithm of 1.105 is $= 0.043,364,3$. Therefore the logarithm $\overline{1.105}^7$ will be $(= 7 \times 0.043,364,3) = 0.303,550,1$, which is the logarithm of 2.0116 , &c. Therefore $\overline{1.105}^7$ will be $= 2.0116$ &c. Therefore $\overline{1.105}^8$ will be $(= 2.0116 \text{ \&c} \times 1.105) = 2.2228$ &c, and $1.179,134 \times \overline{1.105}^7$ will be $(= 1.179,134 \times 2.0116 \text{ \&c}) = 2.3719$ &c, and the binomial quantity $1.179,134 \times \overline{1.105}^7 - \overline{1.105}^8$ will be $(= 2.3719 \text{ \&c} - 2.2228 \text{ \&c}) = 0.1491$. Therefore, if R be $= 1.105$, the binomial quantity $1.179,134R^7 - R^8$ will be $= 0.1491$. This quantity is considerably less than $0.179,134$, or the absolute term of the equation $1.179,134R^7 - R^8 = 0.179,134$; and consequently the quantity 1.105 must be considerably greater than the true value of R , or the greater root of that equation.

We will next substitute the number 1.047 , (resulting from the expression $M + \frac{d}{2}$), instead of R in the binomial quantity $1.179,134R^7 - R^8$.

Now, if R is $= 1.047$, we shall have $R^7 = \overline{1.047}^7$, and $R^8 = \overline{1.047}^8$, and $1.179,134R^7 - R^8 = 1.179,134 \times \overline{1.047}^7 - \overline{1.047}^8$. We must therefore find the values of $\overline{1.047}^7$ and $\overline{1.047}^8$; which may be done as follows.

The logarithm of 1.047 is $= 0.019,946,7$. Therefore the logarithm of $\overline{1.047}^7$ is $(= 7 \times 0.019,946,7) = 0.139,626,9$, which is the logarithm of 1.3792 —. Therefore $\overline{1.047}^7$ will be $= 1.3792$ —, and consequently $\overline{1.047}^8$ will be $(= 1.3792 \times 1.047) = 1.4440$, and $1.179,134 \times \overline{1.047}^7$ will be $(= 1.179,134 \times 1.3792) = 1.6262$ &c. Therefore $1.179,134 \times \overline{1.047}^7 - \overline{1.047}^8$ will be $= 1.6262 - 1.4440 = 0.1822$. Therefore, if R be $= 1.047$, the binomial quantity $1.179,134R^7 - R^8$ will be $= 0.1822$. This quantity is somewhat greater than $0.179,134$, or the absolute term of the equation $1.179,134R^7 - R^8 = 0.179,134$; and consequently the quantity 1.047 must be somewhat less than the true value of R , or the greater root of that equation. But it will be much nearer to the said true value than the former number 1.105 is.

Now let the number 1.105 be denoted by the small letter b , and the quantity 0.1491 (which is the value of the binomial quantity $1.179,134R^7 - R^8$ resulting

resulting from the substitution of 1.105, or b , in it's terms instead of R ,) be denoted by the capital letter B. And let the number 1.047 be denoted by the small letter c , and the quantity 0.1822, (which is the value of the binomial quantity $1.179,134R^7 - R^8$ resulting from the substitution of 1.047, or c , in it's terms, instead of R ,) be denoted by the capital letter C. And, by the differential method of approximation, we may obtain another near value of R that shall be more exact than either of the two former, by making the following proportion; As $C - B$ is to $C - 1.Q$, so (very nearly,) will $b - c$ be to $R - c$, that is, as $0.1822 - 0.1437$ is to $0.1822 - 0.179,134$, so (very nearly) will $1.105 - 1.047$ be to $R - 1.047$, or as 0.058 is to $0.003,066$, so (very nearly,) will 0.058 be to $R - 1.047$. Therefore $R - 1.047$ will be, nearly, $= \frac{0.003,066 \times 0.058}{0.0332} (= \frac{0.000,177,828}{0.0332}) = 0.005,35$, and consequently R will be, nearly, $(= 0.005,35 + 1.047) = 1.052,35$; or the next near value of R , or the greater root of the proposed equation $1.179,134R^7 - R^8 = 0.179,134$, will be 1.052,35. Q. E. I.

Now let this new value of R , to wit, 1.052,35, or (dropping the last figure 5, to save trouble in the calculation,) 1.0523, be substituted instead of R in the binomial quantity $1.179,134R^7 - R^8$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or less, than 0.179,134, or the absolute term of the equation $1.179,134R^7 - R^8 = 0.179,134$, and consequently to determine whether the said number 1.0523 will be less, or greater, than R , or the greater root of that equation. This may be done as follows.

If R is $= 1.0523$, we shall have $R^7 = \overline{1.0523}^7$, and $R^8 = \overline{1.0523}^8$, and $1.179,134 \times R^7 - R^8 = 1.179,134 \times \overline{1.0523}^7 - \overline{1.0523}^8$. We must therefore find the values of $\overline{1.0523}^7$ and $\overline{1.0523}^8$; which may be done as follows.

The logarithm of 1.0523 is 0.022,139,6. Therefore the logarithm of $\overline{1.0523}^7$ will be $(= 7 \times 0.022,139,6) = 0.154,977,2$, which is the logarithm of 1.4288. Therefore $\overline{1.0523}^7$ is $= 1.4288$, and consequently $\overline{1.0523}^8$ will be $(= 1.4288 \times 1.0523) = 1.503,526,24$, and $1.179,134 \times \overline{1.0523}^7$ will be $(= 1.179,134 \times 1.4288) = 1.684,746,659,2$. Therefore $1.179,134 \times \overline{1.0523}^7 - \overline{1.0523}^8$ will be $(= 1.684,746,659,2 - 1.503,526,24) = 0.181,220,419,2$; that is, if R is $= 1.0523$, the binomial quantity $1.179,134R^7 - R^8$ will be $= 0.181,220,419,2$. This quantity is somewhat greater than 0.179,134, or the absolute term of the equation $1.179,134R^7 - R^8 = 0.179,134$. Therefore 1.0523 must be less than the true value of R , or the greater root of that equation. Q. E. I.

In

In order therefore to obtain a more exact value of R , we will suppose R to be $= 1.0523 + z$, and substitute this binomial quantity instead of it in the equation $1.179,134R^7 - R^8 = 0.179,134$, and resolve the transformed equation resulting from such substitution, as if it were a mere simple equation, according to the directions of Mr. Raphson's method of approximation.

Now, since R is $= 1.0523 + z$, we shall have $R^7 = \overline{1.0523 + z}^7 = \overline{1.0523}^7 + 7 \times \overline{1.0523}^6 \times z + \&c (= \overline{1.0523}^7 + 7 \times \frac{\overline{1.0523}^7}{1.0523} \times z + \&c = 1.4288 + 7 \times \frac{1.4288}{1.0523} \times z + \&c = 1.4288 + 7 \times 1.357,787 \times z + \&c) = 1.4288 + 9.504,509 \times z + \&c$, and $R^8 = \overline{1.0523 + z}^8 = \overline{1.0523}^8 + 8 \times \overline{1.0523}^7 \times z + \&c (= 1.4288 \times 1.0523 + 8 \times 1.4288 \times z + \&c) = 1.503,526,24 + 11.4304 \times z + \&c$. And $1.179,134 \times R^7$ will be $= 1.179,134 \times \overline{1.0523 + z}^7 (= 1.179,134 \times [1.4288 + 9.504,509 \times z + \&c] = 1.179,134 \times 1.4288 + 1.179,134 \times 9.504,509 \times z + \&c) = 1.684,746,659,2 + 11.207,089,715,206 \times z + \&c$. Therefore the binomial quantity $1.179,134R^7 - R^8$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 1.684,746,659,2 + 11.207,089,715,206 \times z + \&c \\ - 1.503,526,24 - 11.430,400,000,000 \times z - \&c \end{array} \right\} \\ = 0.181,220,419,2 - 0.223,310,284,794 \times z \&c.$$

But the binomial quantity $1.179,134R^7 - R^8$ is $= 0.179,134$.

Therefore the compound quantity $0.181,220,419,2 - 0.223,310,284,794 \times z \&c$ will also be $= 0.179,134$.

Therefore $0.181,220,419,2$ will be $= 0.179,134 + 0.223,310,284,794 \times z \&c$, and consequently $0.223,310,284,794 \times z \&c$ will be $(= 0.181,220,419,2 - 0.179,134) = 0.002,086,419,2$, and z will be $(= \frac{0.002,086,419,2}{0.223,310,284,794}) = 0.009,3$. Therefore R , or $1.0523 + z$ will be $=$

$1.0523 + 0.009,3 = 1.0616$, or the new near value of R in the equation $1.179,134R^7 - R^8 = 0.179,134$, obtained by this last process of Mr. Raphson's method of approximation, will be 1.0616 . Q. E. I.

This last number 1.0616 is sufficiently near to 1.06 to make it reasonable to conjecture that the true value of R in this equation will be $= 1.06$; which, upon trial, will be found to make the binomial quantity $1.179,134R^7 - R^8$ be equal to $0.179,134$, and consequently must be equal to the greater root of the equation $1.179,134R^7 - R^8 = 0.179,134$. It seems therefore to be unnecessary to carry the approximation to the value of R , or the greater root of this equation, any further.

Art. 12. The third Example of an affected equation resulting from a Question concerning Compound Interest, given by Mr. John Ward in the parts of his works that have been here reprinted, occurs above in page 718, in the Treatise intitl'd *Clavis Ufuræ*, or *A Key to Interest, both Simple and Compound*, Chapter IV, Section III, and is expressed in the following words;

“ Example I. In Yearly Rents.

“ Suppose there were paid 3205*l.* 5*s.* for an annuity, or a lease, of 250*l.* *per annum*, to continue for 21 years; what rate of Interest per cent &c is “ allowed to the Purchaser?”

This Question Mr. Ward does not solve in the same manner as he did the two former, to wit, by reducing them to an affected equation, and investigating the value of R , or the greater root of this equation, by supposing it to be equal to $1 + e$, and substituting $1 + e$ instead of R in the said equation, with an omission of all the terms in the serieses that are equal to the powers of R or $1 + e$, that involve any higher powers of e than it's square, or e^2 , and resolving the imperfect, transformed, equation that results from such a substitution: but he solves it by mere conjectural assumptions of different quantities for the value of R , and by computing, by means of such assumed values of R , the value of the annuity which might, at those supposed rates of Interest, be purchased for the given sum of 3205*l.* 5*s.*, till he finds that the annuity corresponding to the last-assumed value of R is the very annuity of 250*l.* *per annum* which was given in the Question. In this way of proceeding he, first, assumes 1.06 for the value of R , and finds the annuity that corresponds to it to be 196.1059*l.*, instead of 250*l.*, and thence concludes that 1.06 is greater than the true value of R ; and he then assumes 1.05 for the said value, and finds the annuity that corresponds to it to be 250.000*l.*, or exactly 250*l.* a year, or the very annuity given in the Question, and thence concludes that this second assumed number, 1.05, is the true value of R , or that the rate of Interest required is 5 per cent.

This way of answering a question of this kind is very clear and satisfactory; but there is some danger of it's becoming tedious by the number of conjectures and trials it may be necessary to make before we arrive at the true value of R , if they are not made with great judgement and sagacity, and, one may almost say, with great good luck. I shall therefore solve the foregoing Question by resolving the affected equation to which it may be reduced, and which comes under the general equation $\frac{U}{P} = \frac{U}{P} \times R' + R' - R'^{+1}$, or $\frac{U}{P} \times R' + 1 \times R' = \frac{U}{P}$, or $\frac{U}{P} + 1 \times R' - R'^{+1} = \frac{U}{P}$, as Mr. Ward has informed

formed us in the preceeding page 717; in which equation the letter U denotes the annuity, or rent, to be purchased; the letter t denotes the time of it's continuance, or the number of separate equal portions of time at the ends of which the annuity becomes payable, during it's continuance; the letter P denotes the present worth of the annuity, or rent, or the price to be paid for it by the purchaser; and the letter R denotes the amount of $1l.$, together with it's interest for one year, or half-year, or quarter of a year, or, in general, for one of the portions of time at the ends of which the interest is to be paid; according to the notation of our author in page 707 at the beginning of Section III.

Art. 13. We must therefore apply the general equation $\frac{U}{P} + 1 \times R^t - R^{t+1} = \frac{U}{P}$ to the particular question now before us; which may be done as follows.

In this Question U , or the annuity, or rent, purchased, is $= 250l. \text{ per annum}$, and it is to be paid at the end of every year; the time t , during which the annuity is to continue, is 21 years, or contains 21 separate equal portions of time, of a year each, at the ends of which the annuity is made payable; the present worth, P , of the annuity, or the price to be paid for it by the purchaser, is $3205l. 5s.$, or 3205.25 ; and R , or the value of $1l.$, together with it's interest for one year, is the unknown quantity which we are required to find. Therefore $\frac{U}{P}$ will be $= \frac{250}{3205.25} = 0.077,997$, and $\frac{U}{P} + 1$ will be $= 1.077,997$, and R^t will be $= R^{21}$, and R^{t+1} will be $= R^{22}$, and the general equation $\frac{U}{P} + 1 \times R^t - R^{t+1} = \frac{U}{P}$ will be converted into the numeral equation $1.077,997 R^{21} - R^{22} = 0.077,997$. This equation we must therefore now endeavour to resolve.

The Resolution of the Equation $1.077,997 R^{21} - R^{22} = 0.077,997$ in the Manner described in the Appendix to Dr. Halley's Discourse on Compound Interest in Pages 414, 415, 416, &c - - - to Page 423.

Art. 14. By comparing this numeral equation $1.077,997 R^{21} - R^{22} = 0.077,997$ with the general equation $Px^m - x^{m+1} = Q$, (under which it falls,) we shall have $P = 1.077,997$, and $Q = 0.077,997$, and $x = R$, and

$5 \text{ L } 2$ $m = 21,$

$m = 21$, and consequently $m + 1 = 22$. Therefore the greater limit of the magnitude of R , or the greater root of this equation, (which greater limit in the general equation $Px^m - x^{m+1} = Q$ is $= P$,) will be $= 1.077,997$, and the lesser limit of R (which in the said general equation is $= \frac{m}{m+1} \times P$,) will, in this numeral equation, be $= \frac{21}{22} \times 1.077,997$ ($= \frac{22.637,937}{22}$) $= 1.028,997$, and the arithmetical mean between these two limits will be ($= \frac{1.077,997 + 1.028,997}{2} = \frac{2.106,994}{2}$) $= 1.053,497$, or, very nearly, 1.0535 . Therefore 1.0535 will be a pretty good first near value of R , or the greater root of the equation $1.077,997R^{21} - R^{22} = 0.077,997$.

Further, it is evident that the lesser root of this equation is 1 . For, if R be taken $= 1$, we shall have $R^{21} = 1$, and R^{22} also $= 1$; and consequently the binomial quantity $1.077,997R^{21} - R^{22}$ will be ($= 1.077,997 \times 1 - 1 = 1.077,997 - 1$) $= 0.077,997$, which is the absolute term of the equation $1.077,997R^{21} - R^{22} = 0.077,997$; and therefore 1 will be a root of this equation. Q. E. D.

Now let M be put $= \frac{m}{m+1} \times P$, or $1.028,997$, the lesser limit of the greater root of the equation; and let d be put for the excess of the said lesser limit of the greater root R above the lesser root 1 . And we shall have d ($= 1.028,997 - 1$) $= 0.028,997$, and $\frac{d}{2} = 0.014,498$, and consequently $M + \frac{d}{2}$ ($= 1.028,997 + 0.014,498$) $= 1.043,495$, or, very nearly, $= 1.0435$. Therefore $1.043,495$, or 1.0435 , will also be a tolerably good first near value of R , as well as $1.053,497$, or 1.0535 , which was the arithmetical mean between its higher and lower limits.

The former of these near values of R , to wit, $1.053,497$, or 1.0535 , (which is the arithmetical mean between its greater and lesser limits $1.077,997$ and $1.028,997$, and is equal to the fraction $\frac{2m+1}{2m+2} \times P$, or $\frac{43}{44} \times 1.077,997$,) comes nearer to the true value of it, (which is 1.05 ,) than the second near value $1.043,495$, or 1.0435 , which was equal to $M + \frac{d}{2}$; the difference of 1.0535 and 1.05 being 0.0035 , and the difference between 1.05 and 1.0435 being 0.0065 ; which circumstance is the reverse of what happened in the former equation $1.179,134R^7 - R^8 = 0.179,134$, in which m , or the index of the power of R in the first term of the equation, was equal to the small number 7 . And either

either of these two near values, 1.0535 and 1.0435, is near enough to the truth to be a very good ground-work for a further approach to the true value of R either by the Differential method of approximation or by Mr. Raphson's method.

Further, we may observe that both in this equation and in the former equation $1.179,134R^7 - R^8 = 0.179,134$, the near value of R derived from the expression $\frac{2m+1}{2m+2} \times P$ is greater than the true value of R , and the other near value of it, derived from the expression $M + \frac{d}{2}$, is less than the said true value. Therefore it will often happen that an arithmetical mean between the two near values of R derived from those expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ will be nearer to the true value of R than either of the said two near values themselves; and this will happen in the present case in a remarkable degree, because the excess of the first near value, 1.0535, above 1.050, or 1.05, (which is the true value of R), is but little less than the excess of 1.050, or 1.05, above the second near value 1.0435. This, indeed, we could not have known with certainty, if we had not already known that 1.05 was the true value of R : but yet there would have been good reason to suspect that it might be so, and therefore to take this new arithmetical mean between 1.0535 and 1.0435, (which will be $= \frac{1.0535+1.0435}{2} = \frac{2.0970}{2} = 1.0485$), for our first practical near value of R , or for the ground-work of a further approach to its true value either by the Differential method of approximation, or by Mr. Raphson's method. We will therefore now take 1.0485 for our first practical near value of R in the equation $1.077,997R^{21} - R^{22} = 0.077,997$, and will substitute it instead of R in the binomial quantity $1.077,997R^{21} - R^{22}$, in order to discover how near the value of the said binomial quantity resulting from such substitution will approach to 0.077,997, or the absolute term of the said equation, and whether the said resulting value of the said binomial quantity will be greater, or less, than the said absolute term, and consequently whether the said number 1.0485 will be less, or greater, than the true value of R , or the greater root of the said equation. This substitution of 1.0485 instead of R in the binomial quantity $1.077,997R^{21} - R^{22}$ may be made in the following manner.

Art. 15. If R is $= 1.0485$, we shall have $R^{21} = \overline{1.0485}^{21}$, and $R^{22} = \overline{1.0485}^{22}$, and $1.077,997R^{21} - R^{22} = 1.077,997 \times \overline{1.0485}^{21} - \overline{1.0485}^{22}$. We must therefore find the values of $\overline{1.0485}^{21}$ and $\overline{1.0485}^{22}$; which may be done as follows.

The

The logarithm of 1.0485 is = 0.020,568,4. Therefore the logarithm of $\overline{1.0485}^{21}$ will be $(= 21 \times 0.020,568,4) = 0.431,936,4$, which is the logarithm of 2.703,562,5. Therefore $\overline{1.0485}^{21}$ is = 2.703,562,5; and consequently $\overline{1.0485}^{22}$ will be $(= \overline{1.0485}^{21} \times 1.0485 = 2.703,562,5 \times 1.0485) = 2.834,685,281,25$; and $1.077,997 \times \overline{1.0485}^{21}$ will be $(= 1.077,997 \times 2.703,562,5) = 2.914,432,264,312,5$. Therefore $1.077,997 \times \overline{1.0485}^{21} - \overline{1.0485}^{22}$ will be $(= 2.914,432,264,312,5 - 2.834,685,281,25) = 0.079,746,983,062,5$. This number is greater than the number 0.077,997, or the absolute term of the equation $1.077,997R^{21} - R^{22} = 0.077,997$. Therefore 1.0485 must be less than the true value of R , or the greater root of that equation.

Having thus found that 1.0485 is less than the true value of R , or the greater root of the equation $1.077,997R^{21} - R^{22} = 0.077,997$, we will, in the next place, suppose R to be equal to 1.0495, (which is a little greater than 1.0485, and therefore probably nearer to the true value of R .) and will substitute 1.0495 instead of R in the binomial quantity $1.077,997R^{21} - R^{22}$, in order to discover the value of the said binomial quantity resulting from such substitution; and, when that value shall be obtained, we will have recourse to a process of the Differential method of approximation in order to find another near value of R that shall be nearer than either 1.0485 or 1.0495 to its true value. This substitution of 1.0495 instead of R in the binomial quantity $1.077,997R^{21} - R^{22}$ may be made in the following manner.

If R is = 1.0495, we shall have $R^{21} = \overline{1.0495}^{21}$, and $R^{22} = \overline{1.0495}^{22}$, and $1.077,997R^{21} - R^{22} = 1.077,997 \times \overline{1.0495}^{21} - \overline{1.0495}^{22}$. We must therefore find the values of $\overline{1.0495}^{21}$ and $\overline{1.0495}^{22}$; which may be done as follows.

The logarithm of 1.0495 is = 0.020,982,4. Therefore the logarithm of $\overline{1.0495}^{21}$ will be $(= 21 \times 0.020,982,4) = 0.440,630,4$, which is the logarithm of 2.758,229,7. Therefore $\overline{1.0495}^{21}$ is = 2.758,229,7; and consequently $\overline{1.0495}^{22}$ will be $(= 2.758,229,7 \times 1.0495) = 2.894,762,070,15$; and $1.077,997 \times \overline{1.0495}^{21}$ will be $(= 1.077,997 \times 2.758,229,7) = 2.973,363,341,910,9$. Therefore $1.077,997 \times \overline{1.0495}^{21} - \overline{1.0495}^{22}$ will be $(= 2.973,363,341,910,9 - 2.894,762,070,15) = 0.078,601,271,760,9$. This number is a little, and but a little, greater than the number 0.077,997, or the absolute term of the equation $1.077,997R^{21} - R^{22} = 0.077,997$. Therefore 1.0495 will be a little, and but a little, less than the true value of R , or the greater root of that equation.

Now,

Now, since 1.0485, 1.0495, and R , are three quantities that are very nearly equal to each other, and the three numbers 0.079,746,9, 0.078,601,2, and 0.077,997, are three numbers corresponding to the three quantities 1.0485, 1.0495, and R , or that result from the substitution of the said three quantities instead of R in the binomial quantity $1.077,997R^{21} - R^{22}$, it follows that the differences of the three former quantities will be to each other in, nearly, the same proportion as the differences of the three latter quantities; that is, $R - 1.0495$ will be to $1.0495 - 1.0485$ in, nearly, the same proportion as 0.078,601,2 - 0.077,997 is to 0.079,746,9 - 0.078,601,2, or $R - 1.0495$ will be to 0.0010 in, nearly, the same proportion as 0.000,604,2 is to 0.001,145,7.

Therefore $R - 1.0495$ will be, nearly, $= \frac{0.0010 \times 0.000,604,2}{0.001,145,7} = 0.000,527,3$.

Therefore R will be, nearly, $(= 0.000,527,3 + 1.0495) = 1.050,027,3$; that is, the greater root of the equation $1.077,997R^{21} - R^{22} = 0.077,997$ will be nearly equal to 1.050,027,3. Q. E. I.

This value of R , to wit, 1.050,027, (which has been obtained by choosing 1.0485, or the arithmetical mean between 1.0535 and 1.0435, or the values of the quantities $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, for the first practical near value of R , or the ground-work of a further approach to it's true value, and afterwards employing one process of the Differential method of approximation,) approaches wonderfully near to it's true value 1.05, or 1.050,000, which it exceeds by only the small quantity 0.000,027, or the 38,888th part of the said true value 1.05; which proves that, in this case at least, the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ are very useful in enabling us to find a first near value of the unknown quantity R , or the greater root of the equation $P \times R^m - R^{m+1} = Q$, or $P \times R^m - R^{m+1} = P - 1$, that will be a very convenient ground-work for a further approach to it's true value by either the Differential method, or Mr. Raphson's method, of approximation.

Art. 16. This equation $1.077,997R^{21} - R^{22} = 0.077,997$ may also be resolved in the manner mentioned above in Art. 5 and 10 by substituting $1 + e$ in it's terms instead of R , and retaining four terms of the two serieses that will be equal to R^{21} and R^{22} , or to $\overline{1 + e}^{21}$ and $\overline{1 + e}^{22}$; as may all equations of this form, $Px^m - x^{m+1} = P - 1$, or in which the co-efficient of x^m in the first term of the equation is greater than the absolute term of it by an unit. But the value of R that will be thereby obtained will not be so near to the true value of R as either of the two values 1.0535 and 1.0435, which

which were obtained above in Art. 14 by means of the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$; and much less will it be so near to it as the number 1.0485, which was an arithmetical mean between the two former values. For it will be $\approx 1.058,730$; as will appear by the following investigation of it.

The Resolution of the foregoing Equation $1.077,997R^{21} - R^{22} = 0.077,997$, in the Manner described above in Art. 5 and 10, by substituting $1 + e$ in it's Terms instead of R , and retaining four Terms of the two Serieses that will be equal to R^{21} and R^{22} , or $1 + e^{21}$ and $1 + e^{22}$, and resolving the Quadratick Equation that will result from that Method of Approximation.

If R is $= 1 + e$, we shall have $R^{21} = 1 + e^{21} =$ (by the binomial theorem,) to the series $1 + 21e + 21 \times \frac{21-1}{2} \times e^2 + 21 \times \frac{21-1}{2} \times \frac{21-2}{3} \times e^3 + \&c$, continued to 22 terms, (or to the series $1 + 21e + 21 \times \frac{20}{2} \times e^2 + 21 \times \frac{20}{2} \times \frac{19}{3} \times e^3 + \&c$, continued to 22 terms, or to the series $1 + 21e + 21 \times 10 \times e^2 + 21 \times 10 \times \frac{19}{3} \times e^3 + \&c$, continued to 22 terms, or to the series $1 + 21e + 210e^2 + 7 \times 10 \times 19 \times e^3 + \&c$, continued to 22 terms,) or to the series $1 + 21e + 210e^2 + 1330e^3 + \&c$, continued to 22 terms; and R^{22} will be $= R^{21} \times R = 1 + e^{21} \times 1 + e (= \text{the series } 1 + 21e + 210e^2 + 1330e^3 + \&c \text{ multiplied into } 1 + e) = \text{the series } 1 + 22e + 231e^2 + 1540e^3 + \&c$, continued to 23 terms; and $1.077,997R^{21}$ will be $= 1R^{21} + 0.077,997 \times R^{21} = \text{the series } 1 + 21e + 210e^2 + 1330e^3 + \&c$, continued to 22 terms, $+ 0.077,997 \times \text{the same series } 1 + 21e + 210e^2 + 1330e^3 + \&c$, continued to 22 terms.

Therefore the binomial quantity $1.077,997 \times R^{21} - R^{22}$ will be $=$ the following compound quantity, to wit, the decimal fraction 0.077,997 multiplied into the series $1 + 21e + 210e^2 + 1330e^3 + \&c$, continued to 22 terms, $+ \text{the same series } 1 + 21e + 210e^2 + 1330e^3 + \&c$, continued to 22 terms, $- \text{the series } 1 + 22e + 231e^2 + 1540e^3 + \&c$, continued to 23 terms, $= \text{the series } 0.077,997 + 0.077,997 \times 21e + 0.077,997 \times 210e^2 + 0.077,997 \times 1330e^3 + \&c$, continued to 22 terms,

+ the

+ the series $1 + 21e + 210e^2 + 1330e^3 + \&c$, to 22 terms,
 — the series $1 + 22e + 231e^2 + 1540e^3 + \&c$, to 23 terms,
 = the series $0.077,997 + 1.637,937e + 16.379,370e^2 + 103.736,010e^3$
 + $\&c$, to 22 terms,
 + the series $1 + 21e + 210e^2 + 1330e^3 + \&c$, to 22 terms,
 — the series $1 + 22e + 231e^2 + 1540e^3 + \&c$, to 23 terms,
 = the series $0.077,997 + 1.637,937e + 16.379,370e^2 + 103.736,010e^3$
 + $\&c$, to 22 terms
 + the series $1 + 21e + 210e^2 + 1330e^3 + \&c$, to 22 terms,
 — the series $1 + 22e + 231e^2 + 1540e^3 + \&c$, to 23 terms,
 = the series $0.077,997 + 1 + 22.637,937e + 226.379,370e^2$
 + $1433.736,010e^3 + \&c$, to 23 terms,
 — the series $1 + 22.000,000e + 231.000,000e^2 + 1540.000,000e^3$
 + $\&c$, to 23 terms,
 = the series $0.077,997 + 0.637,937e - 4.620,630e^2 - 106.263,990e^3$
 — $\&c$, continued to 22 terms.

But the binomial quantity $1.077,997R^{21} - R^{22}$ is $= 0.077,997$.

Therefore the series $0.077,997 + 0.637,937e - 4.620,630e^2 - 106.263,990e^3 - \&c$, continued to 22 terms, will also be $= 0.077,997$.

Therefore $0.077,997 + 0.637,937e$ will be $= 0.077,997 + 4.620,630e^2$
 + $106.263,990e^3 + \&c$, and (subtracting $0.077,997$ from both sides,) $0.637,937e$ will be $= 4.620,630e^2 + 106.263,990e^3 + \&c$, and (dividing all the terms by e ,) $0.637,937$ will be $= 4.620,630e + 106.263,990e^2 + \&c$,
 and (dividing all the terms by $106.263,990$, the co-efficient of e^2 ,) $\frac{0.637,937}{106.263,990}$
 will be $= \frac{4.620,630e}{106.263,990} + e^2 + \&c$, that is, $0.006,003$ will be $= 0.043,482e$
 + $e^2 + \&c$, or (ranging the terms of the equation in the more usual manner,
 with the absolute term $0.006,003$ on the right-hand side of the mark of
 equality,) $e^2 + 0.043,482e + \&c$ will be $= 0.006,003$. Therefore (adding
 $0.021,741^2$ to both sides) we shall have $e^2 + 0.043,482e + 0.021,741^2$
 + $\&c = 0.006,003 + 0.021,741^2 (= 0.006,003 + 0.000,472,671,081) =$
 $0.006,475,671,081$, and (extracting the square-roots of both sides,) we shall
 have $e + 0.021,741 + \&c (= \sqrt{0.006,475,671,081}) = 0.080,471$, and $e (=$
 $0.080,471 - 0.021,741 - \&c) = 0.058,730 - \&c$. Therefore R , or
 $1 + e$, will be $(= 1 + 0.058,730 - \&c) = 1.058,730 - \&c$, or the greater
 root of the equation $1.077,997R^{21} - R^{22} = 0.077,997$ will be nearly equal
 to, but less than, $1.058,730$. Q. E. I.

From this example we may conclude that, when m , or the index of the
 power of R , or x , in an equation of this form, $P \times R^m - R^{m+1} = P - 1$,
 or $Px^m - x^{m+1} = P - 1$, is 21, or any greater number, this manner of
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resolving it by substituting $1 + e$ in it's terms instead of R , will not give us so good a first near value of it as either of the two expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$. And I believe the same observation will be true even when m is less than 21, if it be greater than 11 or 12. But, if it do not exceed 12, and more especially if it do not exceed 7, this method of approximation will be found to give us a very good first near value of R , as we have seen above in art. 10, where it enabled us to find the number 1.061,293 for a near value of R , or the greater root of the equation $1.179,134R^7 - R^8 = 0.179,134$, of which root the true value is 1.06. It is, however, in all cases attended with much more labour of calculation than computing the two expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$, and finding the arithmetical mean between them.

Art. 17. The fourth Example of a Problem, or Question, concerning Compound Interest, that produces an affected equation, given us by Mr. John Ward, occurs above in page 719, or the next page to that which contained the last Question, which produced the equation $1.077,997R^{21} - R^{22} = 0.077,997$, which we have been just now resolving. And it produces an equation of the same form with this last equation. It is expressed in the following words.

“ Example II. In Half-Yearly Payments.

“ Admit 3244*l.* 5*s.* were given for an annuity, or lease, of 250*l.* *per annum*, “ to be paid half-yearly, (viz. 125*l.* every half-year,) and to continue 21 years; “ what rate of Interest per cent is allowed the purchaser ?”

This Question Mr. Ward solves as he did the last, that is, by mere conjectural assumptions of different quantities for the value of R , and by making trials of the quantities so assumed by computing, by means of them, the value of the annuity, or, rather, half-yearly pension, which might, at those supposed rates of interest, be purchased for the given sum of 3244*l.* 5*s.*, till he finds that the half-yearly pension corresponding to the last-assumed value of R is the very half-yearly pension of 125*l.* which was given in the Question. In this way of proceeding he first assumes 1.06 for the value of R , or of 1*l.* together with it's interest for a year, and finds the half-yearly pension that corresponds to it to be 98.9283*l.*, instead of 125*l.*; and he thence concludes that the true value of R must be less than 1.06. He then, for a second trial, assumes 1.05 for the value of R , and computes the half-yearly pension that corresponds to this value of R , and finds it to be exactly 125*l.*, or the very half-yearly pension supposed in the Question, and thence concludes that this second assumed value of R , to wit, 1.05*l.*, is it's true value, or that the rate of interest required by the Question is 5 *per cent.* But, if we mean to solve this Question by first reducing

reducing it to an equation, and then resolving the equation so obtained, we must apply the general equation given us by Mr. Ward in page 717, to wit, $\frac{U}{P} = \frac{U}{P} \times R^t + R^t - R^{t+1}$, or $\left(\frac{U}{P} + 1\right) \times R^t - R^{t+1} = \frac{U}{P}$, to the particular Question we are to solve; which may be done as follows.

Since the annuity is to be paid every half-year by equal payments of 125*l.* each, the letter *U* will not represent a whole years' annuity, but one of these half-yearly payments, or 125*l.* And the letter *P* (which denotes the price paid for the annuity, or lease, of 250*l.* a year, or 125*l.* every half-year, during 21 years,) will be 3244*l.* 5*s.*, or 3244.25*l.*; and *t*, or the number of half-years, or equal portions of time at the ends of which the several payments of 125*l.* each are to be made, contained in the whole time during which the lease is to continue, (which is 21 years,) will be 42, and consequently *t* + 1 will be 43. Therefore the general equation $\left(\frac{U}{P} + 1\right) \times R^t - R^{t+1} = \frac{U}{P}$ will

in this Question be converted into the numeral equation $\left(\frac{125}{3244.25} + 1\right) \times R^{42} - R^{43} = \frac{125}{3244.25}$, or $0.038,529 + 1 \times R^{42} - R^{43} = 0.038,529$, or $1.038,529R^{42} - R^{43} = 0.038,529$.

This therefore is the equation which we are now to resolve. But it must be observed that the letter *R* in this equation will not (as before,) denote the value of one pound together with it's interest for a whole year, but the value of one pound together with it's interest for half a year, which will be a geometrical mean proportional between 1 pound and 1 pound together with it's interest for a whole year, or will be equal to the square-root of the sum of 1 pound and it's interest for a whole year, and will be somewhat less than an arithmetical mean between 1 pound and 1 pound together with it's interest for a whole year. Thus, if 1 pound together with it's interest for a whole year is = 1.06*l.*, *R* will in this equation be = $\sqrt{1.06}$, or 1.029, &c, or somewhat less than 1.03. This being premised, the equation $1.038,529R^{42} - R^{43} = 0.038,529$ may be resolved in the manner following.

The Resolution of the Equation $1.038,529R^{42} - R^{43} = 0.038,529$ in the Manner described above in the "Appendix to Dr. Halley's Discourse on Compound Interest," in Pages 414, 415, 416, &c - - to Page 523.

Art. 18. If we compare this equation with the general equation $Px^m - x^{m+1} = Q$ examined above in pages 414, 415, &c - - 423 of the present Volume,

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we

we shall have $P = 1.038,529$, and $x = R$, and $m = 42$, and $m + 1 = 43$, and $Q = 0.038,529$, and consequently $\frac{m}{m+1} \times P = \frac{42}{43} \times 1.038,529 (= \frac{43.618,218}{43}) = 1.014,377$. Therefore R , or the greater root of the equation $1.038,529R^{42} - R^{43} = 0.038,529$ will be less than $1.038,529$, but greater than $1.014,377$, and will be nearly equal to $\frac{2m+1}{2m+2} \times P$, or to $\frac{84+1}{84+2} \times 1.038,529$, or to $\frac{85}{86} \times 1.038,529$, or to $1.026,453$, or to an arithmetical mean between the said two limits $1.038,529$ and $1.014,377$.

Further, it is evident that the lesser root of this equation is 1. For, if R is $= 1$, we shall have $R^{42} = 1$, and $R^{43} = 1$, and consequently $1.038,529R^{42} - R^{43} (= 1.038,529 \times 1 - 1 = 1.038,529 - 1) = 0.038,529$; which is the absolute term of the equation $1.038,529R^{42} - R^{43} = 0.038,529$. Therefore 1 is a root of the said equation. Q. E. D.

Let this root 1 be subtracted from $1.014,377$, or $\frac{m}{m+1} \times P$, or M ; and let the difference be called d . And we shall have $d (= 1.014,377 - 1.000,000) = 0.014,377$, and $\frac{d}{2} = 0.007,188$, and $M + \frac{d}{2} (= 1.014,377 + 0.007,188) = 1.021,565$. Therefore this number $1.021,565$ will also be a near value of R , or the greater root of the equation $1.038,529R^{42} - R^{43} = 0.038,529$, as well as the number $1.026,453$. And, if we add these two near values of R together, and take the half of their sum, (which will be an arithmetical mean between them, and which will be $= \frac{1.026,453 + 1.021,565}{2} = \frac{2.048,018}{2} = 1.024,009$) it is probable that the said arithmetical mean $1.024,009$ will be still nearer than either of the two former numbers to the true value of R , or the greater root of the equation $1.038,529R^{42} - R^{43} = 0.038,529$. We will therefore take this number $1.024,009$, (or (neglecting the last figure 9,) 1.024 , for our first practical near value of R , and substitute it instead of R in the binomial quantity $1.038,529R^{42} - R^{43}$, in order to discover whether the value of the said binomial quantity resulting from such substitution will be greater, or less, than $0.038,529$, or the absolute term of the said equation, and consequently whether the said number 1.024 is less, or greater, than the true value of R , or the greater root of the said equation.

Now, if R be supposed to be $= 1.024$, we shall have $R^{42} = \overline{1.024}^{42}$, and $R^{43} = \overline{1.024}^{43}$, and $1.038,529R^{42} = 1.038,529 \times \overline{1.024}^{42}$; and consequently

quently the binomial quantity $1.038,529R^{42} - R^{43}$ will be $= 1.038,529 \times \overline{1.024}^{42} - \overline{1.024}^{43}$. We must therefore find the values of $\overline{1.024}^{42}$ and $\overline{1.024}^{43}$; which may be done as follows.

The logarithm of 1.024 is $= 0.010,300,0$. Therefore the logarithm of $\overline{1.024}^{42}$ will be $(= 42 \times 0.010,300,0) = 0.432,600,0$, which is the logarithm of 2 707,700,0. Therefore $\overline{1.024}^{42}$ is $= 2.707,700,0$; and consequently $\overline{1.024}^{43}$ will be $(= \overline{1.024}^{42} \times 1.024 = 2.707,700,0 \times 1.024) = 2.772,684,8$; and $1.038,529 \times \overline{1.024}^{42}$ will be $(= 1.038,529 \times 2.707,700,0) = 2.812,024,973,3$. Therefore the binomial quantity $1.038,529 \times \overline{1.024}^{42} - \overline{1.024}^{43}$ will be $= 2.812,024,973,3 - 2.772,684,8 = 0.039,340,173,3$; that is, the binomial quantity $1.038,529R^{42} - R^{43}$ will be $= 0.039,340,173,3$. This number 0.039,340,173,3 is a little greater than 0.038,529, the absolute term of the equation $1.038,529R^{42} - R^{43} = 0.038,529$. Therefore 1.024 must be somewhat less than the true value of R , or the greater root of the said equation. But it will be near enough to the said true value of R to be an excellent ground-work for a further approach to the said true value by Mr. Raphson's method of approximation.

Let us therefore put z for the excess of the true value of R above the quantity 1.024.

And we shall then have $R = 1.024 + z$, and consequently $R^{42} = \overline{1.024 + z}^{42} =$ (by the binomial theorem,) $\overline{1.024}^{42} + 42 \times \overline{1.024}^{41} \times z + \&c (= \overline{1.024}^{42} + 42 \times \frac{\overline{1.024}^{42}}{1.024} \times z + \&c = 2.707,700,0 + 42 \times \frac{2.707,700,0}{1.024} \times z + \&c = 2.707,700,0 + 42 \times 2.644,238,2 \times z + \&c) = 2.707,700,0 + 111.058,004,4 \times z + \&c$, and $R^{43} = \overline{1.024 + z}^{43} (= \overline{1.024}^{43} + 43 \times \overline{1.024}^{42} \times z + \&c = 2.772,684,8 + 43 \times 2.707,700,0 \times z + \&c) = 2.772,684,8 + 116.431,100,0 \times z + \&c$, and $1.038,529 \times R^{42} = 1.038,529 \times$ the compound quantity $2.707,700,0 + 111.058,004,4 \times z + \&c (= 1.038,529 \times 2.707,700,0 + 1.038,529 \times 111.058,004,4 \times z + \&c) = 2.812,024,973,3 + 115.336,958,251,527,6 \times z + \&c$.

Therefore the binomial quantity $1.038,529R^{42} - R^{43}$ will be $=$ the compound quantity

$$\left\{ \begin{array}{l} 2.812,024,973,3 + 115.336,958,251,527,6 \times z + \&c \\ - 2.772,684,8 - 116.431,100,000,000,0 \times z - \&c \end{array} \right\}$$

$$= 0.039,340,173,3 - 1.094,141,748,472,4 \times z - \&c.$$

But

But the binomial quantity $1.038,529R^{42} - R^{42}$ is $= 0.038,529$.

Therefore the compound quantity $0.039,340,173,3 - 1.094,141,748,472,4 \times z - \&c$ will also be $= 0.038,529$.

Therefore $0.039,340,173,3$ will be $= 0.038,529 \cdot + 1.094,141,748,472,4 \times z + \&c$, and $1.094,141,748,472,4 \times z + \&c$ will be $(= 0.039,340,173,3 - 0.038,529) = 0.000,811,173,3$, and z will be $= \frac{0.000,811,173,3}{1.094,141, \&c} = 0.000,741$. Therefore R , or $1.024 + z$, will be $(= 1.024 + 0.000,741) = 1.024,741$, that is, the true value of the greater root of the equation $1.038,529R^{42} - R^{42} = 0.038,529$ will be $= 1.024,741$. But R is the value of 1 pound together with it's interest for half a year. Therefore $1.024,741$ will be the value of 1 pound together with it's interest for half a year; and consequently $(1.024,741)^2$, or $1.050,094,117,081$, will be the value of 1 pound together with it's interest for a whole year; which number differs so little from 1.05 that it may be considered as equal to it: so that the rate of Interest allowed to the purchaser of this half-yearly pension of 125*l*, which is to continue for 42 half-years, or 21 years, is 5 per cent, as Mr. Ward has found it to be by his method of conjectures and trials. Q. E. I.

In this example it has appeared that the number 1.024, (which is an arithmetical mean between the two numbers 1.026,453 and 1.021,565, which are derived from the expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$), is a very excellent first near value of R , or the greater root of the proposed equation $1.038,529R^{42} - R^{42} = 0.038,529$, since it agrees with it's more accurate value, 1.024,741, in all the four figures 1.024. And I believe that the like arithmetical mean between the two expressions $\frac{2m+1}{2m+2} \times P$ and $M + \frac{d}{2}$ will often be found to be a very good first near value of R in an equation of the general form, $P \times R^m - R^{m+1} = P - 1$.

Art. 19. The fifth Example of a Problem, or Question, concerning Compound Interest, that produces an affected equation; given us by Mr. John Ward, occurs above in page 721, and is expressed in the words following.

“Example III. Of Quarterly Payments.

“There is [a sum of] 1600 pounds paid for an annuity of 100*l*. per annum, [which is] to continue 99 years, and to be paid quarterly, viz. 25*l*. every every quarter of a year during that time. It is required to find what rate of Interest per cent [per annum] is allowed the purchaser.”

This Question is solved by Mr. Ward in the same manner as the two last, that is, by mere conjectural assumptions of different quantities for the value of R ,

R , and by making trials of the quantities so assumed by computing, by means of them, the value of the annuity, or, rather, of the quarterly pension, which might, at those supposed rates of interest, be purchased for the given sum of 1600*l.*, till he finds that the quarterly pension corresponding to the last-assumed value of R is very nearly equal to the very quarterly pension of 25*l.*, which was given in the Question. In this way of proceeding Mr. Ward first assumes 1.06 for the value of R , or of 1*l.* together with it's interest for a year, and finds the quarterly pension, that corresponds to this value, to be such as to make him conclude that the true value of R must be greater than 1.06. He then, for a second trial, assumes 1.064 for the value of R , and finds the quarterly pension that corresponds to this second value of R , to be such as to make him conclude that the true value of R must be a little less than 1.064. He then in the third place supposes R to be equal to 1.0638, and finds the quarterly pension that corresponds to this third value of R to be such as to make him conclude that the true value of R must be a little greater than 1.0638. And therefore he assumes the number 1.063,855 for a fourth near value of R ; and he finds that the quarterly pension, that corresponds to this fourth value of R , is exceedingly near to the given pension of 25*l.* per quarter; and hence he concludes that 1.063,855 is very nearly equal to the true value of R , and consequently that the rate of interest allowed to the purchaser of the said quarterly pension of 25*l.*, for 99 years, or (4×99 , or) 396 quarters of a year, is 6.3855 *per cent per annum*.

Mr. Ward then confirms this conclusion by supposing the rate of interest to be 6.3855*l.* per cent, or R to be $= 1.063,855$, and the quarterly pension purchased to be 25*l.*, and the time, during which it is to continue, to be 99 years, or 396 quarters of a year, and computing from these *data* the value of P , or the price that ought to be paid for such quarterly pension; which he finds to be 1599.9*l.*, or 1599*l.* 18*s.*; which differs from the sum of 1600*l.*, (which was the price supposed to be paid for it in the Question,) by only the small sum of 2 shillings. And hence he justly concludes that the number 1.063,855 must be extremely near to the true value of R , and that the rate of interest allowed to the purchaser of these quarterly payments of 25*l.* for 99 years, is very nearly $= 6.3855\text{i}l.$, or 6*l.* 7*s.* 8½*d.* *per cent per annum*.

Art. 20. But the foregoing Question may also be solved by first reducing it to an equation that will be of the same form as the two equations $1.038,529R^{42} - R^{43} = 0.038,529$ and $1.077,997R^{21} - R^{22} = 0.077,997$, which resulted from the former Questions proposed by Mr. John Ward in pages 718 and 719, and then resolving the equation thereby obtained. All these equations come under the general equation given us by Mr. Ward in page 717 for the solution

of questions of this nature, to wit, the equation $\frac{U}{P} + 1 \times R^t - R^{t+1} = \frac{U}{P}$; in which the letter U denotes the pension to be paid at the end of every year,

year, or half-year, or quarter of a year, or other given portion of time agreed upon between the seller and the purchaser of such annuity, or other pension; and P denotes the purchase-money, or price to be paid for such pension; and t denotes the number of years, or half-years, or quarters of a year, or other equal portions of time at the ends of which the several payments become due, contained in the whole time during which the annuity, or half-yearly pension, or quarterly pension, or other regularly-returning pension, is to continue; and R denotes the value of one pound together with it's interest for one year, or one half-year, or one quarter of a year, or other given portion of time at the end of which a payment of the pension becomes due. Now, if we apply this

general equation $\left(\frac{U}{P} + 1\right) \times R^t - R^{t+1} = \frac{U}{P}$ to the Question proposed by

Mr. Ward in page 721, we shall have the given pension U , (which is to be paid to the purchaser at the end of every quarter of a year for the space of 99 years, or 396 quarters of a year,) = 25 pounds; and P (or the price to be paid for the said pension by the purchaser of it,) = 1600*l.*; and t (or the number of quarters of a year in the whole space of 99 years during which the said pension is to continue,) = 396; and R will denote the value of 1 pound together with it's interest at the end of a quarter of a year; and consequently

the general equation $\left(\frac{U}{P} + 1\right) \times R^t - R^{t+1} = \frac{U}{P}$ will be converted into the

numeral equation $\left(\frac{25}{1600} + 1\right) \times R^{396} - R^{397} = \frac{25}{1600}$, or $0.015,625 + 1 \times R^{396} - R^{397} = 0.015,625$, or $1.015,625R^{396} - R^{397} = 0.015,625$, which may be resolved in the manner described above in the Appendix to Dr. Halley's Discourse on Compound Interest by proceeding as follows.

The Resolution of the Equation $1.015,625R^{396} - R^{397} = 0.015,625$ in the Manner described above in the "Appendix to Dr. Halley's Discourse on Compound Interest," in Pages 414, 415, 416, &c. - - to Page 423.

Art. 21. This equation has evidently two roots, of which the lesser is = 1. For, if R be = 1, we shall have R^{396} also = 1, and R^{397} also = 1, and consequently $1.015,625 \times R^{396} - R^{397}$ (= $1.015,625 \times 1 - 1 = 1.015,625 - 1$) = 0.015,625, which is the absolute term of the equation; or, in other words, 1 will be a root of the equation. But this root is not the root which we are seeking, or that will answer the question from which the equation was derived; because in that question R stands for the value of 1 pound together with it's interest for one quarter of a year, and consequently must be greater than

than 1; so that the value of R which we are to find will be the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$.

Now this greater root R is evidently less than 1.015,625, or the co-efficient of R^{396} in the first term, $1.015,625R^{396}$, of the equation: for otherwise that first term would not be greater than the second term R^{397} , which is subtracted from it. Therefore 1.015,625 is the higher limit of this greater root R . And by comparing this equation $1.015,625R^{396} - R^{397} = 0.015,625$ with the general

equation $Px^m - x^{m+1} = Q$ examined above in pages 414, 415, &c. - - 423 of the present Volume, (of which general equation the numeral equation $1.015,625R^{396} - R^{397} = 0.015,625$ is a particular case,) we shall have $P = 1.015,625$, and $x = R$, and $m = 396$, and $m + 1 = 397$, and $Q = 0.015,625$; and consequently

$\frac{m}{m+1} \times P$ (which is there shewn to be the lower limit of the magnitude of the greater root of the said general equation,) will be $= \frac{396}{397} \times 1.015,625 (= \frac{402.187,500}{397}) = 1.013,066$; which therefore is the lower limit of the magni-

tude of R in the present equation. Now these two limits of the magnitude of R , to wit, 1.015,625 and 1.013,066, differ from each other by only the small quantity 0.002,559, which is the 397th part of the greater limit 1.015,625; and therefore the difference of the root R (which lies between these two limits,) from either of these limits must be less than 0.002,559. Therefore, if we take an arithmetical mean between these two limits, it is probable that such middle quantity will differ from the true value of R by much less than the small quantity 0.002,559. Now an arithmetical mean

between 1.015,625 and 1.013,066 will be $(= \frac{1.015,625 + 1.013,066}{2} = \frac{2.028,691}{2}) = 1.014,345$. Therefore the number 1.014,345, or (neglecting the two last figures 45, as probably not exact,) the number 1.0143 will probably be a pretty near value of R , or the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$.

Art. 22. Now let this number 1.0143 be substituted instead of R in the binomial quantity $1.015,625R^{396} - R^{397}$, in order to discover whether the value of the said binomial quantity resulting from this substitution will be greater, or will be less, than 0.015,625, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$, and consequently to determine whether the said number 1.0143 will be less, or will be greater, than R , or the greater root of the said equation. This substitution may be made as follows.

If R is $= 1.0143$, we shall have $R^{396} = \overline{1.0143}^{396}$, and $R^{397} = \overline{1.0143}^{397}$, and $1.015,625R^{396} = 1.015,625 \times \overline{1.0143}^{396}$; and the binomial quantity $1.015,625R^{396} - R^{397}$ will be $= 1.015,625 \times \overline{1.0143}^{396} - \overline{1.0143}^{397}$. We must

must therefore endeavour to find the values of $\overline{1.0143}^{396}$ and $\overline{1.0143}^{397}$; which may be done as follows.

The logarithm of 1.0143 is = 0.006,166,4. Therefore the logarithm of $\overline{1.0143}^{396}$ will be ($= 396 \times 0.006,166,4$) = 2.441,894,4; which is the logarithm of 276.626,878,9. Therefore $\overline{1.0143}^{396}$ will be = 276.626,878,9; and consequently $\overline{1.0143}^{397}$ will be ($= \overline{1.0143}^{396} \times 1.0143$) = 280.582,643,2; and $1.015,625 \times \overline{1.0143}^{396}$ will be = $1.015,625 \times 276.626,878,9$ = 280.949,173,8. Therefore $1.015,625 \times \overline{1.0143}^{396} - \overline{1.0143}^{397}$ will be ($= 280.949,173,8 - 280.582,643,2$) = 0.366,530,6; which is greater than 0.015,625, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$. Therefore 1.0143 will be less than the true value of R , or the greater root of this equation. Q. E. I.

But we know that this greater root is less than 1.015,625. Therefore it must be of some intermediate magnitude between 1.015,625 and 1.014,3. We will therefore suppose it to be an arithmetical mean between these two quantities, or to be equal to ($\frac{1.015,625 + 1.0143}{2}$, or to $\frac{2.029,915}{2}$, or to) 1.014,962, and will substitute this new quantity, 1.014,962, instead of R in the binomial quantity $1.015,625R^{396} - R^{397}$, in order to discover how nearly the value of the said binomial quantity resulting from this substitution will approach to 0.015,625, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$; and, when we shall have discovered this value of the said binomial quantity, we will make a further approach to the true value of R by the Differential method of approximation.

Art. 23. Now, if R is = 1.014,962, we shall have $R^{396} = \overline{1.014,962}^{396}$, and $R^{397} = \overline{1.014,962}^{397}$. We must therefore endeavour to find the values of $\overline{1.014,962}^{396}$ and $\overline{1.014,962}^{397}$; which may be done as follows.

The logarithm of 1.014,962 is = 0.006,049,7. Therefore the logarithm of $\overline{1.014,962}^{396}$ will be ($= 396 \times 0.006,049,7$) = 2.554,081,2; which is the logarithm of 358.163,388,4. Therefore $\overline{1.014,962}^{396}$ is = 358.163,388,4. Therefore $\overline{1.014,962}^{397}$ will be ($= \overline{1.014,962}^{396} \times 1.014,962$) = 363.522,229,0. And $1.015,625 \times \overline{1.014,962}^{396}$ will be ($= 1.015,625 \times 358.163,388,4$) = 363.759,691,3, and $1.015,625 \times \overline{1.014,962}^{396} - \overline{1.014,962}^{397}$ will be ($= 363.759,691,3 - 363.522,229,0$) = 0.237,462,3; which is greater than 0.015,625, or the absolute term

term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$, but differs less from it than $0.366,530,6$ does. Therefore $1.014,962$, as well as 1.0143 , is less than the true value of R , or the greater root of that equation. But now, having obtained the two values of the binomial quantity $1.015,625R^{396} - R^{397}$ resulting from the substitution of the two quantities 1.0143 and $1.014,962$ in it's terms instead of R , we may, by means of their differences, obtain another near value of R that shall be nearer than either 1.0143 or $1.014,962$ to it's true value. This may be done in the following manner.

We have now two near values of R that differ but little from each other, and from it's true value in the equation $1.015,625R^{396} - R^{397} = 0.015,625$, and which are both less than the said true value, to wit, the numbers $1.014,3$ and $1.014,962$; and we have also the two values of the binomial quantity $1.015,625R^{396} - R^{397}$ corresponding to these two numbers, to wit, $0.366,530,6$ and $0.237,462,3$. Therefore the difference of the numbers 1.0143 and $1.014,962$ will be to the difference of the number $1.014,962$ and R in nearly the same proportion as the difference of the numbers $0.366,530,6$ and $0.237,462,3$ is to the difference of the number $0.237,462,3$ and $0.015,625$, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$; that is, $1.014,962 - 1.014,3$ will be to $R - 1.014,962$ in nearly the same proportion as $0.366,530,6 - 0.237,462,3$ is to $0.237,462,3 - 0.015,625$, or $0.000,662$ will be to $R - 1.014,962$ in nearly the same proportion as $0.129,068,3$ is to $0.221,837,3$. Therefore $R - 1.014,962$ will be nearly $= \frac{0.000,662 \times 0.221,837,3}{0.129,068,3}$
 $= \frac{0.000,146,856,792,6}{0.129,068,3} = 0.001,137$, and R will be nearly $(= 0.001,137 + 1.014,962) = 1.016,099$.

Art. 24. This value of R is evidently too great, because it is greater than $1.015,625$, which is the higher limit of R . We will therefore suppose R (which has been shewn to be greater than $1.014,962$.) to be nearly $= 1.015,3$, and will substitute this number 1.0153 instead of R in the binomial quantity $1.015,625R^{396} - R^{397}$, in order to discover how nearly the value of the said binomial quantity resulting from this substitution will approach to the value of $0.015,625$, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$, and whether it will be greater, or will be less, than the said absolute term, and consequently whether the said number 1.0153 will be less, or will be greater, than R , or the greater root of that equation.

Now, if R is $= 1.0153$, we shall have $R^{396} = \overline{1.0153}^{396}$, and $R^{397} = \overline{1.0153}^{397}$, and $1.015,625R^{396} = 1.015,625 \times \overline{1.0153}^{396}$, and $1.015,625R^{396} - R^{397} = 1.015,625 \times \overline{1.0153}^{396} - \overline{1.0153}^{397}$. We must therefore endeavour to find the values of $\overline{1.0153}^{396}$ and $\overline{1.0153}^{397}$; which may be done as follows.

The logarithm of 1.0153 is = 0.006,594,4. Therefore the logarithm of $\overline{1.0153}^{396}$ will be ($= 396 \times 0.006,594,4$) = 2.611,382,4, which is the logarithm of 408.679,056,6. Therefore $\overline{1.0153}^{396}$ is = 408.679,056,6; and consequently $\overline{1.0153}^{397}$ will be ($= \overline{1.0153}^{396} \times 1.0153$) = 408.679,056,6 $\times$ 1.0153 = 414.931,846,1; and $1.015,625 \times \overline{1.0153}^{396}$ will be ($= 1.015,625 \times 408.679,056,6$) = 415.064,666,8. Therefore the binomial quantity $1.015,625 \times \overline{1.0153}^{396} - \overline{1.0153}^{397}$ will be ($= 415.064,666,8 - 414.931,846,1$) = 0.132,820,7.

This number 0.132,820,7 is greater than 0.015,625, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$; and therefore the number 1.0153 (from the substitution of which instead of R in the binomial quantity $1.015,625R^{396} - R^{397}$ it arises,) must be less than the true value of R , or the greater root of that equation.

Art. 25. We are therefore now certain that R is greater than 1.0153, but less than 1.015,625; which is a very considerable degree of exactness. But, in order to obtain a still nearer value of it, we will now employ one process of Mr. Raphson's method of approximation.

Let us therefore put z for the unknown excess of the true value of R above 1.0153, and substitute the binomial quantity $1.0153 + z$ instead of R in the equation $1.015,625R^{396} - R^{397} = 0.015,625$.

Now, since R is = $1.0153 + z$, we shall have $R^{396} = \overline{1.0153 + z}^{396}$ ($= \overline{1.0153}^{396} + 396 \times \overline{1.0153}^{395} \times z + \&c = \overline{1.0153}^{396} + 396 \times \frac{\overline{1.0153}^{396}}{1.0153} \times z + \&c = 408.679,056,6 + 396 \times \frac{408.679,056,6}{1.0153} \times z + \&c$) = 408.679,056,6 + 396 $\times$ 402.520,493,0 $\times$ z + $\&c$) = 408.679,056,6 + 159,398.115,228,0 $\times$ z + $\&c$, and $R^{397} = \overline{1.0153 + z}^{397}$ ($= \overline{1.0153}^{397} + 397 \times \overline{1.0153}^{396} \times z + \&c = 414.931,846,1 + 397 \times 408.679,056,6 \times z + \&c$) = 414.931,846,1 + 162,245.585,470,2 $\times$ z + $\&c$, and $1.015,625R^{396}$ ($= 1.015,625 \times$ the series $408.679,056,6 + 159,398.115,228,0 \times z + \&c = 1.015,625 \times 408.679,056,6 + 1.015,625 \times 159,398.115,228,0 \times z + \&c$) = 415.064,666,8 + 161,888.710,778,4 $\times$ z + $\&c$. Therefore the binomial quantity $1.015,625R^{396} - R^{397}$ will be = the compound quantity

$$\left\{ \begin{array}{l} 415.064,666,8 + 161,888.710,778,4 \times z + \&c \\ - 414.931,846,1 - 162,245.585,470,2 \times z - \&c \end{array} \right\}$$

$$= 0.132,820,7 - 356.874,691,8 \times z \&c.$$

But

But the binomial quantity $1.015,625R^{396} - R^{397}$ is $= 0.015,625$.

Therefore the compound quantity $0.132,820,7 - 356.874,691,8 \times z$ will also be $= 0.015,625$.

Therefore (adding $356.874,691,8 \times z$ to both sides,) we shall have $0.132,820,7 = 0.015,625 + 356.874,691,8 \times z$, and (subtracting $0.015,625$ from both sides,) we shall have $356.874,691,8 \times z = 0.117,195,7$, and consequently $z = \frac{0.117,195,7}{356.874,691,8} = 0.000,32$. Therefore R , or $1.0153 + z$, will be $(= 1.0153 + 0.000,32) = 1.015,62$, or the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$ will be nearly equal to $1.015,62$.

Q. E. I.

We will now substitute this last number, $1.015,62$ (which differs very little from $1.015,625$, or the higher limit of the magnitude of R), instead of R in the binomial quantity $1.015,625R^{396} - R^{397}$, in order to discover how near the value of the said binomial quantity resulting from the said substitution will approach to $0.015,625$, or the absolute term of the said equation, and likewise whether it will be greater, or will be less, than the said absolute term, and consequently whether the said number $1.015,62$ will be less, or will be greater, than the true value of R , or the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$.

Now, if R is $= 1.015,62$, we shall have $R^{396} = \overline{1.015,62}^{396}$, and $R^{397} = \overline{1.015,62}^{397}$, and $1.015,625R^{396} = 1.015,625 \times \overline{1.015,62}^{396}$; and consequently the binomial quantity $1.015,625R^{396} - R^{397}$ will be $= 1.015,625 \times \overline{1.015,62}^{396} - \overline{1.015,62}^{397}$. We must therefore endeavour to find the values of $\overline{1.015,62}^{396}$ and $\overline{1.015,62}^{397}$; which may be done as follows.

The logarithm of $1.015,62$ is $= 0.006,731,2$. Therefore the logarithm of $\overline{1.015,62}^{396}$ will be $(= 396 \times 0.006,731,2) = 2.665,555,2$, which is the logarithm of $462.972,553,2$. Therefore $\overline{1.015,62}^{396}$ will be $= 462.972,553,2$; and consequently $\overline{1.015,62}^{397}$ will be $(= \overline{1.015,62}^{396} \times 1.015,62 = 462.972,553,2 \times 1.015,62) = 470.204,184,4$; and $1.015,625 \times \overline{1.015,62}^{396}$ will be $(= 1.015,625 \times 462.972,553,2) = 470.206,499,3$. Therefore $1.015,625 \times \overline{1.015,62}^{396} - \overline{1.015,62}^{397}$ will be $= 470.206,499,3 - 470.204,184,4 = 0.002,314,9$.

This number $0.002,314,9$ is less than $0.015,625$, or the absolute term of the equation $1.015,625R^{396} - R^{397} = 0.015,625$; and therefore the number $1.015,62$ (from the substitution of which, instead of R , in the binomial quantity

quantity $1.015,625R^{396} - R^{397}$ it arose,) must be greater than the true value of R , or the greater root of that equation. We therefore now are certain that the true value of R is greater than 1.0153 , but less than $1.015,62$. And by one process of the Differential method of approximation we shall find a still nearer value of R to be $= 1.015,588$. This process will be as follows.

Art. 26. The numbers 1.0153 , R , and $1.015,62$ are three quantities that differ very little from each other; and the numbers $0.132,820,7$, $0.015,625$, and $0.002,314,9$ are the three values of the binomial quantity $1.015,625R^{396} - R^{397}$ corresponding to the three former numbers respectively, or resulting from the substitution of them instead of R in the said binomial quantity. Therefore the difference of the first and third of the former three numbers will be to the difference of the second and third of the said former three numbers in nearly the same proportion as the difference of the first and third of the latter three numbers is to the difference of the second and third of the said latter three numbers; that is, $1.015,62 - 1.0153$ will be to $1.015,62 - R$ in nearly the same proportion as $0.132,820,7 - 0.002,314,9$ is to $0.015,625 - 0.002,314,9$, or $0.000,32$ will be to $1.015,62 - R$ in nearly the same proportion as $0.130,505,8$ is to $0.013,310,1$; and consequently $1.015,62 - R$ will be nearly $(= \frac{0.013,310,1 \times 0.000,32}{0.130,505,8} = \frac{0.000,004,259,232}{0.130,505,8}) = 0.000,032$.

Therefore (adding R to both sides,) we shall have $0.000,032 + R = 1.015,62$, and $R (= 1.015,620 - 0.000,032) = 1.015,588$. We may therefore conclude that $1.015,588$ is a very near value of R , or the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$, or is the value of 1 pound together with it's interest for a quarter of a year according to the rate of interest allowed to the purchaser of the said quarterly pension of $25l.$, which is to continue for 396 quarters of a year, or 99 whole years, and which is supposed in Mr. Ward's Question to have been sold to him for 1600 pounds. Q. E. I.

And, since $1.015,588$ is the value of 1 pound together with it's Interest for one quarter of a year, it follows that the value of one pound together with it's interest for two quarters of a year, or for half a year, must be $= \overline{1.015,588}^2 = 1.031,418,985$, &c, or, nearly, $1.031,419$; and, in like manner, it follows that the value, or amount, of one pound together with it's interest for two half-years, or for a whole year, must be $= \overline{1.031,419}^2 = 1.063,825$, &c. And consequently the rate of interest per cent *per annum* allowed to the purchaser of the said annuity of 100*l.* a year for 99 years, payable at the end of every quarter of a year during the said term, when the said annuity was sold to him for 1600 pounds was $6.3825l.$, or $6l. 7s. 7\frac{3}{4}d.$, per cent *per annum*; which number $6.3825l.$ agrees in the three highest figures, 6.38 , with the number found, for the rate of interest allowed in the said bargain, by Mr. Ward, which was $6.3855l.$, or $6l. 7s. 8\frac{1}{2}d.$, per cent *per annum*.

Art. 27.

Art. 27. This last number, 1.015,588, is sufficiently near to the true value of R , or the greater root of the equation $1.015,625R^{396} - R^{397} = 0.015,625$ to make it unnecessary to pursue the investigation of it any further. But, if I had suspected this root to be so very near to its greater limit, 1.015,625, as it now appears to be, it might have been obtained with much less trouble than in the foregoing investigation of it, and to a still greater degree of exactness, by deriving it immediately from that greater limit by a single process of Mr. Raphson's method of approximation. This may be done in the following manner.

Let z be put for the excess of 1.015,625, or the higher limit of R , above its true value. And we shall then have $R = 1.015,625 - z$.

Therefore R^{396} will be $= \overline{1.015,625 - z}^{396} =$ (by the binomial theorem,) to the series $\overline{1.015,625}^{396} - 396 \times \overline{1.015,625}^{395} \times z + \&c$; and R^{397} will be $= \overline{1.015,625 - z}^{397} =$ (by the same theorem) to the series $\overline{1.015,625}^{397} - 397 \times \overline{1.015,625}^{396} \times z + \&c$. We must therefore endeavour to find the values of $\overline{1.015,625}^{395}$, and $\overline{1.015,625}^{396}$, and $\overline{1.015,625}^{397}$; which may be done as follows.

The logarithm of 1.015,625 is $= 0.006,733,4$. Therefore the logarithm of $\overline{1.015,625}^{395}$ will be $(= 396 \times 0.006,733,4) = 2.666,426,4$; which is the logarithm of 463.912,150,5. Therefore $\overline{1.015,625}^{396}$ will be $= 463.912,150,5$; and consequently $\overline{1.015,625}^{395}$ will be $(= \frac{\overline{1.015,625}^{396}}{1.015,625} = \frac{463.912,150,5}{1.015,625}) = 456.775,040,4$, and $\overline{1.015,625}^{397}$ will be $(= \overline{1.015,625}^{396} \times 1.015,625 = 463.912,150,5 \times 1.015,625) = 471.160,777,8$.

Therefore R^{396} (which is equal to the series $\overline{1.015,625}^{396} - 396 \times \overline{1.015,625}^{395} \times z + \&c$.) will be $(= 463.912,150,5 - 396 \times 456.775,040,4 \times z + \&c) = 463.912,150,5 - 180,882.915,998,4 \times z + \&c$; and R^{397} (which is $=$ to the series $\overline{1.015,625}^{397} - 397 \times \overline{1.015,625}^{396} \times z + \&c$.) will be $(= 471.160,777,8 - 397 \times 463.912,150,5 \times z + \&c) = 471.160,777,8 - 184,173.123,748,5 \times z + \&c$; and $1.015,625 \times R^{396}$ will be $(= 1.015,625 \times \text{the series } 463.912,150,5 - 180,882.915,998,4 \times z + \&c = 1.015,625 \times 463.912,150,5 - 1.015,625 \times 180,882.915,998,4 \times z + \&c) = 471.160,777,8 - 183,709.211,560,8 \times z + \&c$.

Therefore the binomial quantity $1.015,625R^{396} - R^{397}$ will be $=$ the compound quantity

471.160,

$$\left\{ \begin{array}{l} 471.160,777,8 - 183,709.211,560,8 \times z + \&c \\ - 471.160,777,8 + 184,173.123,748,5 \times z - \&c \\ 0 + 463.912,187,7 \times z \&c. \end{array} \right\} =$$

But the binomial quantity $1.015,625R^{396} - R^{397}$ is $= 0.015,625$.

Therefore the compound quantity $463.912,187,7 \times z \&c$ will also be $= 0.015,625$. And consequently z will be $= \frac{0.015,625,000,000}{463.912,187,7} = 0.000,033,6$.

Therefore R , or $1.015,625 - z$, will be $(= 1.015,625,0 - 0.000,033,6 = 1.015,591,4$; that is, $1.015,591,4$ will be a very near value of R , or the greater root of the proposed equation $1.015,625R^{396} - R^{397} = 0.015,625$, or of the amount of 1 pound together with it's interest for a quarter of a year, according to the rate of interest allowed to the purchaser of the said quarterly pension of 25 pounds, which is to continue for 396 quarters of a year, or for 99 whole years, when the said pension was sold to him for the sum of 1600 pounds. Q. E. I.

And, since the amount of 1 pound together with it's interest for one quarter of a year is $= 1.015,591,4$ l., it follows, from the nature of Compound Interest, that the amount of 1 pound and it's interest for two quarters of a year, or for half a year, must be $= 1.015,591,4^2$, or $1.031,425,891$ l., &c, and, in like manner, that the amount of 1 pound and it's interest for two half-years, or a whole year, must be $= 1.031,425,8^2$, or $1.063,839,180$ l., &c, or $1.063,839$ l. Therefore the rate of interest per cent *per annum* allowed to the purchaser in the aforesaid bargain will be $6.383,9$ l., or 6 l. 7 s. 8 d. . 136 , or 6 l. 7 s. $8\frac{1}{2}$ d., nearly; which differs from the rate of interest assigned by Mr. Ward, to wit, 6.3855 l., or 6 l. 7 s. $8\frac{1}{2}$ d., by less than one half-penny.

This Question relates to the annuities of 100l. *per annum* for 99 years, payable quarterly at the Exchequer, which were established by act of Parliament in Queen Anne's reign in the year 1708, and were sold for 1600 pounds.

End of the Resolution of the Equation $1.015,625R^{396} - R^{397} = 0.015,625$, relating to the Rate of Interest of Money allowed to the Purchasers of Exchequer-Annuities for a Term of 99 Years established by Act of Parliament in the Year 1708.

Art. 28. Mr. Ward has given us another affected equation resulting from Questions concerning Compound Interest. This equation occurs above in page 727, and is a trinomial equation, or involves three different powers of the unknown quantity R , (or of the amount of 1 pound together with it's interest for one year, or one half-year, or one quarter of a year, or other portion of time

time at the end of which the annuity, or pension, purchased becomes payable,) and is as follows; to wit, $UR^t - U = PR^T \times R^t \times \overline{R-1}$, (or $UR^t - U = PR^{t+T} \times \overline{R-1}$, or $UR^t - U = PR^{t+T+1} - PR^{t+T}$, or $UR^t = PR^{t+T+1} - PR^{t+T} + U$, or $UR^t + PR^{t+T} - PR^{t+T+1} = U$), or $\frac{U}{P} \times R^t + R^{t+T} - R^{t+T+1} = \frac{U}{P}$; in which equation U denotes a rent, or periodical payment, to be received at the end of every year, or every half-year, or every quarter of a year, or other given portion of time agreed-upon between the feller and purchaser of such annuity, during a certain time that is to commence at some distant point of time; and T denotes the time that is to elapse before the said annuity, or pension, shall commence; and t denotes the time during which the said annuity, or pension, when it shall have commenced, is to continue; and P denotes the present worth, or value, of the said distant annuity, or pension, or the price which is to be paid for it at the present time by the person who buys it. But Mr. Ward does not give us any numeral example of this equation. He has, however, given us an example of one of the easier questions relating to the same subject, namely, of the finding the number of years in the time, or term, t , during which the annuity, or rent, ought to be enjoyed, after the expiration of a former term of years denoted by T , to the end that it may be worth a given sum of money denoted by P , to be paid down at present, upon a supposition that the annuity is to be paid only once a year, or at the end of every year during its continuance, and that the interest of money is given, or agreed-upon, or that R , or the amount of 1 pound together with its interest for a year is a known quantity. This Question is proposed above in the bottom of page 726 in the following words.

“ Example in Yearly Rents.

“ Suppose a debt of 816*l.* 18*s.* 9*d.* were proposed to be paid-off by making over a lease of 175*l.* *per annum* in reversion, that is not to be entered-upon by the creditor untill nine years are past. The Question is, How many years the creditor must enjoy that lease, in order to have his debt cleared, and to be allowed the rate of 6 per cent *per annum*?”

Mr. Ward's answer to this Question is, “ that 11 years will be the time required; supposing the rents to be paid but once a year.”

In this Question, U , (the annuity, or rent, to be enjoyed by the creditor,) is = 175*l.* *per annum*, and T , (the number of years at the end of which the said annuity is to be enjoyed by the creditor,) is 9 years; and P , (or the

present debt due to the creditor, which is to be annihilated, or paid-off, by this bargain, and therefore must be considered as the present worth, or price, of the said distant annuity,) is = 816*l.* 18*s.* 9*d.*, or 816.937*l.*; and R , (or the amount of 1 pound together with it's interest for a year, according to the rate of interest *per annum* allowed in this bargain, to wit, the rate of 6 per cent,) is = 1.06; and t denotes the number of years during which the creditor is to enjoy the said annuity of 175*l.* *per annum*, in consideration of the annihilation of the present debt of 816*l.* 18*s.* 9*d.*, or 816.937*l.*, that is due to him, and is the unknown quantity required by the Question, and which is found by Mr. Ward's answer to the Question to be 11 years. See above, page 727. But, if we make a small variation in the conditions of this Question, by supposing t , or the number of years during which the said distant annuity of 175*l.* *per annum* is to be enjoyed by the creditor, to be known, and to be 11 years, and the rate of interest, allowed in the bargain, to be unknown, and to be the quantity required to be found, and consequently the quantity denoted by the letter R , or the amount of 1 pound together with it's interest for a year, to be unknown, the new Question produced by this change in the conditions of the former Question will produce a numeral equation of the form of the foregoing general trinomial equation

$$UR^t + PR^{t+T} - PR^{t+T+1} = U, \text{ or } \frac{U}{P} \times R^t + R^{t+T} - R^{t+T+1} = \frac{U}{P}.$$

This new Question may be expressed as follows.

A Question concerning Compound Interest, which produces a trinomial Equation, or an Equation involving three different Powers of the unknown Quantity R .

Suppose a debt of 816*l.* 18*s.* 9*d.* were proposed to be paid-off by making-over to the creditor a lease of a farm producing a clear rent of 175*l.* a year, that is not to be entered-upon by the creditor till the end of 9 years, and that, when entered-upon at the end of the said 9 years, is to be enjoyed by him during the space of eleven years; and that the rent of the said farm were to be paid to the said creditor only once a year, to wit, at the end of each of the said eleven years. It is required to determine what rate of Interest would be allowed to the said creditor by the said bargain for the interest of the said sum of 816*l.* 18*s.* 9*d.*, or 816.937*l.*, which he would have remitted to his debtor in consideration of the lease which should have been so assigned to him.

SOLUTION.

In this Question we shall have U , (or the annuity, or annual rent, to be paid to the creditor during the said distant term of 11 years,) = 175*l.*, and t (or the number of years during which the said rent is to be enjoyed by the said creditor,) = 11, and T (or the number of years that must elapse before the said rent can be received by the said creditor,) = 9, and P (or the debt now due to the said creditor, and which is to be remitted by him to his debtor in consideration of the distant annuity of 175*l.* so made-over to him,) = 816.937*l.*; and R will denote the amount of 1 pound together with it's interest for one year according to the rate of interest allowed to the creditor by the present bargain, which rate is supposed to be unknown, and is required to be found by the present Question. Therefore $\frac{U}{P}$ will be = $\frac{175}{816.937} = 0.214,214$, and $t + T$ will be (= 11 + 9) = 20, and $t + T + 1$ will be (= 20 + 1) = 21; and consequently the general equation $\frac{U}{P} \times R^t + R^{t+T} - R^{t+T+1} = \frac{U}{P}$ will be converted into the numeral equation $0.214,214 \times R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore, in order to solve the present Problem, we must resolve the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. This resolution may be performed in the following manner.

The Resolution of the trinomial Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, in the Manner described above in the "Appendix to Dr. Halley's Discourse on Compound Interest," in Pages 436, 437, 438, &c, - - - 455.

Art. 29. This trinomial equation is of that form which admits of two roots, or, in the language of modern algebraists, two real and affirmative roots; and one of these roots is evidently equal to 1. For, if R is = 1, we shall have $R^{11} = 1$, and R^{20} also = 1, and R^{21} also = 1; and consequently $0.214,214R^{11} + R^{20} - R^{21} (= 0.214,214 \times 1 + 1 - 1) = 0.214,214$; that is, the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be = $0.214,214$, which is the absolute term of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and therefore this value of R , to wit, 1, is a root of the said equation.

Q. E. D.

But this is not the root that will answer the foregoing Question, from which this equation was derived; because in that Question R denotes the amount of 1 pound together with it's interest for a year, and therefore must be greater than 1 $\frac{1}{2}$, or 1. We must therefore endeavour to find the other and greater root of the equation $0.214,214R^{11} + R^{10} - R^{21} = 0.214,214$.

Art. 30. Now this numeral, or particular, equation comes under the general trinomial equation $Px^m + x^{m+n} - x^{m+n+1} = Q$, which has been examined above in pages 440, 441, &c - - 446; where it has been shewn that the highest possible value of x in that equation, or the higher limit of it's greater root, is the value of x in the binomial equation $x^{n+1} - x^n = P$, and that the lower limit of the said greater root, or that value of x which makes the trinomial quantity $Px^m + x^{m+n} - x^{m+n+1}$ be of the greatest magnitude possible, is the value of x in the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$. The former of these quantities, or the root of the binomial equation $x^{n+1} - x^n = P$, is there denoted by the letter A; and the latter of them, or the root of the binomial equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$, is there denoted by the letter M.

By applying this general equation $Px^m + x^{m+n} - x^{m+n+1} = Q$ to the numeral equation $0.214,214R^{11} + R^{10} - R^{21} = 0.214,214$, we shall have $P = 0.214,214$, and $x = R$, and $m = 11$, and $m + n = 20$, and consequently $n (= 20 - m = 20 - 11) = 9$, and $m + n + 1 (= 11 + 9 + 1) = 21$, and $Q = 0.214,214$. Therefore $\frac{m+n}{m+n+1}$ will be $(= \frac{20}{21}) = 0.952,380$, and $\frac{m}{m+n+1}$ will be $(= \frac{11}{21}) = 0.523,809$, and the equation $x^{n+1} - x^n = P$ will become $R^{10} - R^9 = 0.214,214$, and the equation $x^{n+1} - \sqrt{\frac{m+n}{m+n+1}} \times x^n = \frac{m}{m+n+1} \times P$ will become $R^{10} - 0.952,380 \times R^9 (= 0.523,809 \times 0.214,214) = 0.112,207$. Therefore A, or the higher limit of the magnitude of R , or the greater root of the trinomial equation $0.214,214 \times R^{11} + R^{10} - R^{21} = 0.214,214$, will be the root of the binomial equation $R^{10} - R^9 = 0.214,214$; and M, or the lower limit of the magnitude of R , or the greater root of the said trinomial equation, will be the root of the binomial equation $R^{10} - 0.952,380R^9 = 0.112,207$. We must therefore, in order to find these limits of R , resolve the two binomial equations $R^{10} - R^9 = 0.214,214$ and $R^{10} - 0.952,380R^9 = 0.112,207$.

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The Resolution of the binomial Equation $R^{10} - R^9 = 0.214,214$.

Art. 31. In the first of these equations, to wit, $R^{10} - R^9 = 0.214,214$, it is evident, in the first place, that R must be greater than 1. For, if R were less than 1, R^9 would be greater than R^{10} , and consequently could not be subtracted from R^{10} , as it is in this equation. In the next place, R must be less than 2. For, if R was $= 2$, R^9 would be $(= 2^9 = 8^3 = 8^2 \times 8 = 64 \times 8) = 512$, and R^{10} would be $(2^{10} = 2^9 \times 2 = 512 \times 2) = 1024$, and consequently $R^{10} - R^9$ would be $(= 1024 - 512) = 512$, instead of being equal to the fraction $0.214,214$. Therefore R must be greater than 1, but very much less than 2. We will therefore suppose it to be nearly $= 1.1$, and will try the truth of that supposition by substituting 1.1 instead of R in the binomial quantity $R^{10} - R^9$.

Now, if R is $= 1.1$, we shall have $R^3 (= 1.1^3) = 1.331$, and $R^9 (= 1.331^3) = 2.3579$ &c and $R^{10} (= R^9 \times R = 2.3579 \times 1.1) = 2.593,69$ &c, and consequently $R^{10} - R^9 (= 2.593,69 - 2.3579 \text{ &c}) = 0.235,79$; which is somewhat greater than $0.214,214$, or the absolute term of the equation $R^{10} - R^9 = 0.214,214$. Therefore 1.1 must be somewhat greater than the true value of R in that equation. We will therefore in the next place suppose R to be $= 1.09$, and try the truth of that supposition in the same manner.

Now, if R is $= 1.09$, we shall have (by logarithms,) $R^9 = 2.171,893,5$, and consequently $R^{10} (= R^9 \times R = 2.171,893,5 \times 1.09) = 2.367,363,9$. Therefore $R^{10} - R^9$ will be $(= 2.367,363,9 - 2.171,893,5) = 0.195,470,4$; which is somewhat less than $0.214,214$, or the absolute term of the equation $R^{10} - R^9 = 0.214,214$. Therefore $0.195,470,4$ will be somewhat less than the true value of R in that equation.

But it has been shewn that 1.1 is greater than the true value of R in that equation. Therefore the said true value of R must be of an intermediate magnitude between 1.1 and 1.09 .

And, if we apply one process of the Differential method of approximation to this equation, we shall obtain a new near value of R that will be as near to its true value as need to be desired. This may be done as follows.

Since the three quantities 1.1 , or 1.10 , R , and 1.09 differ but little from each other, and the three numbers $0.235,790$, $0.214,214$, and $0.195,470,4$ are the three values of the binomial quantity $R^{10} - R^9$ that correspond to the three former quantities 1.10 , R , and 1.09 , or result from the substitution of those

those quantities in it's terms instead of R , the differences of the former three quantities will be to each other in, nearly, the same proportion as the corresponding differences of the latter three quantities; that is, $1.10 - 1.09$ will be to $R - 1.09$ in, nearly, the same proportion as $0.235,790 - 0.195,470$ is to $0.214,214 - 0.195,470$, or 0.01 will be to $R - 1.09$ in, nearly, the same proportion as $0.040,320$ is to $0.018,744$; and consequently $R - 1.09$ will be, nearly, $(= \frac{0.01 \times 0.018,744}{0.040,320} = \frac{0.000,187,44}{0.040,320}) = 0.0048$ &c, and R will be, nearly, $(= 0.0048$ &c $+ 1.09) = 1.0948$. We may therefore consider 1.0948 as a pretty near value of the quantity R , or the root of the binomial equation $R^{10} - R^9 = 0.214,214$, or of A , or the higher limit of the magnitude of R , or the greater root of the trinomial equation $0.214,214R^{11} + R^{10} - R^{21} = 0.214,214$. Q. E. I.

Art. 32. We will therefore now proceed to investigate the value of M , or the lower limit of the magnitude of R in the said trinomial equation $0.214,214R^{11} + R^{10} - R^{21} = 0.214,214$, by resolving the second binomial equation, $R^{10} - 0.952,380R^9 = 0.112,207$, to the root of which the said lower limit of R is equal.

The Resolution of the binomial Equation $R^{10} - 0.952,380R^9 = 0.112,207$.

Since the root of the former binomial equation $R^{10} - R^9 = 0.214,214$ is $= 1.0948$, it seems reasonable to suppose that the root of the latter binomial equation $R^{10} - 0.952,380R^9 = 0.112,207$, (of which the absolute term, $0.112,207$, is not much above half the absolute term, $0.214,214$, of the former binomial equation,) will be much less than 1.0948 ; and therefore I shall suppose it to be nearly equal to $1 + \frac{0.0948}{2}$, or to 1.0474 , and try the justness of the supposition by substituting 1.0474 instead of R in the binomial quantity $R^{10} - 0.952,380R^9$, in order to see how nearly the result will approach to the absolute term $0.112,207$ of the said second binomial equation, and whether it will be greater, or whether it will be less, than the said absolute term, and consequently to discover whether 1.0474 will be greater than the true value of R in the said second equation, or will be less than the said true value.

Now, if we suppose R to be $= 1.0474$, we shall (by logarithms,) have $R^9 = 1.485,981,5$, and consequently $R^{10} (= R^9 \times R = 1.485,981,5 \times 1.0474) = 1.556,417,0$, and $0.952,380R^9 = 0.952,380 \times 1.485,981,5 = 1.415,219,6$. Therefore the binomial quantity $R^{10} - 0.952,380R^9$ will be $(= 1.556,417,0 - 1.415,219,6) = 0.141,197,4$; which is greater than

0.112,

0.112,207, or the absolute term of the equation $R^{10} - 0.952,380R^9 = 0.112,207$. Therefore 1.0474 must be greater than the true value of R in the said equation.

We will therefore, in the next place, suppose R to be $= 1.04$, and try the justness of this conjecture by substituting 1.04 instead of R in the binomial quantity $R^{10} - 0.952,380R^9$, and comparing the result with the absolute term 0.112,207.

Now, if R is $= 1.04$, we shall have (by logarithms,) $R^9 = 1.423,310,7$, and consequently $R^{10} (= R^9 \times R = 1.423,310,7 \times 1.04) = 1.480,243,1$, and $0.952,380R^9 (= 0.952,380 \times 1.423,310,7) = 1.355,532,6$. Therefore $R^{10} - 0.952,380R^9$ will be $(= 1.480,243,1 - 1.355,532,6) = 0.124,710,5$; which is greater than 0.112,207, or the absolute term of the equation $R^{10} - 0.952,380R^9 = 0.112,207$. Therefore 1.04 will also be greater than the true value of R in the said equation.

Having now obtained the two numbers 0.141,197 and 0.124,710, which result from the substitution of the two numbers 1.0474 and 1.04 instead of R in the binomial quantity $R^{10} - 0.952,380R^9$, we may find a nearer value of R by the Differential method of approximation by proceeding as follows. The difference of 1.0474 and 1.04 will be to the difference of 1.04 and R , in, nearly, the same proportion as the difference of the numbers 0.141,197 and 0.124,710 (which correspond to the numbers 1.0474 and 1.04,) to the difference of the numbers 0.124,710 and 0.112,207, which correspond to the numbers 1.04 and R ; that is, $1.0474 - 1.04$ will be to $1.04 - R$ in, nearly, the same proportion as $0.141,197 - 0.124,710$ is to $0.124,710 - 0.112,207$, or 0.0074 will be to $1.04 - R$ in nearly the same proportion as 0.016,487 is to 0.012,503. Therefore $1.04 - R$ will be, nearly, $(= \frac{0.0074 \times 0.012,503}{0.016,487} = \frac{0.000,092,522,2}{0.016,487}) = 0.0056$; and consequently 1.04 will be, nearly, $= 0.0056 + R$, and R will be, nearly, $(= 1.0400 - 0.0056) = 1.0344$. We may therefore conclude that the root of the second binomial equation $R^{10} - 0.952,380R^9 = 0.112,207$ will be pretty nearly $= 1.0344$, and consequently that M , or the lower limit of R , or of the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, will also be nearly $= 1.0344$. Q. E. I.

The Application of the Numbers 1.0948 and 1.0344, (which have been found to be the Limits of the Magnitude of R, or the greater Root of the trinomial Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$,) to the Investigation of the Value of the said greater Root, or to the Resolution of the said trinomial Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$.

Art. 33. Having now discovered that the limits of the magnitude of R, or the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, are nearly equal to the numbers 1.0948 and 1.0344, we may proceed to make further approaches to the value of the said greater root in the following manner.

Since R, or the greater root of this equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, is less than 1.0948, but greater than 1.0344, it seems reasonable to conjecture that it will be nearly equal to an arithmetical mean between them, (or to $\frac{1.0948 + 1.0344}{2}$, or to $\frac{2.1292}{2}$), or to 1.0646. We will try the justness of this supposition by substituting 1.0646 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, and comparing the result of this substitution with 0.214,214, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$.

Now, if R is = 1.0646, we shall have, (by logarithms,) $R^{11} = 1.941,116,0$, and consequently $R^{10} (= \frac{R^{11}}{R} = \frac{1.941,116,0}{1.0646}) = 1.823,328,9$, and consequently $R^{20} (= R^{10} \times R^{10} = 1.823,328,9 \times 1.823,328,9) = 3.324,528,2$, and $R^{21} (= R^{20} \times R = 3.324,528,2 \times 1.0646) = 3.539,292,7$, and $0.214,214R^{11} (= 0.214,214 \times 1.941,116,0) = 0.415,814,2$. Therefore the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be $(= 0.415,814,2 + 3.324,528,2 - 3.539,292,7 = 3.740,342,4 - 3.539,292,7) = 0.201,049,7$; which is tolerably near to, but somewhat less than, 0.214,214, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore 1.0646 will be somewhat greater than the true value of R, or the greater root of the said equation.

In order to obtain a more exact value of R it will now be expedient to have recourse to Mr. Raphson's method of approximation, by proceeding as follows.

Let z be put for the excess of the number 1.0646 above R, so that R shall be = $1.0646 - R$.

Then will R^{11} be $(= \overline{1.0646 - R})^{11} = \overline{1.0646}^{11} - 11 \times \overline{1.0646}^{10} \times z + \&c = 1.941,116,0 - 11 \times 1.823,328,9 \times z + \&c) = 1.941,116,0 -$
3 20.056,

$20.056,617,9 \times z + \&c$, and $0.214,214 \times R^{11}$ will be $(= 0.214,214 \times 1.941,116,0 - 20.056,617,9 \times z + \&c = 0.214,214 \times 1.941,116,0 - 0.214,214 \times 20.056,617,9 \times z + \&c) = 0.415,814,2 - 4.296,408,3 \times z + \&c$; and R^{20} will be $(= \overline{1.0646 - z})^{20} = \overline{1.0646}^{20} - 20 \times \overline{1.0646}^{19} \times z + \&c = \overline{1.0646}^{20} - 20 \times \frac{\overline{1.0646}^{20}}{1.0646} \times z + \&c = 3.324,528,2 - 20 \times \frac{3.324,528,2}{1.0646} \times z + \&c = 3.324,528,2 - 20 \times 3.122,795,6 \times z + \&c) = 3.324,528,2 - 62.455,912,0 \times z + \&c$; and R^{21} will be $(= \overline{1.0646 - z})^{21} = \overline{1.0646}^{21} - 21 \times \overline{1.0646}^{20} \times z + \&c = 3.539,292,7 - 21 \times 3.324,528,2 \times z + \&c) = 3.539,292,7 - 69.815,092,2 \times z + \&c$.

Therefore the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be equal to the compound quantity

$$\left\{ \begin{array}{l} 0.415,814,2 - 4.296,408,3 \times z + \&c \\ + 3.324,528,2 - 62.455,912,0 \times z + \&c \\ - 3.539,292,7 + 69.815,092,2 \times z - \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{l} 3.740,342,4 - 66.752,320,3 \times z + \&c \\ - 3.539,292,7 + 69.815,092,2 \times z - \&c \end{array} \right\} =$$

$$0.201,049,7 + 3.062,771,9 \times z \&c.$$

But the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ is $= 0.214,214$.

Therefore the compound quantity $0.201,049,7 + 3.062,771,9 \times z \&c$ will also be $= 0.214,214$; and consequently $3.062,771,9 \times z$ will be $(= 0.214,214 - 0.201,049,7) = 0.013,164,3$, and z will be $(= \frac{0.013,164,3}{3.062,771,9}) = 0.004,298$. Therefore R , or $1.0646 - z$, will be $(= 1.064,600 - 0.004,298) = 1.060,302$; and consequently the interest allowed to the creditor in the Question above-proposed, for the debt of $816.937l.$, or $816l. 18s. 9d.$, remitted by him in consideration of the remote annuity of $175l. \text{ per annum}$ that has been granted to him, is, nearly, $6.0302l.$, or $6l. 0s. 7\frac{1}{4}d.$ per cent *per annum*.
Q. E. I.

The value of R obtained by this process of Mr. Raphson's method of approximation, to wit, $1.060,302$, is so very little greater than $1.060,000$, or 1.06 , that it would naturally induce us to conjecture that it's true value was 1.06 , and to try whether it is so, or not, by substituting 1.06 instead of R in the binomial quantity $0.214,214R^{11} + R^{20} - R^{21}$; by which we should find that the value of the said binomial quantity resulting from such substitution

would be accurately equal to 0.214,214, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and consequently that 1.06 is the true value of R , or the greater root of that equation. It seems therefore unnecessary to carry this investigation of the value of R in this equation any further.

Another Method of deriving from the Number 1.646, (which has been obtained above in Art. 33 for a first near Value of R ,) a second and much more accurate Value of R by the Differential Method of Approximation.

Art. 34. But, as it may be of use to shew that, in resolving equations of these high orders, we may sometimes take several different methods of obtaining a pretty good near value of the root sought, which may afterwards, if greater exactness should be required, be made the basis, or ground-work, of a further approach to it's true value by a process, or two, of Mr. Raphson's method of approximation, I will now proceed to shew how (after having obtained the number 1.0646 in art. 33, for a near value of R , or the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and having found, upon trial, that the value of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ resulting from the substitution of the said number 1.0646 in it's terms instead of R , is equal to 0.201,049,7, which is but a little less than 0.214,214, or the absolute term of the said trinomial equation, and having thence concluded that the said number 1.0646 must be somewhat greater than the true value of R , or the greater root of the said equation,) we may, by means of one process of the Differential method of approximation, obtain another near value of R that shall approach much nearer to it's true value than the said former number 1.0646. This may be done in the following manner.

Since 1.0646 is greater than the true value of R , or the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, let us subtract from 1.0646 the quantity $\frac{1.0646}{200}$, or 0.0053; and the remainder 1.0593 will probably be nearer than 1.0646 to the true value of R . This number 1.0593 must now be substituted instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to obtain the value of the said trinomial quantity corresponding to it, or resulting from that substitution, and to discover whether this said result will be greater, or will be less, than 0.214,214, or the absolute term of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and consequently whether 1.0593 will be less, or will be greater, than the true value of R in that equation.

Now, if R is $= 1.0593$, we shall have $R'' = \overline{1.0593}''$, and $R^{20} = \overline{1.0593}^{20}$, and $R^{21} = \overline{1.0593}^{21}$, and $0.214,214R'' + R^{20} - R^{21} = 0.214,214 \times \overline{1.0593}'' + \overline{1.0593}^{20} - \overline{1.0593}^{21}$. We must therefore find the values of $\overline{1.0593}''$, $\overline{1.0593}^{20}$, and $\overline{1.0593}^{21}$; which may be done most conveniently by means of a table of logarithms, as follows.

The logarithm of 1.0593 is $= 0.025,019,0$. Therefore the logarithm of $\overline{1.0593}''$ will be $(= 11 \times 0.025,019,0) = 0.275,209,0$; which is the logarithm of the number $1.884,555,8$. Therefore $\overline{1.0593}''$ is $= 1.884,555,8$. Therefore $\overline{1.0593}^{22}$ will be $(= \overline{1.0593}''^{22} = \overline{1.884,555,8}^{22}) = 3.551,550,563,313,64$, and consequently $\overline{1.0593}^{21}$ will be $(= \frac{\overline{1.0593}^{22}}{\overline{1.0593}} = \frac{3.551,550,563,313,64}{1.0593}) = 3.352,733,468,6$, and $\overline{1.0593}^{20}$ will be $(= \frac{\overline{1.0593}^{21}}{\overline{1.0593}} = \frac{3.352,733,468,6}{1.0593}) = 3.165,046$; and $0.214,214 \times \overline{1.0593}''$ will be $(= 0.214,214 \times 1.884,555,8) = 0.403,698,2$ &c. Therefore the trinomial quantity $0.214,214 \times \overline{1.0593}'' + \overline{1.0593}^{20} - \overline{1.0593}^{21}$ will be $(= 0.403,698 + 3.165,046 - 3.352,733 = 3.568,744 - 3.352,733) = 0.216,011$; which is greater than $0.214,214$, or the absolute term of the trinomial equation $0.214,214R'' + R^{20} - R^{21} = 0.214,214$. Therefore the number 1.0593 must be less than the true value of R , or the greater root of that equation. Q. E. I.

We have now three quantities that are nearly equal to each other, to wit, 1.0646 , R , and 1.0593 , and we know that the values of the trinomial quantity $0.214,214R'' + R^{20} - R^{21}$ corresponding to these three quantities, or arising from the substitution of the said three quantities in it's terms instead of R , are $0.201,049$, $0.214,214$, and $0.216,011$. It follows therefore, according to the principles of the Differential Method of approximation, that the difference of the first and third of the first set of quantities will be to the difference of the second and third of the same, or first, set of quantities in, nearly, the same proportion as the difference of the first and third of the second set of quantities is to the difference of the second and third of the same, or second, set of quantities, that is, $1.0646 - 1.0593$ will be to $R - 1.0593$ in, nearly, the same proportion as $0.216,011 - 0.201,049$ is to $0.216,011 - 0.214,214$, or 0.0053 will be to $R - 1.0593$ in, nearly, the same proportion as $0.014,962$ is to $0.001,797$. Therefore $R - 1.0593$ will be, nearly, $(= \frac{0.0053 \times 0.001,797}{0.014,962} = \frac{0.000,009,524,1}{0.014,962}) = 0.000,636$, and consequently R will be, nearly, $(= 0.000,636 + 1.0593) = 1.059,936$. Q. E. I.

This number 1.059,936 differs from the true value of R , (which is 1.06,) by only the small quantity 0.000,074, which is less than the 14,324th part of the said true value. This is so great a degree of exactness that it seems to be quite unnecessary to seek for a more exact value of R , or the greater root of the said trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. But, if we chose to prosecute the resolution of this equation further, and were, for that purpose, to employ one process of Mr. Raphson's method of approximation, by taking 1.059,936 for the basis, or ground-work, of such process, or supposing R to be equal to the binomial quantity $1.059,936 + z$, and substituting the said binomial quantity instead of R in the said equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and resolving the transformed equation thereby obtained as if it were a mere simple equation, we should thereby obtain the value of R exact to at least eight places of figures.

Another Method of resolving the foregoing Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, derived from our Knowledge of it's lesser Root 1, and of the Number 1.0344, which is the lesser Limit of the Magnitude of it's greater Root R .

Art. 35. We may observe further that we might at first have taken another method of obtaining a tolerably good first near value of R , or the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, instead of taking an arithmetical mean between the two numbers 1.0948 and 1.0344, which are the limits of the magnitude of R . For we might have conjectured with a good degree of probability, that the number 1.0344 (which is the lower limit of the magnitude of R ,) would be nearly an arithmetical mean between R , or the greater root of the said trinomial equation, and it's lesser root, which we know to be 1; from which it would follow that R would be nearly $(= 1.0344 + 0.0344) = 1.0688$. This would have been a pretty good first near value of R , though not quite so good a one as the number 1.0646, which was obtained above in art. 33 by taking an arithmetical mean between the numbers 1.0948 and 1.0344, or the two limits of the magnitude of R . And, if we had taken this number 1.0688 for the said first near value of R , we must have substituted it instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to discover whether the value of the said trinomial quantity resulting from such substitution would be greater, or would be less, than 0.214,214, or the absolute term of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and consequently whether the said number 1.0688 would be less, or would be greater, than R . And, upon making this substitution, we should have found that the value of the said trinomial quantity thence resulting (which we will call A ,) would have been less than the absolute term 0.214,214, and consequently that the number 1.0688 must be greater than R . And, after this discovery, we might have subtracted

subtracted from 1.0688 the fraction $\frac{1.0688}{200}$, or 0.005,344, by which we should have obtained 1.063,456, or (neglecting the two last figures,) 1.0634, for another near value of R that would probably be nearer than 1.0688 to it's true value. Then we must have substituted this new number 1.0634 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to discover whether the value of the said trinomial quantity resulting from such substitution (which value we will call B .) would be greater, or would be less, than the absolute term 0.214,214 of the said equation; and we should have found that it would have been less than the said absolute term, and consequently we must have concluded that the said number 1.0634 (as well as the former number 1.0688,) would be greater than the true value of R . And, having thus three quantities that are nearly equal to each other, to wit, the quantities 1.0688, 1.0634, and R , and three different values of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ that correspond to the said three former quantities respectively, or result from the substitution of the said three former quantities in the said trinomial quantity, to wit, the three values A , B , and 0.214,214, we might have found a third near value of R that would have been nearer than either 1.0688 or 1.0634 to it's true value, by means of the following proportion according to the principles of the Differential method of approximation; As $1.0688 - 1.0634$, or 0.0054, is to $1.0634 - R$, so (very nearly,) will $B - A$ be to $0.214,214 - B$; whence it follows that $1.0634 - R$ will be (very nearly) $= \frac{0.0054 \times \frac{0.214,214 - B}{B - A}}$, and consequently that 1.0634 will be $= \frac{0.0054 \times \frac{0.214,214 - B}{B - A}}{B - A} + R$, and that R will be $= 1.0634 - \frac{0.0054 \times \frac{0.214,214 - B}{B - A}}{B - A}$. I have not made this computation of the value of R ; but I am persuaded that it would be almost as exact as the number 1.059,936, which was found above by a similar application of the Differential method of approximation, by taking 1.0646, (or the arithmetical mean between 1.0948 and 1.0344, the two limits of the magnitude of R .) for the first near value, or basis of the investigation.

Another Method of resolving the Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, derived from our knowledge of the Limits of the Magnitude of R , or the greater root of that equation.

Art. 36. But there is also another method of obtaining a tolerably good first near value of R , or the greater root of the trinomial equation $0.214,214R^{11} +$

$+ R^{20} - R^{21} = 0.214,214$, by means of our knowledge of the two numbers 1.0948 and 1.0344, which are the limits of the magnitude of the said greater root. This method is as follows.

Let 1.0344, or the lower limit of the magnitude of R , be substituted instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to discover the value of the said trinomial quantity resulting from such substitution; which value will be the greatest possible magnitude of the said trinomial quantity. This substitution may be made as follows.

If R is $= 1.0344$, we shall have $R^{11} = \overline{1.0344}^{11}$, and $R^{20} = \overline{1.0344}^{20}$, and $R^{21} = \overline{1.0344}^{21}$, and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be $= 0.214,214 \times \overline{1.0344}^{11} + \overline{1.0344}^{20} - \overline{1.0344}^{21}$. We must therefore find the values of $\overline{1.0344}^{11}$, and $\overline{1.0344}^{20}$ and $\overline{1.0344}^{21}$; which may be done most conveniently by the help of a table of logarithms in the manner following.

The logarithm of 1.0344 is $= 0.014,688,5$. Therefore the logarithm of $\overline{1.0344}^{11}$ will be $(= 11 \times 0.014,688,5) = 0.161,573,5$; which is the logarithm of the number 1.450,686,3. Therefore $\overline{1.0344}^{11}$ is $= 1.450,686,3$. And consequently $\overline{1.0344}^{22}$ will be $(= \overline{1.0344}^{11})^2 = 1.450,686,3^2) = 2.104,490,741,007,69$, and $\overline{1.0344}^{21}$ will be $(= \frac{\overline{1.0344}^{22}}{\overline{1.0344}} = \frac{2.104,490,741,007,69}{1.0344}) = 2.034,503,899,45$, and $\overline{1.0344}^{20}$ will be $(= \frac{\overline{1.0344}^{21}}{\overline{1.0344}} = \frac{2.034,503,899,45}{1.0344}) = 1.966,844,4$; and $0.214,214 \times \overline{1.0344}^{11}$ will be $(= 0.214,214 \times 1.450,686,3) = 0.310,757,315,068,2$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0344}^{11} + \overline{1.0344}^{20} - \overline{1.0344}^{21}$ will be $(= 0.310,757,3 + 1.966,844,4 - 2.034,503,8 = 2.277,601,7 - 2.034,503,8) = 0.243,097,9$. Therefore 0.243,097,9 is the value of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ when R is $= 1.0344$, and is the greatest possible magnitude of the said trinomial quantity.

Having thus obtained this value, 0.243,097,9, of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ which corresponds to the number 1.0344, or results from the substitution of 1.0344 in the said trinomial quantity, we may now apply it to the finding a first near value of R , or the greater root of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, in the following manner.

Art. 37. Since, when R is $= 1.0344$, the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ is equal to 0.243,097,9; and, when R is $= 1.0948$, the said trinomial

trinomial quantity is $= 0$; it is evident that, while R increases from 1.0344 to 1.0948, or receives an increment equal to $1.0948 - 1.0344$, or to 0.0604, the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ suffers a decrement equal to it's whole magnitude, or 0.243,097,9. Now let us suppose that the increment received by R while the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ decreases from 0.243,097,9 to 0.214,214 is to it's whole increment, 0.0604, received while the said trinomial quantity decreases from 0.243,097,9 to 0, in, nearly the same proportion as the decrement suffered by the said trinomial quantity in the former time, to wit, $0.243,097,9 - 0.214,214$, or 0.028,883,9, is to the decrement suffered by the said trinomial quantity in the latter time, that is, to it's whole magnitude 0.243,097,9; and we shall have the increment of R in the former time, (which we will call I), nearly, equal to $(\frac{0.0604 \times 0.028,883,9}{0.243,097,9} = \frac{0.001,744,587,56}{0.243,097,9}) = 0.0071$. Therefore $1.0344 + I$, or the value of R , when the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ shall have decreased from 0.243,097,9 to 0.214,214, will be nearly $= 1.0344 + 0.0071 = 1.0415$, or the value of R , or the greater root of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$ will be nearly $= 1.0415$. And thus we may obtain without much labour the number 1.0415 for a first near value of R .

This number 1.0415 is not so exact a value of R as either 1.0646, or 1.0688, which were obtained by the two former methods. But yet it is near enough to the truth to be a good ground-work for a further approximation to the true value of R by proceeding as follows.

Art. 38. Let 1.0415 be substituted instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to discover whether the value of the said quantity resulting from such substitution will be greater, or will be less, than 0.214,214, or the absolute term of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, and consequently whether 1.0415 will be less, or will be greater, than the true value of R , or the greater root of that equation.

Now, if R is $= 1.0415$, we shall have $R^{11} = \overline{1.0415}^{11}$, and $R^{20} = \overline{1.0415}^{20}$, and $R^{21} = \overline{1.0415}^{21}$; and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be $= 0.214,214 \times \overline{1.0415}^{11} + \overline{1.0415}^{20} - \overline{1.0415}^{21}$. We must therefore endeavour to find the values of $\overline{1.0415}^{11}$, and $\overline{1.0415}^{20}$, and $\overline{1.0415}^{21}$; which may be most conveniently done by means of a table of logarithms in the following manner.

The logarithm of 1.0415 is $= 0.017,659,3$. Therefore the logarithm of $\overline{1.0415}^{11}$ will be $(= 11 \times 0.017,659,3) = 0.194,252,3$; which is the
logarithm

logarithm of 1.564,056,1. Therefore $\overline{1.0415}^{11}$ is = 1.564,056,1. Therefore $\overline{1.0415}^{22}$ is $(= \overline{1.0415}^{11})^2 = \overline{1.564,056,1}^2) = 2.446,271,483,947,21$, and $\overline{1.0415}^{21}$ is $(= \frac{\overline{1.0415}^{22}}{\overline{1.0415}} = \frac{2.446,271,483,947,21}{1.0415}) = 2.348,796,432,01$, and $\overline{1.0415}^{20}$ is $(= \frac{\overline{1.0415}^{21}}{\overline{1.0415}} = \frac{2.348,796,432,01}{1.0415}) = 2.255,205,4$; and $0.214,214 \times \overline{1.0415}^{11}$ will be $(= 0.214,214 \times 1.564,056,1) = 0.335,042,713,405,4$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0415}^{11} + \overline{1.0415}^{20} - \overline{1.0415}^{21}$ will be $= 0.335,042,7 + 2.255,205,4 - 2.348,796,4 = 2.590,248,1 - 2.348,796,4 = 0.241,451,7$; which is considerably greater than 0.214,214, or the absolute term of the proposed equation. Therefore 1.0415 must be considerably less than the true value of R , or the greater root of the said equation.

Art. 39. Since the number 1.0415 now appears to be considerably less than the true value of R , we will add to it, not the 200th part of it, as in some former instances, but the 100th part of it, or the fraction $\frac{1.0415}{100}$, or 0.0104; and then we shall have 1.0415 + 0.0104, or 1.0519, for another conjectural value of R , which will probably be much nearer than 1.0415 to it's true value; and we then will substitute this new number 1.0519 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$. This may be done as follows.

If R is = 1.0519, we shall have $R^{11} = \overline{1.0519}^{11}$, and $R^{20} = \overline{1.0519}^{20}$, and $R^{21} = \overline{1.0519}^{21}$, and $0.214,214R^{11} = 0.214,214 \times \overline{1.0519}^{11}$, and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214 \times \overline{1.0519}^{11} + \overline{1.0519}^{20} - \overline{1.0519}^{21}$. We must therefore endeavour to find the values of $\overline{1.0519}^{11}$, and $\overline{1.0519}^{20}$, and $\overline{1.0519}^{21}$; which may be done most conveniently by the help of a table of logarithms in the following manner.

The logarithm of 1.0519 is = 0.021,974,5. Therefore the logarithm of $\overline{1.0519}^{11}$ will be $(= 11 \times 0.021,974,5) = 0.241,719,5$; which is the logarithm of 1.744,694,7. Therefore $\overline{1.0519}^{11}$ will be = 1.744,694,7. And consequently $\overline{1.0519}^{22}$ will be $(= \overline{1.0519}^{11})^2 = \overline{1.744,694,7}^2) = 3.043,959,596,208,09$, and $\overline{1.0519}^{21}$ will be $(= \frac{\overline{1.0519}^{22}}{\overline{1.0519}} = \frac{3.043,959,596,208,09}{1.0519}) = 2.893,772,788,485$, and $\overline{1.0519}^{20}$ will be $(= \frac{\overline{1.0519}^{21}}{\overline{1.0519}} = \frac{2.893,772,788,485}{1.0519}) =$

2.750,

2.750,996,09; and $0.214,214 \times \overline{1.0519}^{11}$ will be ($= 0.214,214 \times 1.744,694,7$) $= 0.373,738,030.465,8$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0519}^{11} + \overline{1.0519}^{20} - \overline{1.0519}^{21}$ will be ($= 0.373,738,0 + 2.750,996,0 - 2.893,772,7 = 3.124,734,0 - 2.893,772,7$) $= 0.230,961,3$; which is greater than $0.214,214$, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore 1.0519 must be less than the true value of R , or the greater root of the said equation.

Art. 40. And, as this last value, $0.230,961,3$, of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ (which results from the substitution of 1.0519 in it's terms instead of R), is considerably greater than $0.214,214$, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, we may conclude that 1.0519 will be considerably less than the true value of R , or the greater root of that equation. And therefore we will increase the said number 1.0519 by adding to it a 200th part of itself, to wit, the fraction $\frac{1.0519}{200}$, or 0.0052 , whereby we shall have $1.0519 + 0.0052$, or 1.0571 , for another near value of R , which will probably be much nearer than 1.0519 to it's true value. And this new value of R , to wit, 1.0571 , must now be substituted instead of R in the terms of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, in order to discover whether the value of the said trinomial quantity resulting from such substitution will be greater, or will be less, than $0.214,214$, or the absolute term of the proposed equation, and consequently whether the number 1.0571 will be less, or will be greater, than R , or the greater root of the said equation. This substitution may be made as follows.

If R is $= 1.0571$, we shall have $R^{11} = \overline{1.0571}^{11}$, and $R^{20} = \overline{1.0571}^{20}$, and $R^{21} = \overline{1.0571}^{21}$, and $0.214,214R^{11} = 0.214,214 \times \overline{1.0571}^{11}$; and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be $= 0.214,214 \times \overline{1.0571}^{11} + \overline{1.0571}^{20} - \overline{1.0571}^{21}$. We must therefore find the values of $\overline{1.0571}^{11}$, $\overline{1.0571}^{20}$, and $\overline{1.0571}^{21}$; which may be most conveniently done by the help of a table of logarithms, as follows.

The logarithm of 1.0571 is $= 0.024,116,1$. Therefore the logarithm of $\overline{1.0571}^{11}$ will be ($= 11 \times 0.024,116,1$) $= 0.265,277,1$; which is the logarithm of $1.841,947,0$. Therefore 1.0571 is $= 1.841,947,0$. Therefore $\overline{1.0571}^{22}$ will be ($= \overline{1.0571}^{11} \times \overline{1.0571}^{11} = 1.841,947,0^2$) $= 3.392,768,750,809,00$, and $\overline{1.0571}^{21}$ will be ($= \frac{\overline{1.0571}^{22}}{1.0571} = \frac{3.392,768,750,809,00}{1.0571}$) $= 3.209,505,960,46$, and $\overline{1.0571}^{20}$ will be ($= \frac{\overline{1.0571}^{21}}{1.0571} =$

$= \frac{3.209,505,960,46}{1.0571} = 3.036,142,2$; and $0.214,214 \times \overline{1.0571}^{11}$ will be $(= 0.214,214 \times 1.841,947,0) = 0.394,570,834,658$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0571}^{11} + \overline{1.0571}^{20} - \overline{1.0571}^{21}$ will be $(= 0.394,570,8 + 3.036,142,2 - 3.209,505,9 = 3.430,713,0 - 3.209,505,9) = 0.221,207,1$; which is greater than $0.214,214$, or the absolute term of the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore 1.0571 must be less than the true value of R , or the greater root of that equation.

Art. 41. Having thus obtained the two numbers 1.0519 and 1.0571 for two successive near values of R , which are both less than its true value, and of which consequently the second, or greater, 1.0571 , is the nearest to the truth, and having found the two numbers $0.230,961,3$ and $0.221,207,1$ which are the values of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ that correspond to the said two numbers 1.0519 and 1.0571 respectively, or that arise from the substitution of those two numbers instead of R in the said trinomial quantity, we may now apply a process of the Differential Method of approximation to these several numbers with success, and we shall thereby obtain another near value of R , or the greater root of the proposed equation, that shall be much nearer than even 1.0571 to its true value. This may be done in the manner following.

Let the three quantities 1.0519 , 1.0571 , and R , which differ but little from each other, be set down one under the other, with the three corresponding numbers, or values of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, to wit, the numbers $0.230,961,3$, $0.221,207,1$, and $0.214,214,0$, in the same lines with them respectively, and consequently under each other, as follows;

$$\begin{array}{rcll} 1.0519, & - & - & - & 0.230,961,3, \\ 1.0571, & - & - & - & 0.221,207,1, \\ R, & - & - & - & 0.214,214,0. \end{array}$$

Then will $R - 1.0571$ be to $\left\{ -\frac{1.0571}{1.0519} \right\}$, very nearly, in the same

proportion as $\left\{ -\frac{0.221,207,1}{0.214,214,0} \right\}$ is to $\left\{ -\frac{0.230,961,3}{0.221,207,1} \right\}$, that is,

$R - 1.0571$ will be to 0.0052 , very nearly, in the same proportion as $0.006,993,1$ is to $0.009,754,2$. Therefore $R - 1.0571$ will be, nearly, $(= \frac{0.0052 \times 0.006,993,1}{0.009,754,2} = \frac{0.000,036,364,12}{0.009,754,2}) = 0.0037$; and consequently R will be, nearly, $(= 0.0037 + 1.0571) = 1.0608$; that is, R , or the greater root of the

the proposed equation $0.214,214R^{11} + R^{10} - R^{11} = 0.214,214$, will be nearly equal to 1.0608.

Q. E. I.

This number 1.0608 exceeds the true value of R (which is 1.06) by only 0.0008, or the 1325th part of the said true value, and therefore would form an excellent basis, or ground-work, for a further approach to the said true value by a process of Mr. Raphson's method of approximation, by putting z for the difference of 1.0608 and the said true value, and substituting the binomial quantity $1.0608 - z$ instead of R in the trinomial equation $0.214,214R^{11} + R^{10} - R^{11} = 0.214,214$, and resolving the new, or transformed, equation thence arising as if it were a mere simple equation: for by such an operation we should obtain another near value of R that would be exact to six places of figures, or of which the first six figures would be either 1.059,99, or 1.060,00. But this last operation seems not to be necessary in order to our discovering that 1.06 is the true value of R ; because in the last near value of it, to wit, 1.0608, the figure 6 is followed by a cypher, or 0, by which we are naturally led to conjecture that the true value of R will be 1.06; and this value, upon trial by substituting it instead of R in the trinomial quantity $0.214,214R^{11} + R^{10} - R^{11}$, will be found to make the said trinomial quantity be equal to 0.214,214, or the absolute term of the proposed equation $0.214,214R^{11} + R^{10} - R^{11} = 0.214,214$. We may therefore consider this last resolution of the proposed equation, which is derived from the knowledge of the two numbers 1.0948 and 1.0344, (which are the limits of the magnitude of R , or the greater root of the said equation,) together with the knowledge of the number 0.243,097,9 (which is the value of the trinomial quantity $0.214,214R^{11} + R^{10} - R^{11}$ that corresponds to the number 1.0344, and is the greatest possible magnitude of the said trinomial quantity,) as being now complet.

This last resolution of the proposed equation $0.214,214R^{11} + R^{10} - R^{11} = 0.214,214$ having consisted of more operations than the former resolutions of it, I will here briefly recapitulate the several steps by which we approached to the last near value of R , or it's greater root, which was 1.0608, that the reader may see them all in one view.

A Recapitulation of the several Processes used in this last Resolution of the Equation

$$0.214,214R^{11} + R^{10} - R^{11} = 0.214,214.$$

Art. 42. Having substituted 1.0344, (the lower limit of the magnitude of R), instead of R in the trinomial quantity $0.214,214R^{11} + R^{10} - R^{11}$, and

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found

found that the value of the said trinomial quantity resulting from that substitution was 0.243,097,9, we made the following proportion; As $R - 1.0344$ is to $1.0948 - 1.0344$, so (nearly,) will $0.243,097,9 - 0.214,214$ be to $0.243,097,9 - 0$, or as $R - 1.0344$ is to 0.0604, so (nearly,) will 0.028,883,9 be to 0.243,097,9; and consequently $R - 1.0344$ will be, nearly, $(= \frac{0.0604 \times 0.028,883,9}{0.243,097,9} = \frac{0.001,744,587,56}{0.243,097,9}) = 0.0071$. Therefore R will be, nearly, $(= 0.0071 + 1.0344) = 1.0415$. This is the first near value of R obtained in this method of resolving the proposed equation.

We then substituted this number 1.0415 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, and found that the value of the said quantity resulting from such substitution was 0.241,451,7, which is considerably greater than 0.214,214, or the absolute term of the proposed equation; and thence we concluded that the number 1.0415 must be considerably less than the true value of R , or the greater root of the proposed equation. We therefore increased the number 1.0415 by adding to it a hundredth part of itself, or the fraction $\frac{1.0415}{100}$, or 0.0104, and took the sum 1.0519 for a second near value of R , which we conjectured would be nearer than 1.0415 to its true value.

We then tried the justness of this conjecture by substituting this new number 1.0519, or second near value of R , instead of R , in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$; and we found the value of the said trinomial quantity resulting from this substitution to be 0.230,961,3; which is considerably less than the number 0.241,451,7 (which resulted from the substitution of the former number 1.0415 instead of R in the said trinomial quantity,) but yet is greater than 0.214,214, or the absolute term of the proposed equation: and we thence concluded that the number 1.0519 would (as well as the former number 1.0415,) be less than the true value of R , or the greater root of the proposed equation, and would therefore be much nearer than the said former number 1.0415 (as we had conjectured that it would be,) to the true value of the said greater root.

Nevertheless, as the number 0.230,961,3 (which resulted from the substitution of 1.0519 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$), is considerably greater than 0.214,214, or the absolute term of the proposed equation, it followed that the number 1.0519 (though much nearer than the former number 1.0415 to the true value of R), must still be considerably less than the true value of R , or the greater root of the said equation. I therefore increased the said number 1.0519 by adding to it a 200th part of itself, or the fraction $\frac{1.0519}{200}$, or 0.0052, and thereby obtained $(1.0519 + 0.0052, \text{ or } 1.0571)$ for a third near value of R , which would probably be nearer

nearer than either 1.0415 or 1.0519, to it's true value. And I then tried the justness of this conjecture by substituting this number 1.0571 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$; and I found the value of the said trinomial quantity resulting from such substitution to be 0.221,207,1; which is greater than 0.214,214, or the absolute term of the proposed equation; and I thence concluded that the said number 1.0571 must be less than the true value of R , or the greater root of the said equation, and therefore nearer than the former number 1.0519 to the said true value.

Having thus obtained the two numbers 1.0519 and 1.0571 for two successive near values of R , which are both less than it's true value, and having found the two numbers 0.230,961,3 and 0.221,207,1, which are the corresponding values of the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, we obtained another, or a fourth, near value of R , to wit, the number 1.0608, (which is much nearer than even the last number 1.0571 to it's true value,) by the following process of the Differential method of approximation; to wit, by supposing that, as $R - 1.0571$ is to $1.0571 - 1.0519$, so (very nearly,) will $0.221,207,1 - 0.214,214,0$ be to $0.230,961,3 - 0.221,207,1$, or that as $R - 1.0571$ is to 0.0052, so (very nearly,) will 0.006,993,1 be to 0.009,754,2; whence it follows that $R - 1.0571$ will be nearly $(= \frac{0.0052 \times 0.006,993,1}{0.009,754,2}) = \frac{0.000,036,364,12}{0.009,754,2}$ and consequently that R will be $(= 1.0571 + 1.0608) = 1.0608$. And thus, in this manner of resolving the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, we obtained the four numbers 1.0415, 1.0519, 1.0571, and 1.0608 for successive near values of R , or it's greater root; of which the last value 1.0608 differs from the true value of R , (which is 1.06,) by only the 1325th part of the said true value.

Another Method of resolving the Equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$ without first finding the Limits of the Magnitude of R , it's greater Root.

Art. 43. In all the foregoing methods of resolving the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$ we have made use of either one or both of the numbers 1.0948 and 1.0344, which are the limits of the magnitude of R , or the greater root of the said equation, and which were themselves obtained above (in art. 31 and 32,) by the resolution of the two subordinate binomial equations $R^{10} - R^9 = 0.214,214$ and $R^{10} - 0.952,380R^9 = 0.112,207$, which were derived from the said trinomial equation. And this way of proceeding was the most regular and scientific that could be taken for the purpose of resolving the trinomial equation $0.214,214R^{11} + R^{20} - R^{21} =$

= 0.214,214 itself, or investigating the value of it's greater root R , which lies between the said limits 1.0948 and 1.0344, so previously discovered: just as, in laying siege to a fortified town, it is, I believe, customary for the besieging army to make themselves masters of the out-works, before they attack the body of the place. But, if this should be thought too tedious a way of proceeding, we may resolve the equation in a more direct manner without first discovering those limits of the magnitude of R , by forming at once a probable conjecture concerning the magnitude of R , or the greater root of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, in the manner following.

Since R is the value of 1*l.* together with it's interest for one year, it will be equal to either 1.01, or 1.02, or 1.03, or 1.04, or 1.05, &c, according as the rate of the interest of money (which in this question is supposed to be unknown,) shall be 1 per cent, or 2 per cent, or 3 per cent, or 4 per cent, or 5 per cent, &c. Therefore, as 5 per cent is a very common rate of the interest of money in bargains made in England, it seems reasonable to conjecture that the interest of money allowed in the present question was 5 per cent, or near it. We will therefore take 1.05 for our first near value of R , and will try the justness of our conjecture by substituting the number 1.05 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, and comparing the value of the said quantity resulting from such substitution with 0.214,214, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. This may be done as follows.

If R is = 1.05, we shall have $R^{11} = \overline{1.05}^{11}$, and $R^{20} = \overline{1.05}^{20}$, and $R^{21} = \overline{1.05}^{21}$, and $0.214,214R^{11} = 0.214,214 \times \overline{1.05}^{11}$; and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be = $0.214,214 \times \overline{1.05}^{11} + \overline{1.05}^{20} - \overline{1.05}^{21}$. We must therefore endeavour to find the values of $\overline{1.05}^{11}$, and $\overline{1.05}^{20}$, and $\overline{1.05}^{21}$; which may be done most conveniently by the help of a table of logarithms in the following manner.

The logarithm of 1.05 is = 0.021,189,3. Therefore the logarithm of $\overline{1.05}^{11}$ will be (= $11 \times 0.021,189,3$) = 0.233,082,3; which is the logarithm of 1.710,339,3. Therefore $\overline{1.05}^{11}$ is = 1.710,339,3. Therefore $\overline{1.05}^{22}$ will be (= $\overline{1.05}^{11}{}^2 = \overline{1.710,339,3}^2$) = 2.925,260,521,124,49, and $\overline{1.05}^{21}$ will be (= $\frac{\overline{1.05}^{22}}{1.05} = \frac{2.925,260,521,124,49}{1.05}$) = 2.785,962,401,070, and $\overline{1.05}^{20}$ will be (= $\frac{\overline{1.05}^{21}}{1.05} = \frac{2.785,962,401,070}{1.05}$) = 2.653,297,5; and $0.214,214 \times \overline{1.05}^{11}$ will be (= $0.214,214 \times 1.710,339,3$) = 0.366,378,622,810,2. Therefore the trinomial quantity $0.214,214 \times \overline{1.05}^{11} + \overline{1.05}^{20} - \overline{1.05}^{21}$ will

will be $(= 0.366,378,6 + 2.653,297,5 - 2.785,962,4 = 3.019,676,1 - 2.785,962,4) = 0.233,713,7$; which is greater than $0.214,214$, or the absolute term of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore the number 1.05 must be less than the true value of R , or the greater root of the said equation.

Art. 44. We will therefore increase the number 1.05 by a 200th part of itself, or by the fraction $\frac{1.05}{200}$, or 0.0052 ; and we shall then have $(1.05 + 0.0052, \text{ or } 1.0552$ for a second near value of R , which will probably be nearer than 1.05 to its true value; and we will try the justness of this new conjecture by substituting 1.0552 instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, and comparing the value resulting from such substitution with $0.214,214$, or the absolute term of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. This may be done as follows.

If R is $= 1.0552$, we shall have $R^{11} = \overline{1.0552}^{11}$, and $R^{20} = \overline{1.0552}^{20}$, and $R^{21} = \overline{1.0552}^{21}$; and $0.214,214R^{11}$ will be $= 0.214,214 \times \overline{1.0552}^{11}$, and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be $= 0.214,214 \times \overline{1.0552}^{11} + \overline{1.0552}^{20} - \overline{1.0552}^{21}$. We must therefore endeavour to find the values of $\overline{1.0552}^{11}$, and $\overline{1.0552}^{20}$, and $\overline{1.0552}^{21}$; which may be done most conveniently by the help of a table of logarithms in the following manner.

The logarithm of 1.0552 is $= 0.023,334,8$. Therefore the logarithm of $\overline{1.0552}^{11}$ will be $(= 11 \times 0.023,334,8) = 0.256,682,8$; which is the logarithm of the number $1.805,854,7$. Therefore $\overline{1.0552}^{11}$ is $= 1.805,854,7$. Therefore $\overline{1.0552}^{22}$ will be $(= \overline{1.0552}^{11})^2 = \overline{1.805,854,7}^2) = 3.261,111,197,512,09$, and $\overline{1.0552}^{21}$ will be $(= \frac{\overline{1.0552}^{22}}{1.0552} = \frac{3.261,111,197,512,09}{1.0552}) = 3.090,514,781,56$, and $\overline{1.0552}^{20}$ will be $(= \frac{\overline{1.0552}^{21}}{1.0552} = \frac{3.090,514,781,56}{1.0552}) = 2.928,842,6$; and $0.214,214 \times \overline{1.0552}^{11}$ will be $(= 0.214,214 \times 1.805,854,7) = 0.386,839,358,705,8$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0552}^{11} + \overline{1.0552}^{20} - \overline{1.0552}^{21}$ will be $(= 0.386,839,3 + 2.928,842,6 - 3.090,514,7 = 3.315,681,9 - 3.090,504,7) = 0.225,167,2$; which is greater than $0.214,214$, or the absolute term of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore the number 1.0552 will be less than the true value of R , or the greater root of that equation.

Art. 45.

Art. 45. It has now appeared that, while R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ increases from 1.05 to 1.0552, or receives the increment 0.0052, the said trinomial quantity decreases from 0.233,713,7 to 0.225,167,2, or suffers a decrement of $(0.233,713,7 - 0.225,167,2)$, or 0.008,546,5. Therefore, if we suppose that, while this trinomial quantity suffers another decrement equal to 0.008,546,5, or decreases further from 0.225,167,2 (to $0.225,167,2 - 0.008,546,5$, or) to 0.216,520,7, the quantity R will receive another increment equal to the last or to 0.0052, (which supposition seems likely to be near the truth,) the value of R when the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ is become = 0.216,520,7, will be = $(1.0552 + 0.0052)$, or 1.0604. But 0.216,520,7 is but little greater than 0.214,214, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore the value of R when the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ is equal to 0.216,520,7 must be nearly equal to R , or the greater root of the said equation; that is, 1.604 must be nearly equal to the said root. We will therefore take 1.604 for our next, or third, near value of R , or the greater root of the proposed equation, and will proceed to try the justness of this conjecture by substituting it instead of R in the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$, and comparing the value of the said quantity resulting from such substitution with 0.214,214, or the absolute term of the proposed equation. This may be done in the following manner.

If R is = 1.0604, we shall have $R^{11} = \overline{1.0604}^{11}$, and $R^{20} = \overline{1.0604}^{20}$, and $R^{21} = \overline{1.0604}^{21}$; and $0.214,214R^{11}$ will be = $0.214,214 \times \overline{1.0604}^{11}$, and the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be = to the trinomial quantity $0.214,214 \times \overline{1.0604}^{11} + \overline{1.0604}^{20} - \overline{1.0604}^{21}$. We must therefore endeavour to find the values of $\overline{1.0604}^{11}$, and $\overline{1.0604}^{20}$, and $\overline{1.0604}^{21}$; which may be done most conveniently by the help of a table of logarithms in the manner following.

The logarithm of 1.0604 is = 0.025,469,7. Therefore the logarithm of $\overline{1.0604}^{11}$ will be $(= 11 \times 0.025,469,7) = 0.280,166,7$; which is the logarithm of the number 1.906,192,1. Therefore $\overline{1.0604}^{11}$ is = 1.906,192,1. Therefore $\overline{1.0604}^{22}$ will be $(= \overline{1.0604}^{11})^2 = \overline{1.906,192,1}^2) = 3.633,568,322,102,41$, and $\overline{1.0604}^{21}$ will be $(= \frac{\overline{1.0604}^{22}}{\overline{1.0604}^{11}} = \frac{3.633,568,322,102,41}{1.906,192,1}) = 3.426,601,586,29$, and $\overline{1.0604}^{20}$ will be $(= \frac{\overline{1.0604}^{21}}{\overline{1.0604}^{11}} = \frac{3.426,601,586,29}{1.906,192,1}) = 3.231,423,6$; and $0.214,214 \times \overline{1.0604}^{11}$ will be $(= 0.214,214 \times 1.906,192,1) = 0.408,333,034,509,4$. Therefore the trinomial quantity $0.214,214 \times \overline{1.0604}^{11}$

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+ $\overline{1.0604}^{10} - \overline{1.0604}^{21}$ will be ($= 0.408,333,0 + 3.231,423,6 - 3.426,601,5 = 3.639,756,6 - 3.426,601,5$) $= 0.213,155,1$; which is less than $0.214,214$, or the absolute term of the equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. Therefore 1.0604 must be a little greater than the true value of R , or the greater root of that equation. But the difference will be very small, and therefore the said number 1.0604 will be an excellent basis, or ground-work for a further approach to the said true value by a process of Mr. Raphson's method of approximation; which may be performed as follows.

Art. 46. Let z be put for the unknown excess of the number 1.0604 above R , or the greater root of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$. And let the binomial quantity $1.0604 - z$ be substituted instead of R in the said equation.

Then, since R is $= 1.0604 - z$, we shall have $R^{11} = \overline{1.0604 - z}^{11} = \overline{1.0604}^{11} - 11 \times \overline{1.0604}^{10} \times z + \&c$ ($= \overline{1.0604}^{11} - \frac{11 \times \overline{1.0604}^{10}}{1.0604} \times z + \&c$) $= 1.906,192,1 - \frac{11 \times 1.906,192,1}{1.0604} \times z + \&c$ $= 1.906,192,1 - \frac{20.968,113,1}{1.0604} \times z + \&c$) $= 1.906,192,1 - 19.774,531,4 \times z + \&c$, and $R^{20} = \overline{1.0604 - z}^{20} = \overline{1.0604}^{20} - 20 \times \overline{1.0604}^{19} \times z + \&c$ ($= \overline{1.0604}^{20} - \frac{20 \times \overline{1.0604}^{19}}{1.0604} \times z + \&c$) $= 3.231,423,6 - \frac{20 \times 3.231,423,6}{1.0604} \times z + \&c$ $= 3.231,423,6 - \frac{64.628,472,0}{1.0604} \times z + \&c$) $= 3.231,423,6 - 60.947,257,6 \times z + \&c$, and $R^{21} = \overline{1.0604 - z}^{21} = \overline{1.0604}^{21} - 21 \times \overline{1.0604}^{20} \times z + \&c$ ($= 3.426,601,5 - 21 \times 3.231,423,6 \times z + \&c$) $= 3.426,601,5 - 67.859,895,6 \times z + \&c$. And $0.214,214 \times R^{11}$ will be $= 0.214,214 \times$ the compound quantity $1.906,192,1 - 19.774,531,4 \times z + \&c$ ($= 0.214,214 \times 1.906,192,1 - 0.214,214 \times 19.774,531,4 \times z + \&c$) $= 0.408,333,0 - 4.235,981,4 \times z + \&c$.

Therefore the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ will be equal to the following compound quantity, to wit,

$$\left\{ \begin{array}{l} 0.408,333,0 - 4.235,981,4 \times z + \&c \\ + 3.231,423,6 - 60.947,257,6 \times z + \&c \\ - 3.426,601,5 + 67.859,895,6 \times z - \&c \end{array} \right\} =$$

$$\left\{ \begin{array}{l} 3.639,756,6 - 65.183,239,0 \times z + \&c \\ - 3.426,601,5 + 67.859,895,6 \times z - \&c \end{array} \right\}$$

$$= 0.213,155,1 + 2.676,656,6 \times z - \&c.$$

But the trinomial quantity $0.214,214R^{11} + R^{20} - R^{21}$ is $= 0.214,214$.

Therefore the compound quantity $0.213,155,1 + 2.676,656,6 \times z - 8c$ will also be $= 0.214,214$. And therefore $2.676,656,6 \times z - 8c$ will be $(= 0.214,214 - 0.213,155,1) = 0.001,058,9$, and $z - 8c$ will be $= \frac{0.001,058,9}{2.676,656,6} = 0.000,395,6$. Therefore R , or $1.0604 - z$, will be $= 1.0604 - 0.000,395,6 = 1.060,004,4$; that is, R , or the greater root of the proposed equation $0.214,214R^{11} + R^{20} - R^{21} = 0.214,214$, is $= 1.060,004,4$.

Q. E. I.

This value of R is exceedingly near to it's true value, which is $1.060,000,0$, or 1.06 .

A S C H O L I U M.

Art. 47. In resolving these high equations by approximation the principal difficulty consists in forming judicious conjectures concerning the first near values of the roots sought, which are to be made the ground-works of further approaches to their true values by means of Mr. Raphson's, or some other, method of approximation. For, when once we have found a near value of the root sought that shall differ from it by less than a 1000th part, or even than a 500th part, of the said root, those methods of approximation will soon enable us to obtain much nearer values of it to as great a degree of exactness as we please. But three, or four, conjectures will sometimes be necessary to be made, and the justness of them must be tried by substitutions of the near values obtained by them in the multinomial quantity that forms the first, or left-hand, side of the equation, and by comparing the values of the said multinomial quantity resulting from such substitutions with the absolute term of the equation, before we can obtain a near value of the root that shall differ from it's true value by so small a quantity as that just now mentioned. And no general rules can be given to direct us in all cases to make these conjectures with success: but much will depend on the sagacity of the person who undertakes to resolve the equation. The knowledge of the limits of the magnitude of the root sought will be of great use to us in forming these conjectures: but then, on the other hand, there is a good deal of labour necessary to discover these limits by the previous resolution of other equations of a lower order. The nature and conditions of the Problem from which the equation was derived will also sometimes enable us to form a good conjecture concerning the magnitude of the root sought; and other circumstances will sometimes be of use to us: and at all events a random guess that should happen to be very wide of the mark, will do to begin with; because, upon the trial

trial of the number so assumed for a near value of the root sought by substituting it in the left-hand side of the equation, and comparing the result of the said substitution with the absolute term of the equation, we may continually correct it by additions and subtractions till we obtain a number that shall be as near the root sought as we desire. But these substitutions require a good deal of labour, as we have seen in the foregoing resolutions of the equation $0.214,214R^{11} + R^{10} - R^{21} = 0.214,214$; and therefore the fewer of them we are obliged to make, the better it will be. Now, in order to avoid an unnecessary number of these troublesome substitutions, it will, as I conceive, be highly useful to persons who wish to acquire a readiness in resolving equations of this kind, to see copious exhibitions of the several different methods that may be taken to form tolerable conjectures concerning the magnitude of the root sought in the beginning of the resolution of an equation, that they may thereby acquire a sort of habitual and experimental sagacity in judging of the probable magnitudes of the roots sought in other high equations of the same kind. And it is with this view that I have here given so many different resolutions of the foregoing equation, and at so great length, after the first resolution of it in art. 33 had discovered the value of it's greater root R to a sufficient degree of exactness by shewing it to be $= 1.060,302$, and with less trouble than was necessary to any of the following resolutions of it. This therefore must be my excuse for the great length and variety of the resolutions of this last equation; and with them I conclude this Appendix to the worthy and learned Mr. John Ward's *Clavis Usuræ*.

FRANCIS MASERES.

INNER TEMPLE, }
Dec. 24, 1801. }

End of the *Appendix to Mr. John Ward's Clavis Usuræ, or Key to Interest, both Simple and Compound.*

CORRECTIONS AND ADDITIONS

TO

THE REVEREND DR. HALES'S TREATISE,

INTITLED

ANALYSIS FLUXIONUM,

Printed in the former Part of this Volume, in Pages 87, 88, 89, &c - - - to
Page 204; lately received from Dr. HALES.

ANALYSIS FLUXIONUM.

CORRIGENDA. PART I.

PRÆFAT. pag. 91, lin. ult. pro § 131, lege § 26, 27.—p. 94, pro § 162, lin. ult. l. § 155 et § 205, not. (p)

PARS PRIMA, p. 96, § 5, pro *Daniel*, l. *Jacobus*.—p. 100, not. 2, pro *Institutionis—celebrantur*, l. *Instituti—celebratur*.—p. 9, lin. 103, pro $A + B$, l. $A \times B$.—p. 104, § 17, pro $\Phi\omega\nu\alpha\nu\tau\alpha$ l. $\Phi\omega\nu\epsilon\nu\tau\alpha$.—not. (d) lin. ult. pro § 162—164, l. § 154—157.—p. 107, not. lin. 4. à fine, pro *Bezart* l. *Bezout*.—p. 112, § 36, lin. 3, pro *semiffis*, l. *semiffis* a^2 .—p. 119, § 50, lin. 9, pro *For it is*, l. *For in GEOMETRY it is*.—p. 125, § 73, lin. 6, pro *nomini*, l. *nomine*.

PARS SECUNDA, p. 129, comma 4, pro *puncti*, l. *puncto*.—p. 133, § 84, lin. 3, pro *latera*, l. *laterum co-efficientia*.—Dele not. (g), et transfer ad p. 56, § 124.—p. 136, § 90, com. 1, lin. 6, pro *vel* $2aA$, l. *vel* $2bA$.—p. 137, § 92, infere *Caf. 2. Fluxio*, &c.—p. 138, § 94.

Caf. 4. Lege, *Fluxio dignitatis cujuscvis negativæ* (A^{-m}) æquatur *indici illius dignitatis* ($-m$), *fluxioni lateris* (a), *ejusque co-efficienti* (A^{-m-1}), *in se continuè ductis*. Seu fluxio $A^{-m} = -maA^{-m-1}$.—§ 94, in demonstratione, lin. 3, post $\frac{-a}{A^2}$, infere $= -aA^{-2}$.—et ibidem, lin. ult. post $\frac{-ma}{A^{m+1}}$, infere $= -maA^{-m-1}$. Q. E. D.

P. 139, § 97, lin. ult. ante § 98, infere,—*Caf. 6. Cor.*—*Iinc fluxio fractionis* $\frac{A^m}{B^n}$, *sive rectanguli* $A^m B^{-n}$, *per Caf. 1 et 4, fit* $maB^{-n}A^{m-1} - nbA^m B^{-n-1}$, $= \frac{maA^{m-1}}{B^n} - \frac{nbA^m}{B^{n+1}}$. Q. E. D.

P. 143, § 105, lin. 7, pro *indicet* l. *induit*.—p. 144, lin. 3, à fine, pro *terminis* l. *termini*.—p. 151, lin. 9, *in*serere (five ob $m = 1$) = $\frac{x}{x}$;—p. 151, lin. ult. pro *co-efficienti* l. *co-efficientem*.

APPENDIX SECUNDA.—P. 170, not. (b), lin. 9, pro *bixapedarum* l. *hexapedarum*.—p. 171, lin. 3, pro *præmissorum* l. *præmissarum*.—p. 178, lin. 16, pro *light* l. *life*.—p. 179, not. lin. 4, à fine, pro *an* l. *cur*.—p. 184, lin. 4, pro *αωθεν* l. *ανωθεν*.—p. 186, lin. 6, pro *epificis* l. *epificio*.—p. 190, not. lin. 6, pro *Νεω* l. *Νεωτη*.—p. 198, not. lin. 10, pro *Ηλως* l. *Ηελ-ως*.—p. 200, lin. 12, pro *demonstrari* l. *demonstrare*.

ADDENDA.

PARS PRIMA.—P. 97, § 7. See last page of this, p. 113, § 38, post—*Newtoni discipulis*: nempe, 1. *Robins*, qui anno 1735, in a *Discourse concerning the Nature and Certainty of Sir Isaac Newton's Methods of Fluxions and of Prime and Ultimate Ratios*;—hæc methodos, luculentius breviusque exposuit; magis-que ad mentem ipsius inventoris sagacissimi, (meo judicio suffragantibus amicis dignissimis, analytisque simul peritissimis, Episcopo *Horsley* et Barone *Masères*), quam quilibet ex interpretibus insequentibus; 2. *Colson*, anno insequente 1736, in *the Method of Fluxions and Infinite Series*, &c. 3. *Maclaurin*, anno 1742, in *A Treatise of Fluxions*; et his omnibus multo recentior, 4. *Le Croix*, anno 1779, in *Traité du Calcul différentiel*, edentibus analytis summis *La Place* et *Le Gendre*; quorum ex operibus excerpta quædam hanc rem illustrantia proferre non pigebit.

1. *Robins's Account of the Methods of Fluxions, and of Prime and Ultimate Ratios.*

“To avoid the imperfection with which the *Method of Indivisibles* was justly charged, (in which all curves are considered as composed of an infinite number of indivisible straight lines, and curvilinear spaces as composed in the like manner of parallelograms; which being an obscure and indistinct perception, was obnoxious to error;) Sir *Isaac Newton* instituted an *Analysis* for these problems concerning the Tangents of Curve-lines, and the mensuration of Curvilinear Spaces, upon other principles.

“Considering magnitudes, not under the notion of being increased by a repeated *accession* of parts; but as generated by a continued motion or *flux*.; he discovered a Method to compare together the *velocities* wherewith homogeneous magnitudes

magnitudes increase ; and thereby has taught an *Analysis* free from all obscurity and indistinctness.

“ Moreover, to facilitate the demonstrations for these kind of Problems, he invented a *synthetic* form of reasoning from the *prime and ultimate ratios* of the contemporaneous augments or decrements of those magnitudes ; which is much more *concise* than the method of demonstrating used by the ancients, yet is equally distinct and conclusive.

“ I. The method employed by the ancient Geometers (since commonly called the *Method of Exhaustions*) consists in describing upon the curvilinear space a rectilinear one, which, though not equal to the other, yet might differ from it less than by any the most minute difference whatsoever, that should be proposed ; and thereby proving the two spaces they would compare, could be neither greater nor less than each other.

“ For example, in order to prove *the equality between the space comprehended within the circumference of a circle ; and a triangle whose base should be equal to the circumference of that circle ; and its altitude to the semidiameter ; Archimedes* takes this method :

“ About the circle he circumscribes a polygon ; and by multiplying the sides of this polygon, he makes it appear, that there may at length be circumscribed such a one as shall exceed the circle less than by any difference that shall be proposed, how minute soever that difference be.—By this means it was easy to prove that the triangle aforesaid is *not greater* than the circle : for were it greater, how small soever be the excess, it were possible to circumscribe about the circle a polygon less than the triangle ; but the circumference of the polygon is greater than the circumference of the circle ; therefore the polygon can never be less, but must be always greater than the triangle : (for the polygon is equal to a triangle, whose altitude is the semidiameter of the circle, and base equal to the circumference of the polygon.) It appears therefore impossible for the triangle to be greater than the circle.

“ Thus far *Archimedes* makes use of the polygon circumscribing the circle, and no further : but, inscribing another within the circle, he proves, by a similar process of reasoning, that it is impossible for the triangle to be *less* than the circle : whereby at length it becomes certain, that the triangle is neither greater nor less than the circle, but *equal* to it. Q. E. D.

“ However the triangle may be proved *not to be less* than the circle, by the *circumscribed* polygon also : for were it less, another triangle whose base is greater than its base, and height equal, might be taken which would not be greater than the circle ; but a polygon can be circumscribed about the circle, the circumference of which shall exceed the circumference of the circle by less than any line that can be named ; consequently by less than the difference

between the two bases; that is, the circumference of the polygon shall be less than the circumference of the circle; and consequently, the polygon less than the given triangle; therefore it is impossible that this triangle should not exceed the circle, since it is greater than the polygon: consequently, the given triangle cannot be less than the circle.

“ Thus the circle and triangle may be proved to be equal by comparing them with *one polygon* only: and Sir *Isaac Newton* has instituted upon this principle, a briefer method of conception and expression for demonstrating this sort of propositions than what was used by the ancients; and his ideas are equally distinct and adequate to the subject with theirs, though more complex. It became the first writers to choose the most simple form of expression and the least compounded ideas possible; but Sir *Isaac Newton* thought he should oblige the mathematicians by using brevity, provided he introduced no mode of conception difficult to be comprehended by those who are not unskilled in the ancient methods of writing.

1. CASE. “ In this method, any fixed quantity, which some varying quantity, by a continual augmentation or diminution, shall perpetually approach but never pass, is considered as the quantity to which the varying quantity will *at last* or *ultimately* become equal; provided the varying quantity can be made in its approach to the other to differ from it by less than by any quantity how minute soever that can be assigned. Princip. Lib. I. Lem. I.

“ Upon this principle, the equality between the forementioned circle and triangle is at once deducible: for since the polygon circumscribing the circle approaches to each according to all the conditions above set down, this polygon is to be considered as ultimately becoming equal to both; and consequently, they must be esteemed equal to each other.—That this is a just conclusion is most evident; for if either of these magnitudes be supposed less than the other, this polygon could not approach to the least within some assignable difference.

1 DEFINITION. “ An *ultimate magnitude* therefore may be defined the limit to which a *varying magnitude* can approach within any degree of nearness whatever, though it can never be made absolutely equal to it.

“ Thus the foregoing circle is to be called the ultimate magnitude of the polygon *circumscribing* it; because this polygon by increasing the number of its sides can be made to differ from the circle less than by any space that can be proposed, how small soever; and yet the polygon can never become equal to the circle or less. In like manner, the circle will be the *ultimate magnitude* of the polygon *inscribed*.

“ Again, the foregoing triangle is the *ultimate magnitude*, of the constructed triangle; because, the new base being always equal to the circumference of the polygon, will constantly be greater than the given base, which is equal to the circumference

circumference of the circle only; and yet the new base may be made to approach the given one nearer than by any difference that can be named.

2 CASE. Ratios also may so vary, as to be confined after the same manner to some determined limit; and such limit of any ratio is here considered as that, with which the varying ratio will ultimately coincide. Princip. Lib. I. Lem. I.

2 DEFINITION. *If there be two quantities therefore, that are (one or both) continually varying, either by being continually augmented or continually diminished; and if the proportion they bear to each other does by this manner perpetually vary, but in such a manner that it constantly approaches nearer and nearer to some determined proportion; and can also be brought at last in it's approach nearer to this determined proportion than to any other that can be assigned; but can never pass it; this determined proportion is then called the ultimate ratio of these varying quantities.*

“The same ratio may be called sometimes the *prime*, at other times the *ultimate ratio* of the same varying quantities; according as these quantities are considered either under the notion of *vanishing*, or of being produced before the imagination by an uninterrupted motion. The doctrine under examination receives it's name from both these ways of expression.

N. B. “The reader will perceive that I am endeavouring to explain Sir Isaac Newton's expressions, *Ratio ultima quantitatum evanescentium*. And I have rendered the Latin participle *evanescens* by the English word “*vanishing*,” and not by the word “*evanescent*,” which having the form of a noun adjective, does not so certainly imply that motion which ought here to be kept carefully in mind: the quantities under consideration become vanishing, from the time we first ascribe to them this perpetual diminution; that is, from the time they are quantities *going to vanish*: and as during their diminution, they have continually different proportions to each other; so the limiting ratio between them, is not to be called merely *Ratio harum quantitatum evanescentium*, but *ultima ratio*. Princip. p. 37.

“I have attempted to explain, in like manner, the phrase *Ratio prima quantitatum nascentium*; but no English participle occurring to me whereby to render the word *nascent*, I have been obliged to use circumlocution. Under the present conception of the varying quantities, they are to be called *nascentes*, not only at the *very instant* of their first production, but (according to the sense in which such participles are used in common speech) after the same manner as when we say of a body which has lain at rest, that it is *beginning to move*, though it may have been *some little time* in motion: on this account, we must not use the simple expression *Ratio quantitatum nascentium*, but to denote the limiting ratio, we must call it, *Ratio prima quantitatum nascentium*. Princip. *ibid.*

II. "Upon these definitions, we may ground the following Propositions :

I PROP. *When varying magnitudes keep constantly the same proportion to each other, their ultimate magnitudes are in the same proportion.*

"Let A and B be two varying magnitudes, which keep constantly in the same proportion to each other ; and let C be the ultimate magnitude of A, and D the ultimate magnitude of B : I say that C is to D in the same proportion as A to B.

Since A is a varying magnitude continually approaching to C, but can never become equal to it, A may be either always greater or always less than C.

In the first place suppose it *greater* : when A is greater than C, A B
in approaching to C, it is continually diminished ; therefore B C D
keeping always in the same proportion to A, B in approaching to E
it's limit D, is also continually diminished :

Now, if possible, let the ratio of C to D be *greater* than that of A to B : (that is, suppose C to have to some magnitude, E, greater than D, the same proportion as A to B).

Since C is the ultimate magnitude of A in the sense of the preceding definition, A can be made to approach nearer to C than by any difference that can be proposed, but can never become equal to it or less : therefore, since C is to E as A to B, B will always exceed E ; consequently, can never approach to D so near as by the excess E : which is absurd : For, since D is supposed the ultimate magnitude of B, it can be approached by B nearer than by any assignable difference.

After the same manner, neither can the ratio of D to C be *greater* than that of B to A.

In the second place, if the varying magnitude A be *less* than C ; it follows in like manner, that neither the ratio of C to D can be *less* than that of A to B ; nor the ratio of D to C *less* than that of B to A. Q. E. D.

COR. *The ultimate magnitudes of the same or equal varying magnitudes are equal.*

"Now from this corollary (which evidently follows from the proposition) the forementioned equality between the circle and triangle will immediately appear : for the circle being the ultimate magnitude of the polygon, and the given triangle, the ultimate magnitude of the constructed triangle ; since the polygon and the constructed triangle are equal, by this corollary, the circle and given triangle will be also equal. Q. E. D.

N. B. "If the preceding proposition and it's corollary be admitted as genuine deductions from the [First] definition upon which it is grounded ; this demonstration

demonstration of this equality of the circle and triangle cannot be excepted to: for no objection can lie against the definition itself, as no inference is there deduced, but only the sense explained of the term [*Ultimate magnitude*] defined in it."

2. PROP. "All the ultimate ratios of the same varying ratio are the same with each other.

"Suppose the ratio of A to B continually varies by the variation of one or both of the terms A and B: if the ratio of C to D be the ultimate ratio of A to B, and the ratio of E to F be likewise the ultimate ratio of the same; I say, the ratio of C to D is the same with the ratio of E to F.—For, if you deny it, the ratio of E to F differing from the ratio of C to D, the ratio of A to B will either pass that of E to F, or can never approach so near it as to the ratio of C to D: insomuch that the ratio of E to F cannot be the ultimate ratio of A to B, contrary to the hypothesis. Q. E. D.

"The two definitions here set down, together with the general propositions annexed to them, comprehend the whole foundation of this method.

III. "We find in former writers some attempts towards so much of this method as depends upon the first definition.

"Lucas Valerius, in a most excellent treatise on *the Center of Gravity of Solid Bodies*, has given a proposition nothing different but in the form of the expression, from that we have subjoined to our first definition: from which he demonstrates with brevity and elegance his propositions concerning the mensuration and center of gravity of the sphere, spheroid, parabolical and hyperbolical conoids. This author writ before the doctrine of *Indivisibles* was proposed to the world.

"And since, Andrew Tacquet, in his treatise on the *Cylindrical and Annular Solids*, has made the same proposition, though something differently expressed, the basis of his demonstrations; at the same time very judiciously exposing the inconclusiveness of the reasoning from *indivisibles*.

"However, the consideration of the *limits of varying proportions*, when the quantities themselves expressing those proportions have no limits, (which renders this *Method of prime and ultimate ratios* much more extensive,) we owe entirely to Sir Isaac Newton. That this method, as thus compleated, is applicable not only to the subjects treated of by the Ancients in the Method of Exhaustions, but to many others also of the greatest importance, appears from our author's immortal Treatise on the *Mathematical Principles of Natural Philosophy*.

"For it must now be manifest, that *mathematical quantities* become the proper object of this *Doctrine of Fluxions*, whenever they are supposed to increase by

by any continued mode of prolongation, dilatation, expansion, or other kind of augmentation; provided such augmentation be directed by some *general rule* whence the measure of the increase of these quantities may *from time to time* be estimated. And when different homogeneous magnitudes increase after this manner together, one may vary *faster* than another. Now the *velocity of increase* in each quantity is the *fluxion* of that quantity. This is the true interpretation of Fluxions; *Incrementorum velocitates*. For this doctrine does not suppose the fluents themselves to have any motion: fluxions are not the velocities with which the fluents or even the increments which these fluents receive are *themselves* moved; but *the degrees* of velocity wherewith those increments are generated—the terms *velocity* and *celerity* are applied in a *figurative* sense, to denote the *degree* [or *rate*] wherewith this augmentation *in every part* proceeds.

“Subjects incapable of *local motion*, such as fluxions themselves, may also have their fluxions. In this we do not ascribe to these fluxions any *actual* motion; (for, to ascribe motion or velocity to what is itself only a velocity would be wholly unintelligible.) The fluxion of another fluxion, is only a *variation* in the velocity which is that fluxion. In short, *light, heat, sound, the motion of bodies, the power of gravity*, and whatever else is *capable of variation*, and of having that *variation assigned*, for this reason, comes under the present doctrine: nothing more being understood by the *fluxion* of *any* subject, than *the degree* [or *rate*] of *its variation*.

“As the *Doctrine of Fluxions* enabled Sir Isaac Newton to investigate the *geometrical* problems, whereby he was conducted in those remote searches into Nature, which have been the subject of universal admiration; so to the *Method of Prime and Ultimate Ratios* is owing the *surprising brevity*, wherewith he has *demonstrated* those discoveries.”

I shall offer no apology for the length of this *Analysis*, in which I have endeavoured to bring together into one comprehensive view the scattered parts of that masterly argument, by which *Robins* has explained the leading principles of the *Doctrine of Prime and Ultimate Ratios*, upon which Newton's *Method of Fluxions* is founded; it is far superior indeed to the subsequent explanations of professed commentators; and it is a high gratification to myself to find, that the mode of explanation, which I adopted of the *Doctrine of Limits*, is precisely the same as *Robins's*; long before I had seen his admirable treatise; which did not fall into my hands until lately, a considerable time after the publication of the *Analysis Fluxionum*. It deserves indeed to be better known and more studied; as containing a full and sufficient refutation of the cavils of gainsayers both ancient and modern, against *the nature and certainty of the Method of Fluxions*.

II. *Colson's Account of the Method of Fluxions, &c.*

III. *Maclaurin's Account of the Method of Fluxions, &c.*

After the end of Maclaurin's Account, &c. p. 23, insert

IV. *La Croix's Account of the Method of Fluxions.*

To the foregoing testimonies of the most eminent *British* mathematicians I am happy to add the following, which reflects high honour on the candor and liberality of a distinguished *French* analyst, *La Croix*, confessing the superiority of the *Method of Fluxions* over the *Differential Calculus*; from a curious and valuable Extract furnished even by the prejudiced MONTHLY REVIEW, 1800. Vol. 31. Append. p. 497.

“*Newton* supposes lines to be generated by the motion of a point; and surfaces, by the motion of a line; and he gave the name of *Fluxions* to the velocities which regulated the motions. These notions, although rigorous, are foreign to Geometry, and their application is difficult. It is true that, by imagining a point which moves on a line, while the line itself is carried forward with an *uniform* velocity, we may represent any curve whatsoever: but the velocity of the describing point being *variable*, at each instant, we can only determine it by recurring to the Method of the Ancients (*Exhaustions*), or to that of *Prime and Ultimate Ratios*.

“It is of this last method that *Newton* almost always avails himself; so that, properly speaking, *Fluxions* were to him only a mean of giving a *sensible existence* to the quantities on which he operated. By the *Method of Prime and Ultimate Ratios* he understood the investigation of the relations of quantities *at the first and last instant of their existence*, when the quantities were generated or vanished together; and he found in the prime ratio of spaces described by the ordinate on the line of the abscissas, and by the describing point of the ordinate (spaces which he called *Moments*), the ratio of the fluxion of the abscissa to that of the ordinate; whence he determined the direction of the tangent. The Calculus was merely that used by *Barrow* in his Method of Tangents, which *Newton* by means of his Formula for the Binomial Theorem, and by his reduction into series, had extended to irrational expressions.

“The advantage of the *Method of Fluxions* over the *Differential Calculus*, in point of *Metaphysique*, consists in this: That, *fluxions* being *finite* quantities, their *moments* are only infinitely small quantities of the *first order*, and their *fluxions* are *finite*: by these means, the consideration of *infinitely small quantities of superior orders* is excluded.”

How

How was it possible for *Monthly Reviewers*, after reciting this luminous and honourable testimony to the superiority of the Method of Fluxions by the ablest expounders of the Differential Calculus, *La Croix* and his illustrious editors *La Place* and *Le Gendre*, redeeming the character of the *French* analysts, which had been impaired by the aspersions of a *D'Alembert* and of a *La Grange* on the immortal *Newton's* fame; how was it possible, I say, that "their eyes could still be so holden," after adducing this testimony, as still to assert, that "*Newton himself was not perfectly satisfied of the stability of the ground on which he had established his Method of Fluxions!*"—to wonder, [how] that [after] having beyond all controversy obtained TRUTH, mathematicians should have been unable to make it SCIENCE; for the method was simple and easy in its application and rigorous in its conclusions!"—"Viewed as a whole it possessed the greatest stability; though its foundations seen through a mist, seemed uncertain and of discordant and unsuitable materials!"—"Had the native insignificance of the Fluxionary Calculus doomed it to perish, the wit and poignant raillery of *Berkeley* had perpetuated its memory, and 'ridiculed it into immortality:' but in spite of these attacks, the English Mathematicians have still persevered in their opinions; deeming it perhaps more meritorious to err with NEWTON than to think justly with OTHER men," [i. e. THE MONTHLY REVIEWERS]! pp. 495—499.—And they will still persevere in their attachment to the Father of British Science.

How widely different are the sentiments of the illustrious *La Place*, which they cite in the next page, 500, from his Letter to *M. La Croix*.

"I see, with much pleasure, that you are engaged in a great work on the *Differential and Integral Calculus*. . . The several methods, by being brought together, will throw mutual light on each other. What they contain in common is most generally their true metaphysique: and this is the cause why the metaphysique is almost always the last thing that is discovered.—It is only by reflecting on the route which OTHERS have followed, that WE are able to generalize methods and to discover their true metaphysics."

This indeed is worthy of a mighty master of the Sciences, and a genuine disciple of *Newton*, treading in his steps, and thereby surpassing his teacher.

Page 123, after § 68, and before § 69, insert:

The following masterly explanation of the term *momentum*, as employed by *Newton*, is given by *Robins*, in the *Conclusion* of his excellent tract, p. 75.

"In determining the ultimate ratios between the contemporaneous differences of quantities, it is often previously required to consider each of these differences *apart*, in order to discover *how much* of those differences is necessary for expressing that ultimate ratio. In this case, *Sir Isaac Newton* distinguishes by the name of *momentum*, *so much of any difference as constitutes the term used in expressing this ultimate ratio*.

"Thus,

“ Thus, if A and B denote varying quantities, and their contemporaneous increments be represented by a and b , the rectangle under any given line M and a is the contemporaneous increment of the rectangle MA ; and $Ab + Ba + ab$ is the like increment of the rectangle AB .—And here, *the whole* increment Ma represents the *momentum* of the rectangle MA ; but *the part*, $Ab + Ba$, only, (and not the whole increment $Ab + Ba + ab$,) is called the *momentum* of the rectangle AB : because *so much only* of this latter increment is required for determining the ultimate ratio of the increment of MA to the increment of AB ; this ratio being the same with the ultimate ratio of Ma to $Ab + Ba$: (for the ultimate ratio of $Ab + Ba$ to $Ab + Ba + ab$ is the ratio of equality. Consequently, the ultimate ratio of Ma to $Ab + Ba$ differs not from the ultimate ratio of Ma to $Ab + Ba + ab$. Q. E. D.

“ These *momenta* equally relate to the *decrements* of quantities as to their *increments*; and the ultimate ratio of increments and of decrements *at the same place* is the same. Therefore the *momentum* of any body may be determined, either by considering the increment or the decrement of that quantity; or even by considering *both* together. And in determining the momentum of the rectangle AB , Sir Isaac Newton has taken the last of these methods; because by this means the superfluous rectangle (ab) is sooner disengaged from the demonstration.

“ Here it must always be remembered, that the only use which ought ever to be made of these *momenta* is to compare them one with another, and for no other purpose than to determine “ *the ultimate or prime proportion between the several increments or decrements from whence they are deduced.*” § 66.

“ Herein the *Method of Prime and Ultimate Ratios* essentially differs from that of *Indivisibles*; for in that method, these *momenta* are considered absolutely as *parts*, whereof their respective quantities are *actually* composed. But though these *momenta* have no *final magnitude*, (which would be necessary to make their *parts* capable of compounding a whole by accumulation,) yet their *ultimate ratios* are as truly assignable, as the ratios between any quantities whatsoever. Therefore none of the objections made against the doctrine of *Indivisibles* are of the least weight against this method: but while we carefully attend to the observation here laid down, we shall be as secure from error, and the mind will receive as full satisfaction, as in any the most unexceptionable demonstration of Euclid.”

P. 176. Insert in the beginning of note (e),

The opinion of the Indian *Brabmins* is thus recorded by *Strabo*, B. 15.

Νομίζουσιν μὲν γὰρ δὴ ΤΟΝ ΕΝΘΑΔΕ ΒΙΟΝ ὡς ἀν ἀκμὴν κυομένον εἶναι, ΤΟΝ ΔΕ ΘΑΝΑΤΟΝ γενέσθιν εἰς τὸν οὕτως βίον, καὶ τὸν εὐδαιμόνα τοῖς Φιλοσοφῆσασιν.

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“For they are accustomed to account *the present life here*, as if it were an embryo only; but *death*, a birth into the real life and the happy, reserved for the seekers of wisdom.”

The following curious and valuable anecdote, &c.

Pars Prima, p. 97, § 7, after l. 7. Vide quoque § 96 hujus.—Insert:

Idque insuper constat, ex ipsius Newtoni testimonio, *Quadrat. Curvar.* sub initio: “*Incidi paulatim annis 1665 et 1666, in Methodum Fluxionum, quâ bis usus sum in Quadraturâ Curvarum.*”—

CORRIGENDA. PART II.

P. 150, § 122, dele l. 8 et 9, et substitue insequentia:

Erítque *ratio modularis* in quovis systemate logarithmorum ratio ista cujus logarithmus est ipse *modulus*. Hæc autem ratio in omni systemate eadem erit:

scilicet ratio seriei infinitæ $1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4}$, &c. ad 1, ejúsve reciproca: seu ratio numeri 2.71828, &c. ad 1, ejúsve reciproca, nempe, ratio fractionis decimalis 0.367879, &c. ad 1.

N. B. Vulgò fertur, et quibusdam auctoribus etiã melioris notæ placet, (scilicet, *Montucla*, &c.) logarithmorum systemata *Briggianum* atque *Neperianum* non nisi diversis hyperbolis, et diversis curvis logarithmicis, seu logisticis, designari posse; sed minùs rectè, ut videtur: nam, si unitas designet quadratum constans *hyperbolæ rectangulæ*, (contentum utique asymptotis et ordinatis externis hîsce parallelis), aræ inter ordinatas uni ex asymptotis parallelas, alteram asymptotam, ipsãque hyperbolam interclusæ, logarithmorum systema *Neperianum* ritè exponent; sin verò area inter duas ordinatas uni ex asymptotis parallelas quarum major sit decupla minoris, et alteram asymptotam, et hyperbolam ipsam, interclusa per unitatem designetur, seu vocetur unitas, aræ asymptoticæ ejusdem hyperbolæ, quæ antea systema *Neperianum* exponebant, nunc æquo jure systema *Briggianum* exponent.

In quâvis autem *curvâ logistica*, seu *logarithmica*, si subtangens curvæ (quæ per totam curvam est semper ejusdem magnitudinis,) per unitatem designetur, abscissæ axis, seu asymptotæ, inter duas ordinatas interceptæ, logarithmos systematis

systematis *Neperiani* exponent; si verò abscissa quævis axis, seu asymptotæ, inter duas ordinatas, quarum major sit decupla minoris, intercepta, (quæ paritèr per totam curvam erit semper ejusdem magnitudinis,) per unitatem designetur, eadem abscissæ axis, sive asymptotæ, quæ antea exhibebant logarithmos systematis *Neperiani*, nunc exhibebunt logarithmos systematis *Briggiani*.

Hinc constat, epitheton "*hyperbolicum*," logarithmis *Neperianis* vulgò tributum, *Briggianis* aut alterius cujuscvis systematis logarithmis æquè competere. Vide Cl. *Masères*, *Dissertation on the Nature of Logarithms*, in his *Elements of Plane Trigonometry*.

123. *Fluxiones logarithmorum, &c.*

P. 151, dele totum *Cas.* 2, et p. 152 totam N. B.; pro quibus substitue quod hic sequitur;

Cas. 2. Fiat $y^x = z$. Erítque $\log. z = \log. y \times x$. Sed $\log. z$ est $= \frac{z}{z}$, per § 127; et $\log. y \times x$ est $= \log. y \times \dot{x} + x \times \frac{\dot{y}}{y}$, per § 84. Ergò $\frac{z}{z}$ erit $= \log. y \times \dot{x} + x \times \frac{\dot{y}}{y}$; et proindè, $\dot{z}$ erit $= \log. y \times \dot{x}z + xz \frac{\dot{y}}{y} = \log. y \times \dot{x}y^x + x\dot{y}y^{x-1}$, restituendo scilicet y^x pro z ; hoc est, *fluxio quantitatis variabilis (y^x) æquatur duabus quantitativus, quarum una ($\log. y \times \dot{x}y^x$) est fluxio ipsius quantitatis exponentialis (y^x) quasi pro constanti habitæ, ut in *Cas.* 1; altera autem, ($x\dot{y}y^{x-1}$) fluxio ejusdem quantitatis, quasi indici constanti designatæ.* Q. E. D.



E R R A T A.

IN THE PREFACE.

IN page xi, line 13, instead of $\frac{1}{1+x} - \frac{1}{1+x} \frac{1}{n}$ read $\frac{1}{1+x} - \frac{1}{n}$.

In page xii, line 8, instead of $-dx^3 + ex^4 - fx^5 + gx^6 - bx^7 + \&c$,
read $+dx^3 - ex^4 + fx^5 - gx^6 + bx^7 - \&c$.

In page xiii, line 13, after the word *of* insert y^5 .

In page xxviii, line 13, instead of $\frac{z}{a} \times r - r^t \times \frac{z}{a} 1$, read $\frac{z}{a} \times r - r^t =$
 $\frac{z}{a} - 1$.

In page xxxi, line 2nd from the bottom, instead of $+ \frac{t-2}{3} \times v^3$, read $\times$
 $\frac{t-2}{3} \times v^3$.

In page xxxii, line 2, instead of $t + \frac{t-1}{2} \times vv$, read $t \times \frac{t-1}{2} \times vv$.

In page xxxiii, last line, instead of $\frac{t-1}{12} \times vv$, read $\frac{t+1}{12} \times vv$.

In page xxxiv, line 4, instead of $\frac{t-1}{12} \times vv$, read $\frac{t+1}{12} \times vv$.

And in the same page xxxiv, line 6th from the bottom, instead of $\frac{12}{t-1}$, read
 $\frac{12}{t+1}$.

In page xxxvi, line 5, instead of $\frac{-21^3 + 4 \times 21^2 - 2 - 6}{1440}$, read
 $\frac{-21^3 + 4 \times 21^2 - 2 - 6}{1440}$.

In page xxxvii, last line, instead of $\frac{6x-3at}{at^3-3at^2+2at}$, read $\frac{6ax-6at}{at^3-3at^2+2at}$.

In page xxxviii, line 10, instead of $\sqrt{gg+ee}$, read $\sqrt{g+ee}$.

In page xlv, line 3, instead of $g + ee^{\frac{1}{2}}$, read $\overline{g + ee^{\frac{1}{2}}}$.

In page liv, line 3d from the bottom, instead of 1.190,909,0 r^{21} , read 1.090,909,0 r^{21} .

In page lxviii, lines 5th and 6th from the bottom, instead of r , read r^{10} .

In page lxix, lines 6 and 11, instead of r , read r^{10} .

In page lxx, line 16, instead of $\frac{1}{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}^m$, read $\frac{1}{az + bz^2 + cz^3 + dz^4 + ez^5 + fz^6 + \&c}^m$.

In page lxxiv, line 4, instead of *ome* read *some*.

And in the same page lxxiv, line 13, instead of $\left(\frac{z}{at}\right)^{\frac{1}{t+1}}$, read $\left(\frac{z}{at}\right)^{\frac{1}{t-1}}$.

In page lxxx, line 1, instead of $\sqrt{GG - H}$ read $b \times \sqrt{GG - H}$.

In page lxxxii, line 5, before the word *quantity* insert the word *the*.

In page lxxxiv, line 4, instead of *computating* read *computing*.

In page cii, line 11, instead of $\frac{m}{m+n+}$ read $\frac{m}{m+n+1}$.

In page cxvii, line 5, instead of $\overline{1.00}^9$ and $\overline{1.00}^{10}$, read $\overline{1.06}^9$ and $\overline{1.06}^{10}$.

In page cxxv, lines 11 and 12, instead of *fifth* read *sixth*.

In page cxxvi, lines 9 and 10, instead of *sixth* read *seventh*.

In page cxxviii, line 17, instead of *sixth* read *seventh*.

And in the same page cxxviii, lines 18 and 19, instead of *seventh* read *eighth*.

In page cxxix, lines 1 and 2, instead of *eighth* read *ninth*.

And in the same page cxxix, lines 10 and 11, instead of *ninth* read *tenth*.

And again in the same page cxxix, lines 5th and 6th from the bottom, instead of *tenth* read *eleventh*.

In page cxxx, line 16, instead of *eleventh* read *twelfth*.

In page cxxxii, line 1, instead of *twelfth* read *thirteenth*.

In page cxxxviii, line 20th, instead of 1.061.293, read 1.061,293.

And in the same page cxxxviii, the last line, instead of *three* read *four*.

In page cxl, line 9, instead of $\frac{8.253,931}{8}$, read $\frac{8.253,938}{8}$.

In page cli, line 4th from the bottom, instead of *thirteenth* read *fourteenth*.

IN THE BODY OF THE WORK.

In page 5, line 8, instead of $\frac{1}{n} - 6 = Gx^7$, read $\frac{1}{n} - 6 \times Gx^7$.

In page 19, line 4th from the bottom, instead of $-bCx^3$, read $+bCx^3$.

In page 25, line 4th from the bottom, instead of $\frac{1}{1+x^{\frac{1}{n}}}$, read $\frac{1}{1+x^{\frac{1}{n}}}$.

In page 48, line 16 from the bottom, dele the two commas after Nx^{12} and Ox^{13} .

In page 49, line 12, instead of $1 + x)^{\frac{-1}{n}}$, read $1 + x)^{\frac{1}{n}}$.

In page 77, line 7th from the bottom, instead of $6nFx^4 - 7nGx^5$, read $6nGx^4 - 7nHx^5$.

In page 87, in the title of the *Analysis Fluxionum*, by Dr. Hales, instead of *auctore Gulielmo Hales, D. D. Rectore de Killeandra*, read *auctore Gulielmo Hales, S. T. P. seu sanctæ Theologiæ Professore, Rectore ecclesiæ de Killeandra*.

N. B. Several corrections of the errors in this Tract, that have been sent over by Dr. Hales himself, are printed at the end of this volume, in pages 847 and 848.

In page 227, line 11 from the bottom, instead of 7.301,351,1, read 7.401,351,1.
And in the same page 227, line 9 from the bottom, the figures of the number 1120970 ought to be placed one figure farther to the left-hand, thus;

$$L. \frac{z+a}{z} = 0.022,419,4$$

$$t = \frac{59}{2017746}$$

$$t. L. \frac{z+a}{z} = \frac{1120970}{1.322,744,6}$$

In page 228, line 6 from the bottom, instead of $b = 0$, read $b = 0.3$.

In page 261, line 6 from the bottom, instead of $-b$, read $-c$.

In page 266, lines 4 and 5, instead of 967.937,676,6, read 967.937,671,6.

In page 274, line 5, instead of $g + ee^{\frac{-1}{2}}$, read $g + ee^{\frac{1}{2}}$.

In page 283, line 4, instead of 741, read 1741,

In page 287, line 8 from the bottom, instead of $t + 1 \times \frac{t}{2} \times c^{t-1} \times w^2$,

read $\overline{t+1} \times \frac{t}{2} \times c^{t-1} \times w^2$.

In page 321, line 2, dele *in*.

In page 327, line 3d, instead of $3884.285,657,3 \times v^6$, read $3844.285,657,3 \times v^6$.

In page 333, line 3d from the bottom, instead of $\overline{bb-2by}^{\frac{1}{2}}$, read $-(\overline{bb-2by})^{\frac{1}{2}}$.

In page 345, lines 3, 7, 13, and 18, instead of TT in the numerator of the fraction $\frac{4+6T+TT}{3T^3}$, read 2TT.

In page 245, line 3, instead of $\frac{TT+6T+4}{24}$, read $\frac{2TT+6T+4}{24}$.

In page 352, line 7, instead of 0.8 5,545,5, read 0.825,545,5.

And in the same page 352, line 13, instead of 39.998,792,3 $\frac{1}{2}$, read 39.997,

792,3 $\frac{1}{2}$, and instead of $\begin{matrix} \text{£.} & \text{s.} & \text{d.} \\ 39, & 19, & 10\frac{1}{4} \end{matrix}$, read $\begin{matrix} \text{£.} & \text{s.} & \text{d.} \\ 39, & 19, & 11\frac{1}{2} \end{matrix}$.

In page 357, line 8, instead of $+r^{t+x+1}$, read $-r^{t+x+1}$.

In page 362, line 10, instead of $\frac{z}{a}$, read $\frac{z}{a} - 1$.

In page 366, line 13, instead of $M \times \frac{m-1}{2}$, read $m \times \frac{m-1}{2}$.

In page 377, lines 9, 11, and 13 from the bottom, instead of 33,297.095,903,04 $\frac{1}{2}$, read 33,297.055,903,04 $\frac{1}{2}$.

In page 378, line 2, instead of 33,297.095,903,04, read 33,297.055,903,04.

N.B. This error affects the calculations in several of the succeeding pages: but, as it enters at the seventh place of figures, it is not of much consequence.

In page 398, at the beginning of line 11, insert the word *We*.

In page 404, line 4 from the bottom, instead of *first near value of 1*, read *first near value of r*.

In page 407, the Algebraïck expression $\overline{bb+2by}^{\frac{1}{2}}$, is not clearly printed.

In page 440, line 13, instead of $+x^{m+n+1}$, read $-x^{m+n+1}$.

In page 461, line 9 from the bottom, instead of 0.218, read 0.0218; and instead of 1.0665, read 1.0655.

And in the same page 461, line 4 from the bottom, instead of 1.0665, read 1.0655.

N.B. This error enters into the calculations of the three following pages, and renders them inaccurate.

In

In page 482, line 5 from the bottom, instead of 0.090,090,9, read 0.090,909,0.

In page 485, line 10 from the bottom, instead of 1.055,472,0, read 1.055,372,0.

In page 487, lines 11, 13, 21, and 23, instead of $\frac{x}{x+t+1}$, read $\frac{x}{x+t+1} \times$

$$\frac{a}{z}.$$

In page 488, lines 2, 4, and 5, instead of $2 \times$, read $2 \vee \times$.

In page 492, lines 18, 20, and 25, instead of 1.0665, read 1.0655.

In page 499, last line, instead of $\frac{r^t - \sqrt{r^t + 1}}{r^t \times r - 1}$, read $\frac{r^t - \sqrt{r^t - 1}}{r^t \times r - 1}$.

In page 507, line 2, instead of $1.001,332,2 - w)^{41}$, read $1.061,332,2 - w)^{41}$.

In page 515, line 16 from the bottom, instead of $\frac{x}{x+t+1}$, read $\frac{x}{x+t+1}$

$$\times \frac{a}{z}.$$

In page 544, line 7 from the bottom, instead of $\sqrt{H - 2G - 1} \times H$,
read $\sqrt{H - 2 \times H - 1} \times H$.

In page 545, line 4, instead of $\sqrt{2D - 1 + E} \times E$, read
 $\sqrt{2 \times D - 1 + E} \times E$.

In page 546, line 3, instead of $\frac{L, m + L, p}{L, r + 1}$, read $\frac{L, m - L, p}{L, r + 1}$.

In page 557, line 3, instead of r , read $r + 1$.

In page 593, line 9th from the bottom, instead of $\frac{1}{2} \times rr$, read $\frac{1}{3} \times rr$.

In page 609, line 10 from the bottom, instead of $RRR : R^4 : R^4 : R^5 : \&c$,
read $RRR : R^4 : : R^4 : R^5 : : \&c$.

In page 616, line 12th, the words "2nd Divisor" are wanting before the
decimal figures ,00549466.

And in the same page 616, line 14, instead of 2d Divisor, read 3d Divisor;
and in line 16, instead of 3d Divisor, read 4th Divisor.

In page 627, line 3, instead of $U - \frac{U}{R}$, read $U - \frac{U}{R^t}$.

In page 660, last line, instead of 79,64, read 75,64.

In page 665, line 2d from the bottom, instead of 8.778151, read 8.778151.

In page 666, line 2d from the bottom, instead of 0.45, read 0.54.

And in the same page, last line, instead of 1.45, read 1.54.

In

In page 676, line 2 from the bottom, instead of the number 633,264, read 633,364; and instead of the Logarithm 2.801585, read 2.801654; and in the same page 676, instead of 1.400,792, read 1.400,827.

In page 678, line 9 from the bottom, instead of 22.95, read 22.59.

In page 681, line 12 from the bottom, instead of 0,0125, read 0,025.

In page 701, line 7, instead of 9.838,521, read $\bar{9}$.838,521.

In page 719, line 3, instead of the word "Sum," read "Product."

In page 720, line 18 from the bottom, instead of the word "Sum," read "Product."

In page 740, line 13, instead of $T = 0.7494$, read $T = 0.747,94$.

In page 781, line 7, in some copies the figure 9 in the number 1.056,809,4 does not appear.

In page 790, line 13, instead of $\frac{8.253,931}{8}$, read $\frac{8.253,938}{8}$.

In page 791, line 4, at the end, after the word "logarithm" insert the word "of".

In page 794, line 11, instead of the words "by reducing them," read "by reducing it."

In the same page 794, at the beginning of the last line, after $1 \times R^2$ insert $- R^{+1}$.

In page 796, line 12, instead of 1.077,977 R^{21} , read 1.077,997 R^{21} .

In page 813, line 5 from the bottom, instead of 470.206,449,3, read 470.206,499,3.

In page 826, line 6, instead of 1.646, read 1.0646.

In page 828, line 2, instead of 0.000,074, read 0.000,064; and instead of "the 14,324th part," read "the 16,562nd part."

And in the same page 828, at the beginning of line 7, dele the word "for."

In page 831, line 13, in the denominator of the second fraction, instead of 0.243,097,6, read 0.243,097,9.

In page 839, line 15, instead of 0.0214,214, read 0.214,214.

And in the same page 839, line 5 from the bottom, instead of 3.090,504,7, read 3.090,514,7.

In page 840, lines 15 and 16, instead of 1.604, read 1.0604.

F I N I S.

